

# 13 European Option Pricing

## Answers to Questions and Problems

1. What is binomial about the binomial model? In other words, how does the model get its name?

The binomial model is binomial because it allows for two possible stock price movements. The stock can either rise by a certain amount or fall by a certain amount. No other stock price movement is possible.

2. If a stock price moves in a manner consistent with the binomial model, what is the chance that the stock price will be the same for two periods in a row? Explain.

There is no chance. In every period, the stock price will either rise or fall. Therefore, in two adjacent periods, the stock price cannot be the same. From this period to the next, the stock price must necessarily rise or fall. However, the stock price can later return to its present price. This depends on the up and down factors for the change in the stock price.

3. Assume a stock price is \$120, and in the next year, it will either rise by 10 percent or fall by 20 percent. The risk-free interest rate is 6 percent. A call option on this stock has an exercise price of \$130. What is the price of a call option that expires in one year? What is the chance that the stock price will rise?

Our data are:

$$C_u = \$2$$

$$C_d = \$0$$

$$US = \$132$$

$$DS = \$96$$

$$R = 1.06$$

$$B^* = \frac{(C_u D - C_d U)}{[(U - D)R]} = \frac{2(0.8) - 0(1.1)}{(1.1 - 0.8)(1.06)} = \$5.03$$

$$N^* = \frac{C_u - C_d}{(U - D)S} = \frac{2 - 0}{(1.1 - 0.8)120} = 0.0556$$

Therefore,  $C = 0.0556(\$120) - \$5.03 = \$1.64$ . The probability of a stock price increase is:

$$(R - D)/(U - D) = (1.06 - 0.8)/(1.1 - 0.8) = 0.8667$$

4. Based on the data in question 3, what would you hold to form a risk-free portfolio?

Because  $C = N^*S - B^*$ , the portfolio of  $C - N^*S + B^*$  should be a riskless portfolio.

5. Based on the data in question 3, what will the price of the call option be if the option expires in two years and the stock price can move up 10 percent or down 20 percent in each year?

Terminal stock prices in two periods are given as follows:  $UUS = \$145.20$ ,  $DDS = \$76.80$ , and  $UDS = DUS = \$105.60$ . The probabilities of these different terminal stock prices are:  $\pi_{uu} = (0.8667)(0.8667) = 0.7512$ ;  $\pi_{ud} = (0.8667)(0.1333) = 0.1155$ ;  $\pi_{du} = (0.1333)(0.8667) = 0.1155$ ; and  $\pi_{dd} = (0.1333)(0.1333) = 0.0178$ . The call price at expiration equals the terminal stock price minus the exercise price of \$100, or zero, whichever is larger. Therefore, we have  $C_{uu} = \$15.20$ ,  $C_{dd} = 0$ ,  $C_{du} = C_{ud} = 0$ .

We have already found that the probability of an increase is 0.8667, so the probability of a down movement is 0.1333. Because the option pays off only with two increases, we need consider only that path. Thus, the value of the call is:

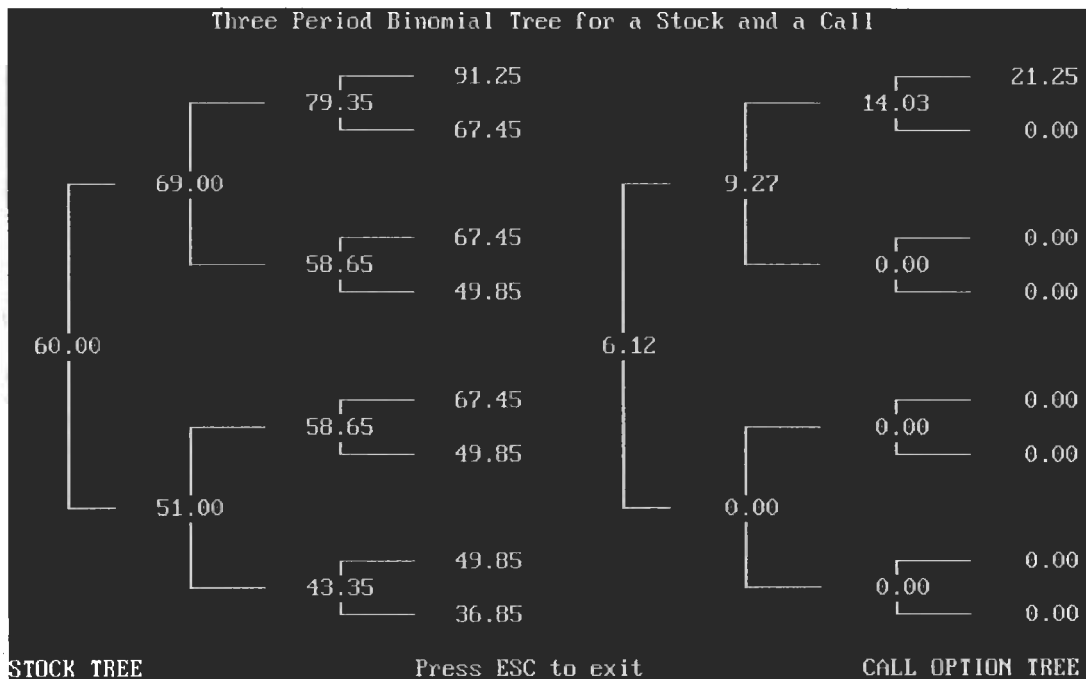
$$C = \pi_{uu}C_{uu}/R^2 = (0.7512)(\$15.20)/(1.06)^2 = \$10.16$$

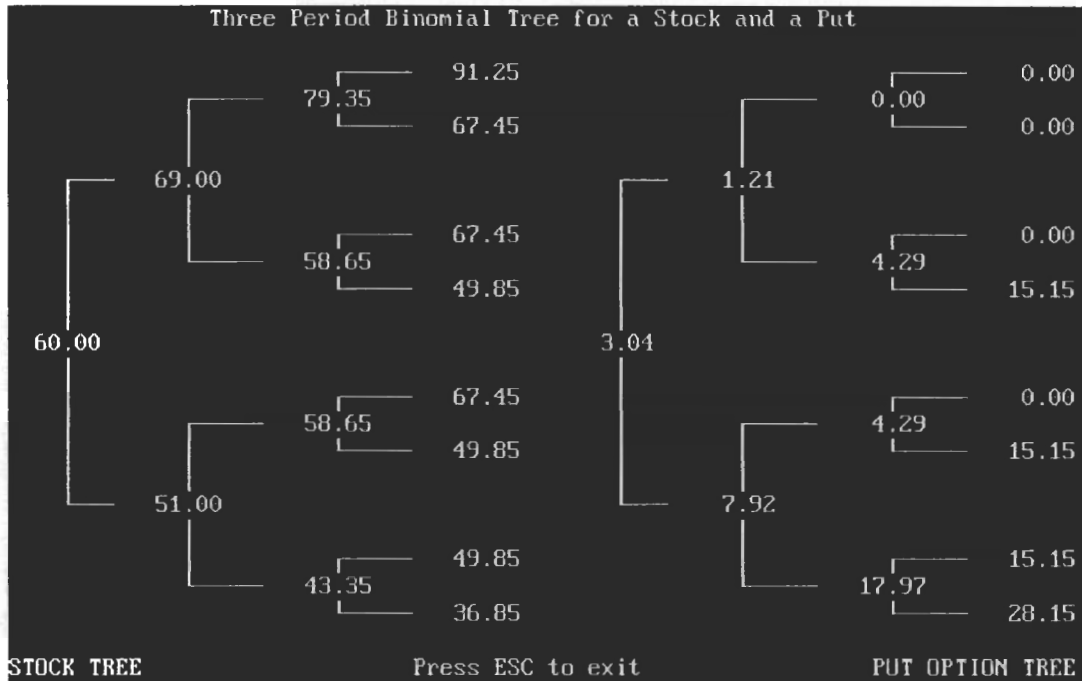
6. Based on the data in question 3, what would the price of a call with one year to expiration be if the call has an exercise price of \$135? Can you answer this question without making the full calculations? Explain.

From question 3, we see that  $US = \$132$ . This is not enough to bring the call into-the-money. Therefore, we know that the call must expire worthless, so its current price is zero.

7. A stock is worth \$60 today. In a year, the stock price can rise or fall by 15 percent. If the interest rate is 6 percent, what is the price of a call option that expires in three years and has an exercise price of \$70? What is the price of a put option that expires in three years and has an exercise price of \$65? (Use **OPTION!** to solve this problem.)

The call is worth \$6.12 and the put is worth \$3.04. The two trees from **OPTION!** are shown here:





8. Consider our model of stock price movements given in Equation 13.8. A stock has an initial price of \$55 and an expected growth rate of 0.15 per year. The annualized standard deviation of the stock's return is 0.4. What is the expected stock price after 175 days?

