Chapter 1

Biographical Introduction to Frege's Philosophy

Gottlob Frege was a nineteenth-century German university professor, little known in his own lifetime, who devoted himself to thinking, teaching and writing. He played no part in public affairs, and much of his life was spent in the classroom and in the library. His books and articles were read by very few of his colleagues, and for a long time, even after his death, his influence in philosophy was exercised mainly through the writings of others. Today he is revered as the founder of modern mathematical logic, and as a philosopher of logic in the same rank as Aristotle. As a philosopher of mathematics he stands out, in the history of the subject, beyond all others.

Frege was born of a Lutheran family in Wismar, on the Baltic coast of Germany, in 1848. His father was the founder of a girls' school. Before Frege graduated from high school, in 1866, his father died. During his education and early academic career, he depended for financial support on his mother, who had succeeded her husband as principal of the girls' school.[1]

Frege entered Jena University in 1869 and spent four semesters there before moving to Göttingen in 1871 for five further semesters, studying philosophy, physics and mathematics. He submitted a dissertation on a geometrical topic and was awarded his Ph.D. by Göttingen in December 1873 (CP, 92).

1. For the details of this biography I have drawn on T. W. Bynum's introduction to CN.

Once the degree was granted, Frege applied for an unsalaried teaching post at Jena University. In support of his application he submitted a paper, 'Methods of Calculation based on an Extension of the Concept of Quantity' (CP, 56–92), which made a novel contribution to mathematical analysis. It was well received by the examiners, and Frege was appointed to the post in spite of the fact that his performance at the oral examination was reported to be 'neither quick-witted nor fluent'.

Frege began lecturing as a *privatdozent* in 1874 and he taught in the mathematics faculty at Jena for forty-four years. He was a clear, conscientious and demanding teacher, and for some years he had to carry the teaching load of a senior colleague who was an invalid. None the less, in the first five years after his appointment, he carried out research which was to lay the foundations of his life's work and which provided the starting point for an entirely new discipline.

Frege began his career as a mathematician at an exciting period in the history of mathematics. Euclidean geometry, which had been regarded as a system of necessary truths for over two millennia, lost its unique status early in the nineteenth century. Euclid had derived the theorems of his system from five axioms: it was now shown that one of these axioms, far from being a necessary truth, could be denied without inconsistency, and non-Euclidean geometries were developed on the basis of alternative axioms. There were also exciting developments in number theory. Imaginary numbers, such as $\sqrt{-1}$, which had been regarded as an eccentric curiosity in the eighteenth century, were shown to serve a purpose in the representation of motion in a plane, and were incorporated along with more familiar kinds of number in a general theory of complex numbers. The Dublin mathematician Sir William Hamilton devised a calculus of hypercomplex numbers (quaternions) to help in representing motion in a plane. At Halle in Germany Georg Cantor was working out, while Frege was a young professor, the theory of infinite numbers which he was to publish in 1883.

Frege early came to believe that the luxuriant expansion of mathematics in his time was inadequately supported. This entire impressive construction, he claimed, rested on shaky foundations. Mathematicians did not really understand what they were about, even at the most basic level. The problem was not a lack of understanding of the true nature of imaginary numbers such as $\sqrt{-1}$, or of irrational numbers such as $\sqrt{2}$ or π, or of fractional numbers like $\frac{2}{3}$ or of negative integers such as -1; the lack of understanding began with the natural numbers such as 1, 2 and 3. Mathematicians, in Frege's view, could not explain the nature of the primary objects of their science or the fundamental basis of the discipline they taught. He resolved to devote his life to remedying this defect: setting out, in a perspicuous manner, the logical and philosophical foundations of arithmetic. A series of publications between his thirtieth and his sixteenth year was devoted to this end.

The first of these was a pamphlet issued in 1879 with the title *Begriffschrift*, which we can render into English as *Concept Script*. The concept script which gave the book its title was a new symbolism designed to bring out with clarity logical relationships which ordinary language concealed. The calculus contained in the book was a significant development in the history of logic.

For generations now the curriculum in formal logic has begun with the study of the propositional calculus. This is the branch of logic that deals with those inferences which depend on the force of negation, conjunction, disjunction, etc. when applied to sentences as wholes. Its fundamental principle is to treat the truth-value (that is, the truth or falsehood) of sentences which contain connectives such as 'and', 'if', 'or' as being determined solely by the truth-values of the component sentences which are linked by the connectives. Frege's *Concept Script* contains the first systematic formulation of the propositional calculus; it is presented in an axiomatic manner in which all laws of logic are derived, by a specified method of inference, from a number of primitive principles. Frege's symbolism, though elegant, is difficult to print, and is no longer used; but the operations which it expresses continue to be fundamental in mathematical logic.

Frege's greatest contribution to logic was his invention of quantification theory: a method of symbolizing and rigorously displaying those inferences that depend for their validity on expressions such as 'all' or 'some', 'any' or 'every', 'no' or 'none'. In *Concept Script*, using a novel notation for quantification, he presented an original calculus to formalize such inferences (a 'functional calculus' or 'predicate calculus' as it was later to be called). This laid the basis for all subsequent developments in logic and formalized the theory of inference in a more rigorous and more general way than the traditional Aristotelian syllogistic which up to the time of Kant was looked on as the be-all and end-all of logic.

In the *Concept Script* Frege was not interested in logic for its own sake. His aim was not simply to show how to conduct logic in a mathematical manner; he wanted to show that logic and mathematics were much more closely linked with each other than had previously been realized.