Substituting values for our problem, and realizing that the expected value of a drawing for a  $N(0,1)$  distribution is zero, gives:

$$S_{t+1} - S_t = \$55(0.15)(175/365) = \$3.96$$

Adding this amount to the initial stock price of \$55 gives \$58.96 as the expected stock price in 175 days.

9. A stock sells for \$110. A call option on the stock has an exercise price of \$105 and expires in 43 days. If the interest rate is 0.11 and the standard deviation of the stock's returns is 0.25, what is the price of the call according to the Black-Scholes model? What would be the price of a put with an exercise price of \$140 and the same time until expiration?

$$d_1 = \frac{\ln\left[\frac{110}{105}\right] + [0.11 + 0.5(0.25)(0.25)]\left[\frac{43}{365}\right]}{0.25\sqrt{\frac{43}{365}}} = .7361$$

$$d_2 = d_1 - \sigma\sqrt{t} = .7361 - 0.25\sqrt{\frac{43}{365}} = .6503$$

From the **OPTION!** software,  $N(0.7361) = 0.769165$  and  $N(0.6503) = 0.742251$ . Therefore,

$$c = \$110(0.769165) - \$105e^{-0.11(43/365)}(0.742251) = \$7.6752$$

For the put option with  $X = \$140$ :

$$d_1 = \frac{\ln\left[\frac{110}{140}\right] + [.11 + .5(.25)(.25)]\left[\frac{43}{365}\right]}{.25\sqrt{\frac{43}{365}}} = -2.6177$$

$$d_2 = d_1 - \sigma\sqrt{t} = -2.6177 - .25\sqrt{\frac{43}{365}} = -2.7035$$

The Black–Scholes put pricing model is:

$$p_t = Xe^{-r(T-t)}N(-d_2) - S_tN(-d_1)$$

$N(2.7035) = 0.996569$  and  $N(2.6177) = 0.995574$ . Therefore,

$$p_t = \$140e^{-0.11(43/365)}(0.996569) - \$110(0.995574) = \$28.21$$

10. Consider a stock that trades for \$75. A put and a call on this stock both have an exercise price of \$70 and they expire in 150 days. If the risk-free rate is 9 percent and the standard deviation for the stock is 0.35, compute the price of the options according to the Black–Scholes model.

$$d_1 = \frac{\ln\left[\frac{75}{70}\right] + [.09 + .5(.35)(.35)]\left[\frac{150}{365}\right]}{.35\sqrt{\frac{150}{365}}} = .5845$$

$$d_2 = d_1 - \sigma\sqrt{t} = .5845 - .35\sqrt{\frac{150}{365}} = .3601$$

$N(0.5845) = 0.720558$ ,  $N(0.3601) = 0.640614$ ,  $N(-0.5845) = 0.279442$ , and  $N(-0.3601) = 0.359386$ .

$$c_t = \$75(0.720558) - \$70e^{-0.09(150/365)}(0.640614) = \$54.04 - \$43.21 = \$10.83$$

$$p_t = \$70e^{-0.09(150/365)}(0.359386) - \$75(0.279442) = \$24.24 - \$20.96 = \$3.29$$

11. For the options in question 10, now assume that the stock pays a continuous dividend of 4 percent. What are the options worth according to Merton's model?

$$d_1^M = \frac{\ln\left[\frac{75}{70}\right] + [.09 - .04 + .5(.35)(.35)]\left[\frac{150}{365}\right]}{.35\sqrt{\frac{150}{365}}} = .5113$$

$$d_2^M = d_1^M - \sigma\sqrt{t} = .5113 - .35\sqrt{\frac{150}{365}} = .2869$$

$N(0.5113) = 0.695430$ ,  $N(0.2869) = 0.304570$ ,  $N(-0.5113) = 0.612906$ , and  $N(-0.2869) = 0.387094$ .

$$c_t^M = e^{-0.04(150/365)}\$75(0.695430) - \$70e^{-0.09(150/365)}(0.612906) = \$9.96$$

$$p_t^M = \$70e^{-0.09(150/365)}(0.387094) - \$75e^{-0.04(150/365)}(0.304570) = \$3.64$$

12. Consider a Treasury bill with 173 days until maturity. The bid and asked yields on the bill are 9.43 and 9.37. What is the price of the T-bill? What is the continuously compounded rate on the bill?

From the text, we have:

$$\begin{aligned} P_{TB} &= 1 - .01 \left( \frac{B + A}{2} \right) \left( \frac{\text{Days until Maturity}}{360} \right) \\ &= 1 - .01 \left( \frac{9.43 + 9.37}{2} \right) \left( \frac{173}{360} \right) \\ &= .9548 \end{aligned}$$

Therefore, the price of the bill is 95.48 percent of par. To find the corresponding continuously compounded rate, we solve the following equation for  $r$ :

$$\begin{aligned} e^{r(T-t)} &= 1/PTB \\ e^{r(173/365)} &= 1/.9548 \\ r &= 0.0962 \end{aligned}$$

Thus, the continuously compounded rate on the bill is 9.62 percent.

13. Consider the following sequence of daily stock prices: \$47, \$49, \$46, \$45, \$51. Compute the mean daily logarithmic return for this share. What is the daily standard deviation of returns? What is the annualized standard deviation?

Let  $P_t$  = the price on day  $t$ ,  $PR_t = P_t/P_{t-1}$ ,  $PR_\mu$  = the mean daily logarithmic return, and  $\sigma$  = the standard deviation of the daily logarithmic return. Then,

$PR_t$	$\ln(PR_t)$	$[\ln PR_t - PR_\mu]^2$
1.0426	0.0417	.000454
0.9388	-0.0632	.006989
0.9783	-0.0219	.001789
1.1333	0.1251	.010962

$$PR_\mu = (0.0417 - 0.0632 - 0.0219 + 0.1251)/4 = 0.0204$$

$$\text{VAR}(PR) = (1/3)(0.000454 + .006989 + .001789 + .010962) = 0.006731$$

$$\sigma = \text{the square root of } 0.006731 = 0.082045.$$

The annualized  $\sigma = \sigma$  times the square root of 250 = 1.2972.

14. A stock sells for \$85. A call option with an exercise price of \$80 expires in 53 days and sells for \$8. The risk-free interest rate is 11 percent. What is the implied standard deviation for the stock? (Use **OPTION!** to solve this problem.)

$\sigma = 0.332383$ . It is also possible to find this value by repeated application of the Black-Scholes formula. For example, with this option data, different trial values of  $\sigma$  give the following sequence of prices:

Call Price	Trial Value of $\sigma$	$\sigma$ is:
\$6.29	.1	too low
9.84	.5	too high
7.67	.3	too low
8.72	.4	too high
8.18	.35	too high
7.98	.33	too low
8.03	.335	too high
8.01	.333	too high
8.00	.332	very close

The **OPTION!** program follows a search like this to zero in on the correct value of  $\sigma$  so that the resulting option price matches the actual price.

15. For a particular application of the binomial model, assume that  $U = 1.09$ ,  $D = 0.91$ , and that the two are equally probable. Do these assumptions lead to any particular difficulty? Explain. (Note: These are specified up and down movements and are not intended to be consistent with the Black–Scholes model.)

Note that  $0.5(1.09) + 0.5(0.91) = 1.0$ , so the expected return on the stock is zero. The expected return on the stock must equal the risk-free rate in the risk-neutral setting of the binomial model. Therefore, these up and down factors imply a zero interest rate.

16. For a stock that trades at \$120 and has a standard deviation of returns of 0.4, use the Black–Scholes model to price a call and a put that expire in 180 days and that have an exercise price of \$100. The risk-free rate is 8 percent. Now assume that the stock will pay a dividend of \$3 on day 75. Apply the known dividend adjustment to the Black–Scholes model and compute new call and put prices.

With no dividends, the call price is \$27.54, and the put price is \$3.67. With the known dividend adjustment, the call price is \$25.14, and the put price is \$4.23.

17. A call and a put expire in 150 days and have an exercise price of \$100. The underlying stock is worth \$95 and has a standard deviation of 0.25. The risk-free rate is 11 percent. Use a three-period binomial model and stock price movements consistent with the Black–Scholes model to compute the value of these options. Specify  $U$ ,  $D$ , and  $\pi_u$ , as well as the values for the call and put.

The call price is \$5.80, and the put price is \$6.38.  $U = 1.0969$ , and  $D = 0.9116$ .

$$\pi_u = \frac{e^{-11[50/365]} - .9116}{1.0969 - .9116} = .5590$$

18. For the situation in problem 17, assume that the stock will pay 2 percent of its value as a dividend on day 80. Compute the value of the call and the put under this circumstance.

Recalling that these are European options, the call is worth \$4.66, and the put is worth \$7.14.

19. For the situation in problem 17, assume that the stock will pay a dividend of \$2 on day 80. Compute the value of the call and the put under this circumstance.

The call is worth \$4.62, and the put is worth \$7.16.

20. Consider the first tree in Figures 13.10 and 13.12. If the stock price falls in both of the first two periods, the price is \$65.59. For the first tree in Figure 13.12, the put value is \$8.84 in this case. Given that the exercise price on the put is \$75, does this present a contradiction? Explain.

The apparent contradiction arises because the intrinsic value of the put is  $\$75 - \$65.59 = \$9.41$ , which exceeds the put price of \$8.84. However, because this is a European put, it cannot be exercised to capture the intrinsic value prior to expiration. Thus, the European put price can be less than the intrinsic value.

21. Consider the second tree in Figures 13.10 and 13.11. If the stock price increases in the first period, the price is \$88.35. For the second tree in Figure 13.11, the call price is \$12.94 in this case. Given that the exercise price on the put is \$75, does this present a contradiction? Explain.

One of the arbitrage conditions we have considered says that the call price must equal or exceed  $S - X$ . In this situation,  $S - X = \$88.35 - \$75.00 = \$13.35$ , which is greater than the call price of \$12.94. Thus, it appears that an arbitrage opportunity exists. The apparent contradiction dissolves when we realize that the call price reflects the dividend that will occur before the option can be exercised.

22. As a cost-cutting measure, your CFO, an accountant, decides to cancel your division's subscription to Bloomberg. You rely on Bloomberg for real-time quotes on Treasury bill prices to value options using the Black-Scholes option pricing model. You discover that you can obtain real-time quotes on commercial paper rates from Reuters, to which you still have a subscription. Discuss the implications of using the yield on AAA rated commercial paper instead of Treasury bill yields to value options using the Black-Scholes model.

It is assumed that the debt of the United States government is free of default risk. The yield on commercial paper, which is an unsecured debt obligation of a corporation, includes a default risk premium. Thus, the return on AAA rated commercial paper,  $r_{cp}$ , will be greater than the return on an equivalent Treasury bill,  $r_{T-bill}$ . The difference in the two returns,  $r_{cp} - r_{T-bill}$ , is equal to the default risk premium. Using the commercial paper rate instead of the Treasury bill rate will lead to the systematic overvaluation of call options, and the undervaluation of put options. The magnitude of the pricing error will be a function of the magnitude of the default risk premium. However, Chapter 14 shows that option prices are not very sensitive to changes in interest rates. Consequently using the commercial paper rate rather than the Treasury bill rate should not produce significant differences in the prices of options valued with the Black-Scholes model.

23. Assume that stock returns follow a random walk with a drift equal to the expected return on the stock. You are modeling stock returns using a binomial process for the purpose of valuing a European call option. Explain why creating an initial position in the stock and the call option that will remain riskless for the entire life of the option is not possible.

Valuing options using the binomial model requires the construction of a synthetic option. The synthetic option is nothing more than a levered stock position. That is, a portfolio with an appropriate investment in the stock underlying the option,  $N^*$ , and Treasury bills,  $B^*$ . The synthetic option is combined with the traded option to create a riskless hedge portfolio. When the price of the underlying stock changes, the value of the traded option changes. Maintaining the riskless hedge requires periodic rebalancing of the synthetic option position as the value of the underlying stock changes. That is, one must alter one's position in the underlying stock and Treasury bill as the value of the traded option changes in concert with changes in the price of the stock.

24. WMM is currently priced at \$117.50 per share. The 50-day options on WMM are currently being traded at three different strike prices, \$110, \$115, and \$120. The 50-day Treasury bill is priced to yield an adjusted annual return of 6 percent compounded continuously. The prices and implied volatilities for the three different options are shown below.

Strike	Call	Implied Volatility
\$110	\$8.50	0.16
115	5.375	0.21
120	3.75	0.26

If the WMM options are priced by the Black-Scholes model, then the implied volatility of each of the 50-day option contracts would be the same.

- A. Which of the three implied volatilities would you use as an estimate for the true volatility?

The recommended practice in the literature is to calculate a weighted average of several estimates of implied volatility. Many factors will influence the decision regarding the appropriate weighting scheme used in calculating a weighted implied volatility, including the trading volume for a particular option. Most weighting systems assign greater weights to options near-the-money.

- B. If you knew that the true volatility of the stock was 0.20, what could you say about the value of the call options? What action would you take upon observing the implied volatilities shown in this table?

If the true volatility of the stock were 0.20, then the \$110 call would be undervalued, the \$120 call overvalued, and the \$115 call would be near its true value. We would expect that the prices of the options would

adjust to a level consistent with an implied volatility of 0.2. Thus, we would expect the price of the \$110 call to rise and the price of the \$120 call to fall. Our trading strategy would be to purchase the \$110 call options and sell the \$120 call options.

25. The dominant asset pricing models in finance maintain that the price of a share of stock depends on the amount of nondiversifiable covariation risk intrinsic in a stock. That is, the market only prices covariation risk that cannot be costlessly diversified away by shareholders. Only the nondiversifiable segment of total risk is priced by the market. The Black–Scholes model argues that the value of an option contract depends on the total variability of a stock's return. Reconcile these apparent inconsistencies in pricing theory. Should option prices depend only on the level of nondiversifiable risk? If not, explain.

When valuing the underlying asset, an investor has the opportunity to reduce risk exposure by constructing a diversified portfolio. This risk reduction by diversification can be achieved at no cost to the investor. Thus, no investor would pay to avoid the diversifiable risk. Consequently, diversifiable risk is not priced in a competitive market. An option is a derivative asset. The value of the option at any point in time depends on the value of the underlying asset at that time. However, the value of the option also varies with changes in the value of the underlying asset that may occur in the future. That is, the value of the option is equal to the present value of the expected payoffs on the option, and the expected payoffs on the option are determined by the volatility of the underlying asset.

26. Solving for the implied volatility of an option “by hand” is a laborious and time-consuming process of trial and error. The process requires you to choose a value for the option's implied volatility and calculate the value of the option using this guess. You compare the calculated value of the option with the value of the option observed in the market, and based on the direction and magnitude of the error, you develop another guess as to the value of the implied volatility. You repeat this process until the calculated value of the option is equal to the observed value of the option. Modern spreadsheets available on desktop computers have taken the labor out of the process of solving for implied volatility. Explain the process that would be used to solve for the implied volatility of an option using the Black–Scholes call option pricing model in a spreadsheet program.

If the Black–Scholes option pricing model is the true pricing model for European call options, then the observed price of traded call options should be determined according to the Black–Scholes model. Given the observed call price, the observed stock price, the observed risk-free interest rate, the strike price of the option, and the time until expiration of the option, one can solve for the volatility implied by these prices. This iterative process is laborious when performed by hand, but very simple when using a spreadsheet. In Excel, for example, one sets up a target cell that equates the value of the option calculated with the Black–Scholes model with the observed price of the option. The Solver function iterates the value of sigma, the firm's volatility, until the calculated value of the option is equal to the observed value of the option. The resulting value for sigma is the implied standard deviation.

27. We generally assume that the price of a share of stock decreases by the dollar amount of a dividend on the day when the stock goes ex-dividend, at least approximately. The ex-dividend date is the date on which the purchaser of a share of stock is not entitled to the next dividend paid by the firm. That is, the stock does not carry the right to the next dividend. Develop an arbitrage based argument why in a competitive market without frictions the price of a stock must fall by the dollar amount of the dividend on the day the stock goes ex-dividend.

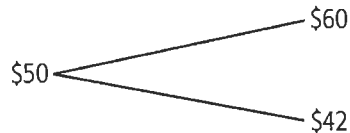
The price of a stock the instant before it goes ex-dividend,  $S_b$ , must be equal to the price of the stock the instant after it goes ex-dividend,  $S_a$ , plus the dollar value of the dividend,  $D$ . If the price of the stock before it goes ex-dividend,  $S_b$ , is greater than the sum of the price of the stock the instant after it goes ex-dividend,  $S_a$ , and the dollar value of the dividend,  $D$ , then it is possible to profit by selling the stock before it goes ex-dividend and purchasing the stock after it goes ex-dividend. Selling the stock obligates the seller to deliver a share of stock for which they are paid,  $\$S_b$ . The seller must deliver the stock and the dollar value of the dividend,  $D$ . However, the cost of the stock when purchased after the ex-dividend date,  $S_a$ , plus the dollar amount of the dividend,  $D$ , is less than the selling price,  $S_b$ , leaving the investor with a profit of



$S_b - (S_a + D) > 0$ . If the price of the stock before it goes ex-dividend,  $S_b$ , is less than the sum of the price of the stock the instant after it goes ex-dividend,  $S_a$ , and the dollar value of the dividend,  $D$ , then it is possible to profit by buying the stock before it goes ex-dividend and selling the stock after it goes ex-dividend. Purchasing the share of stock entitles the investor to the next dividend to be paid to shareholders. Selling the stock obligates the seller to deliver a share of stock for which they are paid  $S_a$ . The purchased stock will be used to deliver against the obligation created by selling the stock. The cost of the stock,  $S_b$ , is less than the sum of the cash received from selling the stock,  $S_a$ , and the dividend paid to the owners of the stock,  $(S_a + D) - S_b > 0$ . Thus, only stock prices that are consistent with  $S_b - D = S_a$  will prevent arbitrage.

28. A quick check of the wire reveals that TMS is trading at \$50 per share. Earlier in the day you were having lunch with a colleague and her husband Will, the CFO of TMS. During the lunch discussion, you talked about TMS's recently introduced new products. Will made it clear that if the products are well received by the market, TMS will be trading at \$60 in six months, and if the market does not respond to the products, TMS will be trading at \$42 in six months. The current six-month risk-free interest rate is 6 percent. Calculate the price of a six-month European call option written on TMS with a \$50 strike price. Show that the price calculated using the one-period risk-neutral pricing model,  $c = (c_U\pi_U + c_D\pi_D)/R$ , is the same as the price calculated using the single-period no-arbitrage binomial pricing model,  $c = N^*S - B^*$ .

The first step in determining the value of this call option is to examine what is expected to happen to the price of TMS's stock over the next six months. The value of the \$50 call option will depend on the movements in the price of TMS's stock. At the end of six months, two outcomes are possible. TMS's stock price will either rise to \$60 with the introduction of successful products or decrease to \$42 if the new products are not successful.



Based on these stock prices, we can then calculate the value of the call option at expiration. The value of the call option at expiration is given by the intrinsic value function,  $\text{MAX}[0, S - X]$ . With successful products, the stock price will be \$60, generating an intrinsic value of \$10,  $c_u = \text{MAX}[0, \$60 - \$50]$ . With unsuccessful products, the stock price will be \$42, generating an intrinsic value of \$0,  $c_d = \text{MAX}[0, \$42 - \$50]$ .

#### Single-period binomial model

$$\begin{aligned}
 U &= \$60/\$50 = 1.2 & D &= \$42/\$50 = 0.84 & R &= 1.06 \\
 B^* &= \frac{c_U D - c_D U}{(U - D)R} = \frac{\$10(0.84) - \$0(1.2)}{(1.2 - 0.84)1.06} = \frac{\$8.4}{0.3816} = \$22.01 \\
 N^* &= \frac{c_U - c_D}{(U - D)S} = \frac{\$10 - \$0}{(1.2 - 0.84)\$50} = \frac{10}{18} = 0.5556 \\
 c &= N^*S - B^* = 0.5556(\$50) - \$22.01 = \$5.77
 \end{aligned}$$

#### Single-period risk-neutral pricing model, $c = (c_U\pi_U + c_D\pi_D)/R$

$$\begin{aligned}
 \pi_U &= \frac{R - D}{U - D} = \frac{1.06 - 0.84}{1.2 - 0.84} = 0.61 & \pi_D &= \frac{U - R}{U - D} = \frac{1.2 - 1.06}{1.2 - 0.84} = 0.39, & \pi_U + \pi_D &= 1 \\
 c &= \frac{(c_U\pi_U + c_D\pi_D)}{R} = \frac{\$10(0.61) + \$0(0.39)}{1.06} = \$5.76
 \end{aligned}$$

The difference in the calculated values of the options is due to rounding error.

29. You are paired with the president of NYB to play golf in a tournament to raise money for the local children's hospital. After playing golf in the tournament, you learn that the president of NYB expects the price of his

firm to increase by 6 percent per quarter if their new stores are successful in Seattle. If the stores are unsuccessful, he expects the stock price of NYB to decrease 5 percent per quarter. Checking Quote.com you find NYB trading at \$30 per share. The current three-month risk-free interest rate is 3 percent, and you expect this rate to remain unchanged for the next six months.

- A. Calculate the price of a six-month European call and put option written on NYB with \$31 strike prices using the two-period risk-neutral pricing model.

$$U = 1.06 \quad D = 0.95 \quad R = 1.03$$

$$\pi_U = \frac{R - D}{U - D} = \frac{1.03 - 0.95}{1.06 - 0.95} = 0.7273 \quad \pi_D = 1 - 0.7273 = 0.2727$$

$$c = \frac{2.708 \times 0.7273^2 + 0.0 \times (0.7273 \times 0.2727) + 0.0 \times (0.2727 \times 0.7273) + 0.0 \times 0.2727^2}{1.03^2} = \$1.35$$

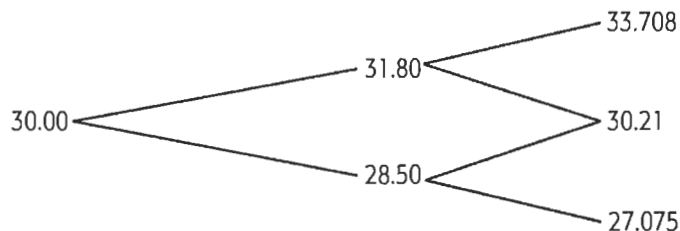
$$p = \frac{0.0 \times 0.7273^2 + 0.79 \times (0.7273 \times 0.2727) + 0.79 \times (0.2727 \times 0.7273) + 3.925 \times 0.2727^2}{1.03^2} = \$0.57$$

- B. Confirm that the put-call parity relationship generates the same price for the put option.

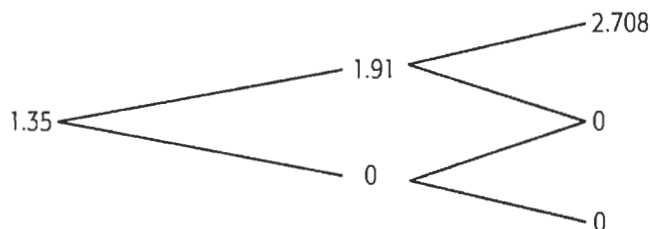
$$p = c - S + X/R^2 = \$1.35 - \$30 + \$31/1.03^2 = \$0.57$$

- C. Construct the two-period stock price tree. Working backward through the stock price tree, use the one-period risk-neutral pricing model,  $c = (c_U \pi_U + c_D \pi_D)/R$ , to calculate the value of a one-period call option, given that the stock price has increased in period one,  $c_{up}$ . Calculate the value of a one-period call option, given that the stock price has decreased in period one,  $c_{down}$ . Calculate the current value of a one-period call option that has a value of  $c_{up}$  if the stock price increases or a value of  $c_{down}$  if the stock price decreases. (This process of valuing an option is known as the recursive valuation process.) Compare the price of the call option calculated using the recursive process with the price of the call option calculated using the two-period risk-neutral pricing model. Do the same for a put.

#### Stock price tree



#### Call price tree

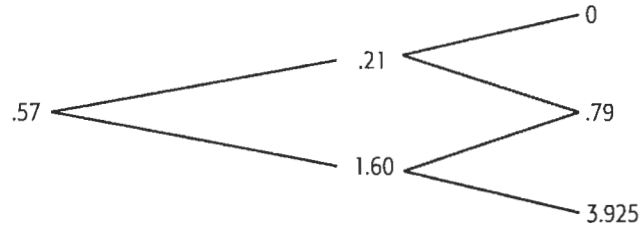


$$c_{up} = (2.708 \times 0.7273 + 0.0 \times 0.2727)/1.03 = \$1.91$$

$$c_{down} = (0.0 \times 0.7273 + 0.0 \times 0.2727)/1.03 = \$0$$

$$c = (c_{up} \times 0.7273 + c_{down} \times 0.2727)/1.03$$

$$= (1.91 \times 0.7273 + 0 \times 0.2727)/1.03 = \$1.35$$

**Put price tree**

$$p_{up} = (0.0 \times 0.7273 + .79 \times 0.2727)/1.03 = \$.21$$

$$p_{down} = (.79 \times 0.7273 + 3.925 \times 0.2727)/1.03 = \$1.60$$

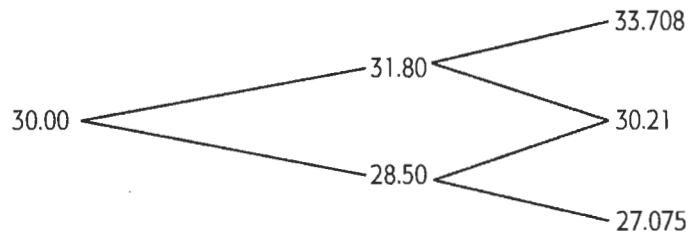
$$p = (p_{up} \times 0.7273 + p_{down} \times 0.2727)/1.03 \\ = (.21 \times 0.7273 + 1.59 \times 0.2727)/1.03 = \$.57$$

The calculated prices for the options are the same.

- D. Repeat the process of part C for a call using the single-period no-arbitrage binomial pricing model,  $c = N^*S - B^*$  to calculate the price of the single-period call option. Discuss what happens to  $N^*$  at each branch in the tree.

$$B^* = \frac{c_u D - c_d U}{(U - D)R}$$

$$N^* = \frac{c_u - c_d}{(U - D)S}$$

**Stock price tree**

If  $S = \$31.80$ ,

$$B_{up} = \frac{2.708 \times 0.95 - 0.0 \times 1.06}{(1.06 - 0.95) 1.03} = 22.71$$

$$N_{up} = \frac{2.708 - 0.0}{(1.06 - 0.95) 31.80} = 0.7742$$

$$c_{up} = N_{up}S - B_{up} = 0.7742 \times 31.80 - 22.71 = \$1.91$$

If  $S = \$28.50$ ,

$$B_{down} = \frac{0.0 \times 0.95 - 0.0 \times 1.06}{(1.06 - 0.95) 1.03} = 0$$

$$N_{down} = \frac{0.0 - 0.0}{(1.06 - 0.95) 28.50} = 0$$

$$c_{down} = 0.0 \times 28.50 - 0 = \$0$$

If  $S = \$30.00$ ,

$$B = \frac{1.91 \times 0.95 - 0.0 \times 1.06}{(1.06 - 0.95) 1.03} = 16.02$$

$$N = \frac{1.91 - 0.0}{(1.06 - 0.95)30} = 0.5788$$

$$c = 0.5788 \times 30 - 16.02 = \$1.34$$

When the price of the underlying stock changes, the value of the traded option changes, and the appropriate investment in the stock underlying the option,  $N^*$ , changes. In the binomial model, we create a mimicking portfolio consisting of a stock position and a Treasury bill position. The outcome we are modeling changes as the underlying stock price changes. Thus, we must change our stock position in the mimicking portfolio. For example, when the stock price is \$30.00, then  $N^* = 0.5788$ , and when the stock price is \$31.80, then  $N^* = 0.7742$ . Thus, as the stock price increases, we increase our holdings of the underlying stock, and as the stock price decreases, we reduce our holdings of the underlying stock.

30. Both put and call options on HWP are traded. Put and call options with an exercise price of \$100 expire in 90 days. HWP is trading at \$95 and has an annualized standard deviation of 0.3. The three-month risk-free interest rate is 5.25 percent per annum.
- A. Use a three-period binomial model to compute the value of the put and call options using the recursive procedure (single-period binomial option pricing model). Be sure to specify the values of  $U$ ,  $D$ , and  $\pi_U$ .

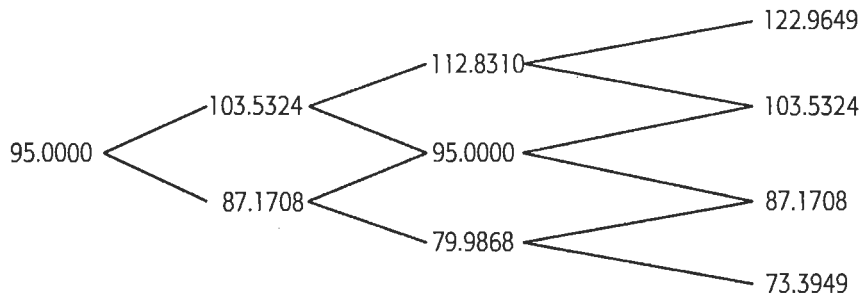
$$U = e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{30/365}} = 1.0898$$

$$D = 1/U = 1/1.0898 = 0.9176$$

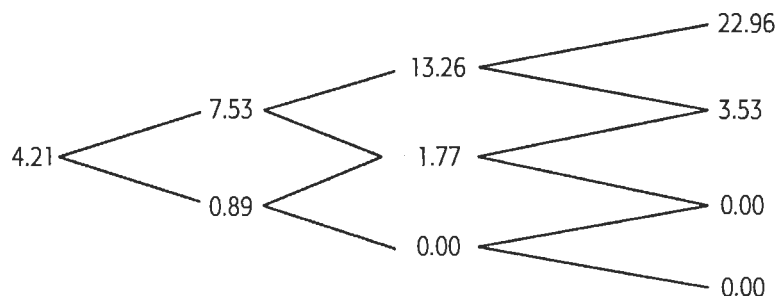
$$\pi_U = \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.0525 \times 30/365} - 0.9176}{1.0898 - 0.9176} = 0.5036 \quad \pi_D = 1 - 0.5036 = 0.4964$$

$$e^{-r\Delta t} = 0.9957$$

#### Stock price tree



#### Call price tree



#### Call price, $c = \$4.21$

- B. What is the risk-neutral probability of a stock price increase in period one, period two, and period three?

A characteristic of the multiplicative process used to model stock price movements in the binomial model is that the probability of a stock price increase in any period is independent of previous stock price changes.

This process produces prices that follow a random walk. Thus,  $\pi_U$  does not change from period to period, and  $\pi_U = 0.5036$ .

31. Consider a stock, ABM, trading at a price of \$70. Analysis of ABM's recent returns reveals that ABM has an annualized standard deviation of return of 0.4. The current risk-free rate of interest is 10 percent per annum.
- A. What is the price of a European call written on ABM with a \$75 strike price that expires in 180 days? Price the option according to the Black-Scholes model.

$$d_1 = \frac{\ln(70/75) + [0.10 + .5(0.4^2)](0.4932)}{.40 \sqrt{(0.4932)}} = 0.0704$$

$$d_2 = 0.0704 - 0.40 \sqrt{(0.4932)} = -0.2105$$

$$N(d_1) = 0.528061 \quad N(d_2) = 0.416638$$

$$c = 70 \times 0.528061 - 75e^{-0.1 \times 0.4932} \times 0.416638 = \$7.22$$

- B. What is the price of a European put written on ABM with a \$75 strike price that expires in 180 days? Price the option with the Black-Scholes model. Calculate the price of the put using put-call parity and compare it with the price calculated using the Black-Scholes model.

$$d_1 = \frac{\ln(70/75) + [0.10 + 0.5(0.4^2)](0.4932)}{0.40 \sqrt{(0.4932)}} = 0.0704$$

$$d_2 = 0.0704 - 0.40 \sqrt{(0.4932)} = -0.2105$$

$$N(-d_1) = 0.471939 \quad N(-d_2) = 0.583362$$

$$p = Xe^{-r(T-t)} N(-d_2) - S N(-d_1) = 75e^{-0.1 \times 0.4932} \times 0.583362 - 70 \times 0.471939 = \$8.61$$

#### Put-call parity

The price of the call calculated using the put-call parity equation,  $p = c - S + Xe^{-r(T-t)}$ , is the same as the put price calculated using the Black-Scholes put option pricing model.

$$p = c - S + Xe^{-r(T-t)} = \$7.22 - \$70 + \$75e^{-0.1 \times 0.4932} = \$8.61$$

32. Consider a stock with a price of \$72 and a standard deviation of 0.4. The stock will pay a dividend of \$2 in 40 days and a second dividend of \$2.50 in 130 days. The current risk-free rate of interest is 10 percent per annum.
- A. What is the price of a European call written on this stock with a \$70 strike price that expires in 145 days? Price the option with the Black-Scholes model.

To calculate the value of this option using the Black-Scholes model, we must adjust the current stock price of \$72 downward by the present value of the dividends to be received prior to the option's expiration. In this problem, both dividends are paid before the option's expiration. The first dividend will be received in 40 days, and the second will be received in 130 days.

$$D_1 Xe^{-r(T-t)} = \$2 \times e^{-0.1 \times (40/365)} = \$1.98$$

$$D_2 Xe^{-r(T-t)} = \$2.5 \times e^{-0.1 \times (130/365)} = \$2.41$$

$$S' = \$72 - \$1.98 - \$2.41 = \$67.61$$

$$d_1 = \frac{\ln(67.61/70) + [0.10 + .5(0.4^2)](0.3973)}{.40 \sqrt{(0.3973)}} = 0.1458$$

$$d_2 = 0.1458 - 0.40 \sqrt{(0.3973)} = -0.1063$$

$$N(d_1) = 0.557958 \quad N(d_2) = 0.457664$$

$$c = 67.61 \times 0.557958 - 70e^{-0.1 \times 0.3973} \times 0.457664 = \$6.93$$

- B. What is the price of a European put written on this stock with a \$70 strike price that expires in 145 days? Price the option with the Black–Scholes model.

$$d_1 = \frac{\ln(67.61/70) + ([.10 + .5(0.4^2)](0.3973))}{.40 \sqrt{(0.3973)}} = 0.1458$$

$$d_2 = 0.1458 - 0.40 \sqrt{(0.3973)} = -0.1063$$

$$N(-d_1) = 0.442042 \quad N(-d_2) = 0.542336$$

$$p = Xe^{-r(T-t)} N(-d_2) - SN(-d_1) = 70e^{-0.1 \times 0.3973} \times 0.542336 - 67.61 \times 0.442042 = \$6.60$$

33. Consider a stock that trades for \$75. A put and a call on this stock both have an exercise price of \$70, and they expire in 145 days. The risk-free rate is 9 percent per annum, and the standard deviation of the stock is 0.35. Assume that the stock pays a continuous dividend of 4 percent.

- A. What is the price of a European call written on this stock according to Merton's model?

$$d_1^M = \frac{\ln(S/X) + ([r - \delta + .5(\sigma^2)](T - t))}{\sigma \sqrt{(T - t)}}$$

$$d_2^M = d_1^M - \sigma \sqrt{(T - t)}$$

$$d_1^M = \frac{\ln(75/70) + ([.09 - 0.04 + .5(0.35^2)](0.3973))}{.35 \sqrt{.3973}} = 0.5131$$

$$d_2^M = 0.5131 - 0.35 \sqrt{.3973} = 0.2925$$

$$N(d_1^M) = .696056 \quad N(d_2^M) = 0.615045$$

$$c^M = e^{-\delta(T-t)} SN(d_1^M) - Xe^{-r(T-t)} N(d_2^M)$$

$$c^M = e^{-0.04(0.3973)} 75 \times 0.696056 - 70e^{-0.09(0.3973)} \times 0.615045 = \$9.84$$

- B. What is the price of a European put written on this stock according to Merton's model?

$$d_1^M = \frac{\ln(75/70) + ([.09 - 0.04 + .5(0.35^2)](0.3973))}{.35 \sqrt{.3973}} = 0.5131$$

$$d_2^M = 0.5131 - 0.35 \sqrt{.3973} = 0.2925$$

$$N(-d_1^M) = .303944 \quad N(-d_2^M) = 0.384955$$

$$p^M = Xe^{-r(T-t)} N(-d_2^M) - e^{-\delta(T-t)} SN(-d_1^M)$$

$$p^M = 70e^{-0.09(0.3973)} \times 0.384955 - e^{-0.04(0.3973)} 75 \times 0.303944 = \$3.56$$

34. Your broker has just told you about TXF. He describes the firm as the real innovator in the entertainment industry. You search the web and discover that TXF has both put and call options trading on the exchange. Put and call options with an exercise price of \$70 expire in 145 days. TXF is currently trading at \$75, and has an annualized standard deviation of 0.35. The three-month risk-free interest rate is 9 percent per annum. A quick, back-of-the-envelope calculation reveals that TXF is paying dividends at a continuous rate of 4 percent.

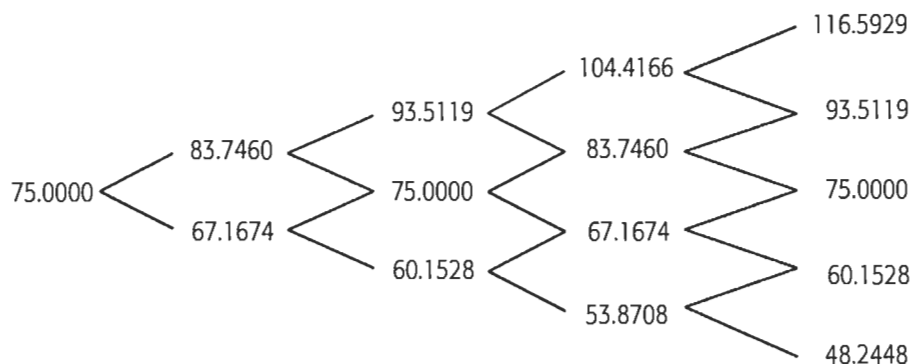
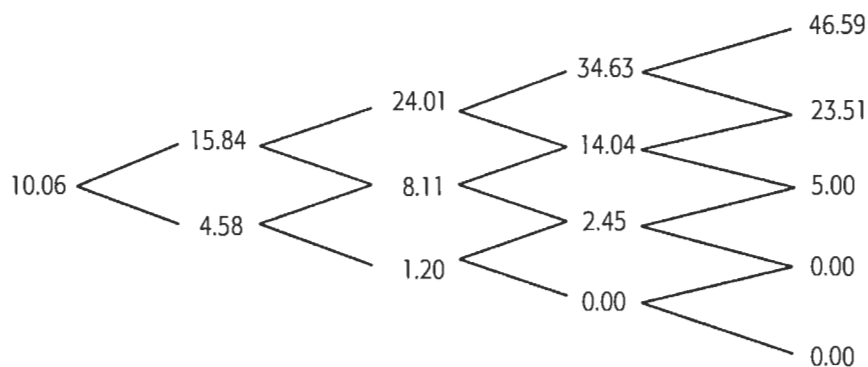
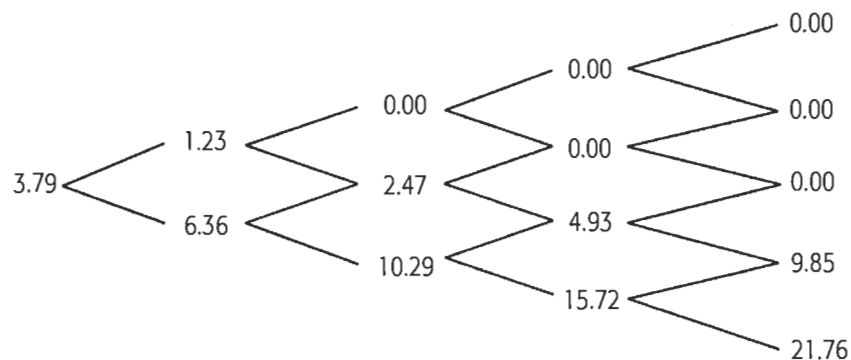
- A. Use a four-period binomial model to compute the value of both the put and call options using the recursive procedure (single-period binomial option pricing model).

$$U = e^{\sigma \sqrt{\Delta t}} = e^{.35 \sqrt{(36.25/365)}} = 1.1166$$

$$D = 1/U = 1/1.1166 = 0.8956$$

$$\pi_U = \frac{e^{(r-\delta)\Delta t} - D}{U - D} = \frac{e^{(0.09 - 0.04) \times 0.0993} - 0.8956}{1.1166 - 0.8956} = 0.4950 \quad \pi_D = 1 - 0.4950 = 0.5050$$

$$e^{-r\Delta t} = 0.9911$$

**Stock price tree****Call price tree****Put price tree**

The call price is \$10.06, and the put price is \$3.79.

- B. Compare the values of the put and call options calculated in this problem with the values of the put and call options calculated using Merton's model. Explain the source of the differences in the calculated values of the options.

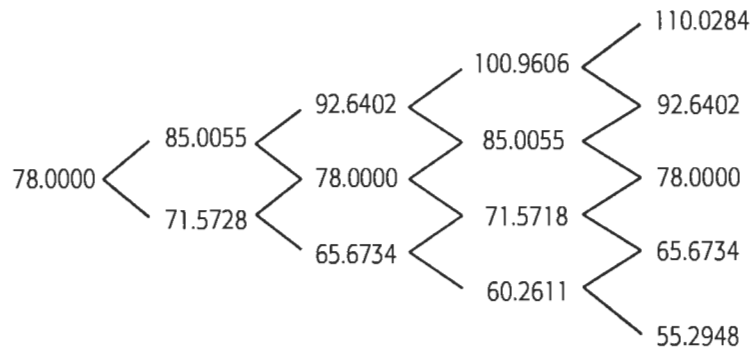
In both cases, the prices for the options calculated using the binomial model are greater than the prices calculated using Merton's model. The prices calculated using the binomial model for the call and put

options are \$10.06 and \$3.79, respectively, while the prices for the same options calculated with Merton's model are \$9.84 and \$3.56. This difference in valuation arises because we modeled a 145-day option using a four-period model. If we were to model the price movements in TXF using more periods in the binomial model, then the prices calculated using the binomial model would be very close to the prices calculated using the Merton model.

35. CSM is trading at \$78 and has an annualized standard deviation of return of 30 percent. CSM is expected to pay a dividend equal to 3 percent of the value of its stock price in 70 days. The current risk-free rate of interest is 7 percent per annum. Options written on this stock have an exercise price of \$80 and expire in 120 days.
- A. Using a four-period binomial model, calculate the value of a European put option written on CSM using the recursive procedure (single-period binomial option pricing model).

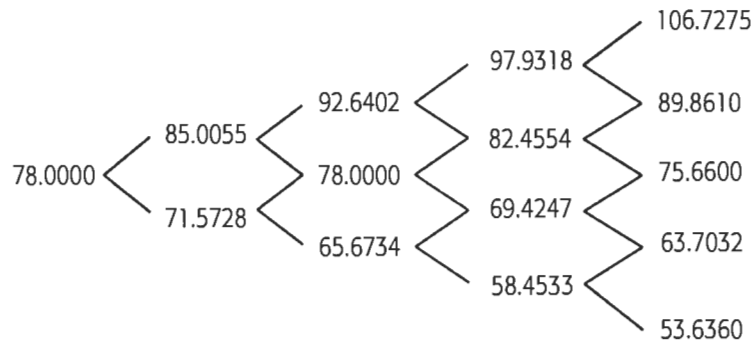
$$\begin{aligned}
 U &= e^{r\sqrt{\Delta t}} = e^{0.3\sqrt{30/365}} = 1.0898 \\
 D &= 1/U = 1/1.0898 = 0.9176 \\
 \pi_U &= \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.07 \times 30/365} - 0.9176}{1.0898 - 0.9176} = 0.5120 \quad \pi_D = 1 - 0.5120 = 0.4880 \\
 e^{-r\Delta t} &= 0.9943
 \end{aligned}$$

#### Stock price tree

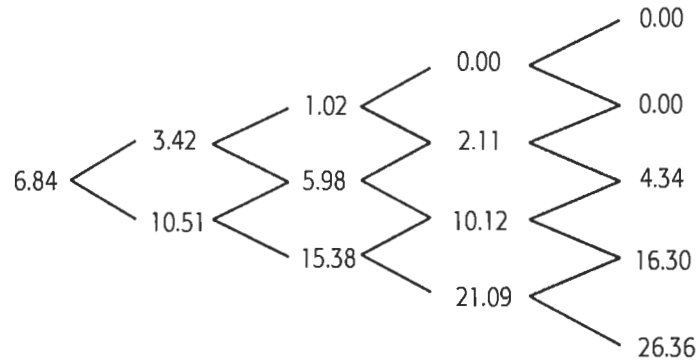


The stock prices in the tree must be adjusted for the dividend to be paid in 70 days before we can calculate the value of the options. Therefore, the stock prices in the tree in periods three and four must be adjusted downward by one minus the dividend yield paid by the firm ( $1 - 3\%$ ).

#### Dividend-adjusted stock price tree

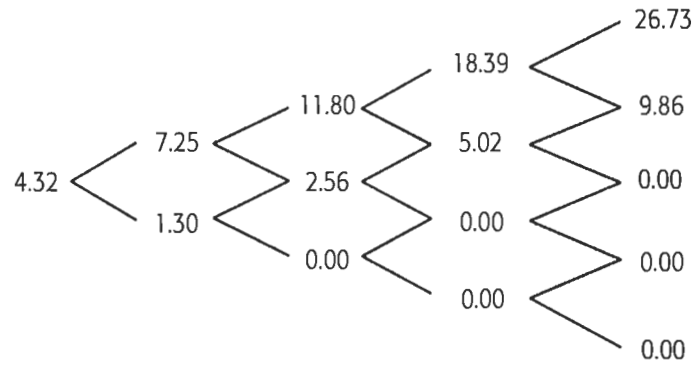




**Put price tree**

The price of the put option is \$6.84.

- B. Using a four-period binomial model, calculate the value of a European call option written on CSM using the recursive procedure (single-period binomial option pricing model).

**Call price tree**

The price of the call option is \$4.32.

36. One year later than the time of the previous question, the management of CSM announces that they are changing their dividend policy. CSM will now pay a fixed dollar dividend each quarter. CSM's management declares that the payout for the current year will be \$6.00 to be paid equally each quarter. CSM now trades at \$85. However, the firm's risk has increased. CSM's annualized standard deviation of return is now 40 percent. CSM will pay the first quarterly dividend in 10 days with the second quarterly dividend coming 90 days after the first dividend. The current risk-free rate of interest is 5.5 percent per annum. Options written on CSM have an exercise price of \$80 and expire in 120 days.
- A. Using a four-period binomial model, calculate the value of a European put option written on CSM using the recursive procedure (single-period binomial option pricing model).

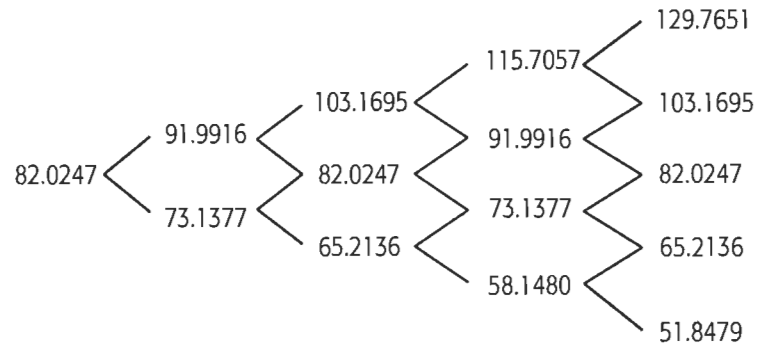
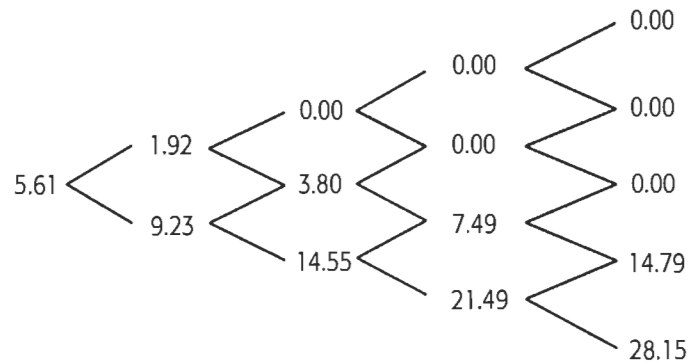
$$\begin{aligned}
 U &= e^{\sigma\sqrt{\Delta t}} = e^{4\sqrt{30/365}} = 1.1215 \\
 D &= 1/U = 1/1.1215 = 0.8917 \\
 \pi_U &= \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.055 \times 30/365} - 0.8917}{1.1215 - 0.8917} = 0.4911 \quad \pi_D = 1 - 0.4911 = 0.5089 \\
 e^{-r\Delta t} &= 0.9955
 \end{aligned}$$

To construct the stock price tree necessary to calculate the value of this option, we must adjust the current stock price of \$85 downward by the present value of the dividends to be received prior to the option's expiration. In this problem, both dividends of \$1.50 will be paid prior to the option's expiration. The first dividend will be received in 10 days, and the second will be received in 100 days.

$$D_1 X e^{-r(T-t)} = \$1.50 \times e^{-0.055 \times (10/365)} = \$1.4977$$

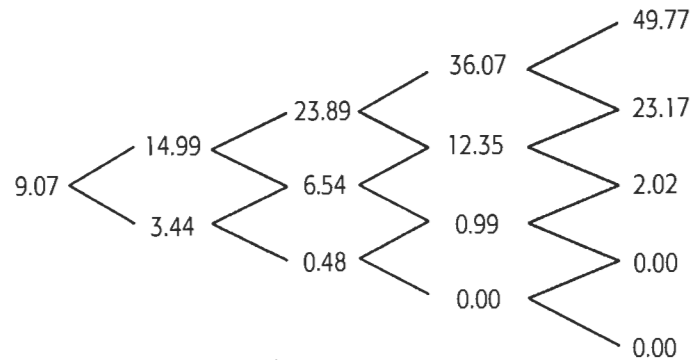
$$D_2 X e^{-r(T-t)} = \$1.50 \times e^{-0.055 \times (100/365)} = \$1.4776$$

$$S' = \$85 - \$1.4977 - \$1.4776 = \$82.0247$$

**Stock price tree****Put price tree**

The price of the put option is \$5.61.

- B. Using a four-period binomial model, calculate the value of a European call option written on CSM using the recursive procedure (single-period binomial option pricing model).

**Call price tree**

The price of the call option is \$9.07.

37. Consider the following data used in all the various parts of this problem. The calculations required to solve these problems can be done in a spreadsheet, or by hand, for the masochist.

Today's date: Friday, December 5, 2003  
 Option's expiration date: January 16, 2004  
 Treasury bill yields: Bid: 5.09% Ask: 5.05%

Date	HCJ Closing Price
October 22, 2006	135.6875
October 23, 2006	135.6250
October 24, 2006	135.3750
October 27, 2006	128.7500
October 28, 2006	133.3750
October 29, 2006	131.1250
October 30, 2006	128.6250
October 31, 2006	130.0000
November 3, 2006	134.1250
November 4, 2006	134.2500
November 5, 2006	133.5625
November 6, 2006	132.0625
November 7, 2006	131.5625
November 10, 2006	130.1875
November 11, 2006	130.6250
November 12, 2006	129.1875
November 13, 2006	131.5625
November 14, 2006	133.3125
November 17, 2006	134.8750
November 18, 2006	134.0000
November 19, 2006	135.0625
November 20, 2006	136.8750
November 21, 2006	137.8750
November 24, 2006	135.5000
November 25, 2006	139.0000
November 26, 2006	141.5625
November 28, 2006	141.5000
December 1, 2006	143.8125
December 2, 2006	142.2500
December 3, 2006	144.6875
December 4, 2006	142.5625

- A. Determine the number of days until the January HCJ options expire.

Determining the number of days until an option expires requires one to count both the weekdays and weekends between the current date and the option's expiration date in January. The options expire in 42 days.

- B. Determine the continuously compounded interest rate on the Treasury bill.

When calculating the continuously compounded interest rate on the Treasury bill, it is important to remember that the Treasury bill matures one day prior to the expiration of the option. That is, the Treasury bill matures on the third Thursday of the month. In this calculation, we must adjust quoted yields on the discount instrument. The price of the Treasury bill is

$$P = 1 - 0.01((5.09 + 5.05)/2) \times (41/360) = 0.994226$$

The continuously compounded interest rate on the Treasury bill is:

$$r = \frac{365}{41} \ln \left( \frac{1}{0.994226} \right) = 0.051553$$

- C. Calculate the annualized standard deviation of return on HCJ's stock using the time series of HCJ stock returns. Assume that there are 252 trading days in a typical year.

To calculate the historical volatility of HCJ, we must convert the price series into a return series. To do this, we construct a price relative series,  $PR_t = P_t/P_{t-1}$ , for using the data for days 1 through 31. The returns series is created by taking the natural logarithm of the 30 price relatives. The daily standard deviation of returns,  $\sigma_d$ , is 0.01762. The annualized standard deviation of returns,  $\sigma_a = \sigma_d \sqrt{252}$ , is 0.27967.

- D. Using the Black–Scholes option pricing model, calculate the current price of the January 140 call and put options written on HCJ. The stock price is 143.125.

Days until expiration	42
Risk-free rate	5.1553%
Historical volatility	0.27967

$$c = S N(d_1) - X e^{-rt} N(d_2)$$

$$d_1 = \frac{\ln(S/X) + (r + .5(\sigma^2))(T - t)}{\sigma \sqrt{(T - t)}}$$

$$d_2 = d_1 - \sigma \sqrt{(T - t)}$$

$$d_1 = \frac{\ln(143.125/140) + (.051553 + .5(0.27967^2))(0.1151)}{.27967 \sqrt{(0.1151)}} = 0.3427$$

$$d_2 = 0.3427 - 0.27967 \sqrt{(0.1151)} = 0.2478$$

$N(d_1)$	0.634074
$N(d_2)$	0.597853
Call price	\$7.55
Put price	\$3.59

- E. Use the following information, as of December 4, to calculate the implied standard deviations for the January 135 and 145 HCJ call options.

Stock price	142.5625	142.5625
Exercise price	135	145
Days until expiration	43	43
Risk-free rate	5.15%	5.15%
Call price	\$12.00	\$4.75

	January 135 call	January 145 call
Implied volatility	0.371367	0.280073
Weighted average (equal weighting)	0.32572	

- F. Use the equally weighted average of the implied standard deviation on the January 135 and 145 call options in the Black–Scholes option pricing model to calculate the current price on the January 140 call and put options. Compare the option prices calculated using the historical volatility and the implied volatility.

Stock price	143.125
Exercise price	140
Days until expiration	42
Risk-free rate	5.1553%
Historical volatility	0.32572

$$d_1 = \frac{\ln(143.125/140) + ([.051553 + .5(0.032572^2)](0.1151))}{.32572\sqrt{0.1151}} = 0.3087$$

$$d_2 = 0.3087 - 0.32572\sqrt{0.1151} = 0.1982$$

$N(d_1)$	0.621238
$N(d_2)$	0.578573
Call price	\$78.39
Put price	\$4.44

The option prices calculated using the implied volatility are higher than the prices calculated using the historical volatility, because the implied volatility is higher than the historical volatility.

- G. Assume that stock prices follow a random walk with a drift. Use the weighted average of the implied volatilities on the January 135 and 145 call options and the continuously compounded return on the Treasury bill calculated in part B above to calculate the parameters in a binomial process,  $U = e^{\sigma\sqrt{\Delta t}}$ ,  $D = 1/U$ , and  $\pi_u = e^{r\Delta t} - D/U - D$ . Use a four-period binomial model to calculate the value of the January 140 call and put options written on HCJ. Compare the option prices calculated using the binomial model with option prices calculated using the Black-Scholes model. Explain the source of the differences in the option prices.

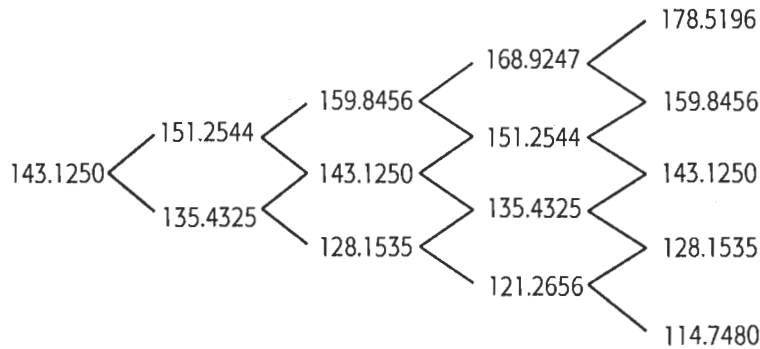
$$U = e^{\sigma\sqrt{\Delta t}} = e^{.32572\sqrt{(10.50/365)}} = 1.0568$$

$$D = 1/U = 1/1.0568 = 0.9463$$

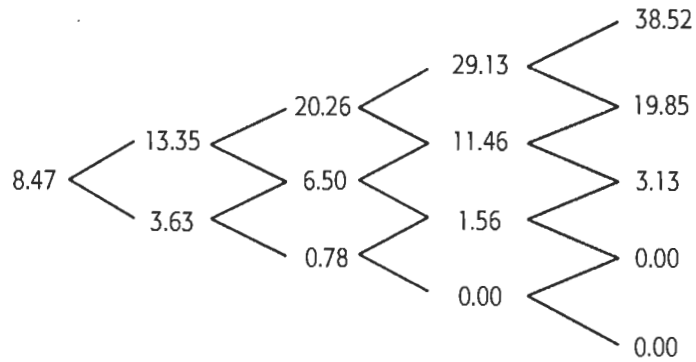
$$\pi_U = \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{.051553 \times 10.50/365} - 0.9463}{1.0568 - 0.9463} = 0.4996 \quad \pi_D = 1 - 0.4996 = 0.5004$$

$$e^{-r\Delta t} = 0.9985$$

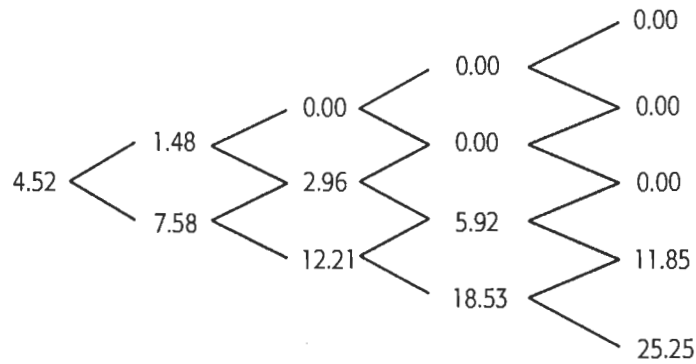
#### Stock price tree



### Call price tree



### Put price tree



The call price is \$8.47, and the put price is \$4.52. These prices are higher than the comparable option prices calculated using the Black–Scholes model. The differences in the prices are due to the fact that we modeled the movement in HCJ stock over a 42-day period using only four periods. Adding more periods to our binomial model produces prices that are very similar to the prices calculated using the Black–Scholes model.