Before Frege addressed the subject, the nature of mathematics was the subject of debate between two schools of philosophical thought. According to Immanuel Kant (1724–1804) our knowledge of both arithmetic and geometry depends on intuition. His *Critique of Pure Reason* set out the position that mathematical truths were, in his terminology, both synthetic and a priori, which means that, while they were genuinely informative, they were known in advance of all experience. John Stuart Mill (1806–1873), on the other hand, thought mathematical truths were known a posteriori, that is to say, on the basis of experience. His *A System of Logic* argued the case that they were empirical generalizations widely applicable and widely confirmed.

The nature of mathematical truth had a central significance in philosophy. It was crucial to the question at issue between empiricist philosophers, who maintained that all our knowledge derived from sense experience, and rationalist philosophers, who maintained that the most universal and important elements of our knowledge derived from some supra-sensible source. Thus Mill says that his *System* 'met the intuition philosophers on ground on which they had previously been deemed unassailable; and it gave its own explanation, from experience and association, of that peculiar character of what are called necessary truths, which is adduced as proof that their evidence must come from a deeper source than experience'.[2]

Frege agreed with Kant against Mill that mathematics was known a priori. But he maintained that the truths of arithmetic were not synthetic at all: he denied that they contained any information not implicit in the nature of thought itself. Unlike geometry – which, he agreed with Kant, rested on a priori intuition – arithmetic was analytic; it was, indeed, nothing more than a branch of logic.

Frege's long-term purpose was to show that arithmetic could be formalized without the use of any non-logical notions or axioms, and that it was based solely upon general laws which were operative in every sphere of knowledge and needed no support from empirical facts. *Concept Script*, in addition to its formalization of propositional and functional calculus, contained some important preparatory work towards this reduction of arithmetic to logic; but the full presentation of Frege's thesis had to wait for the publication of his book *The Foundations of Arithmetic* in 1884.

On the basis, partly, of *Concept Script* Frege was promoted to a salaried professorship in 1879. The book, however, was not well received by the logical or mathematical world in general. Frege's notation was two-dimensional and tabular; this appeared to reviewers to be cumbersome and futile. Several writers compared the book unfavourably with George Boole's *An Investigation of the Laws of Thought*, which had appeared in 1854 and had regimented logic into formulas which resembled familiar arithmetical equations. Frege's publications between 1879 and 1884

2. J. S. Mill, *Autobiography*, Oxford University Press, 1971, p. 135.

consisted mainly of responses to hostile reviews and explanations of how his purposes and methods differed from those of Boole.

Perhaps because of the unfavourable reception of *Concept Script*, Frege wrote *The Foundations of Arithmetic* in a very different style. Symbols appear comparatively rarely, and there is a constant attempt to relate the discussion to the work of other writers. The thesis that arithmetic is derivable from logic – the thesis later to be known by the name of 'logicism' – is set out in this book fully and clearly, but for the most part quite informally.

Almost half the book is devoted to an attack on the ideas of Frege's predecessors and contemporaries, including Kant and Mill. In the course of these attacks the ground is prepared for the logicist position. In the main body of the work Frege showed how to replace the general arithmetical notion of number with logical notions such as the notion of a concept, the notion of an object's falling under a concept, the notion of equivalence between concepts and the notion of the extension of a concept. He offered definitions, in purely logical terms, of the numbers zero and one, and of the relation which each number has to its predecessor in the number series. From these elements, along with the general laws of logic, he offered to derive the whole of number theory.

The Foundations of Arithmetic is a very remarkable book; but when it appeared it received an even poorer reception than *Concept Script*. Only three reviews appeared, all of them hostile, and for almost twenty years the book went virtually unremarked. Frege was disappointed, but not deterred from further work on his great project.

In the *Foundations* there are two theses to which Frege attached great importance. The first is that each individual number is a self-subsistent object. The second is that the content of a statement assigning a number is an assertion about a concept, so that, for instance, the statement 'The Earth has one moon' assigns the number 1 to the concept *moon of the Earth*.

At first sight these theses may seem to conflict, but if we understand what Frege meant by 'concept' and 'object' we see that they are complementary. In saying that a number is an object, Frege is not suggesting that a number is something tangible like a tree or a table. Rather, he is doing two other things. First, he is denying that number is a property belonging to anything, whether to an individual or to a collection. Secondly, he is also denying that it is anything subjective, any mental item or any property of a mental item. Concepts are, for Frege, mind-independent, and so there is no contradiction between the thesis that numbers are objective, and the thesis that number-statements are statements about concepts. These two principles were to remain at the heart of Frege's thinking for many years to come, while he strove to perfect a symbolic and rigorous presentation of the logicist thesis.

It will be seen that Frege's philosophy of mathematics is closely linked to his understanding of several key terms of logic and of philosophy; and indeed in *Concept Script* and *Foundations* Frege not only founded modern logic, but also gave a fresh start to the philosophy of logic. He did so by making a sharp distinction between the philosophical treatment of logic and two other disciplines with which it had often been intermingled. He separated it, on the one hand, from psychology (with which it had often been confused by philosophers in the empiricist tradition) and, on the other hand, from epistemology (with which it is sometimes conflated by philosophers in the tradition stemming from Descartes).

For the nine years after the publication of *Foundations* Frege worked principally on his logicist project of deriving arithmetic from logic. His publications during this period, however, are especially concerned with problems in the philosophy of language. Three papers appeared in 1891–2: 'Function and Concept', 'Sense and Reference', 'Concept and Object'. Each of these authoritative essays presented philosophical ideas of fundamental importance with astonishing brevity and clarity. They were seen, no doubt, by Frege as ancillary to the logicist project, but at the present time they are regarded as founding classics of modern semantic theory.

One of the most significant developments in Frege's thoughts at this time was a new distinction which he now introduced between *sense* and *reference*. Where other philosophers talked ambiguously of the *meaning* of an expression, Frege invited us to mark a difference between the *reference* of an expression (the object to which it refers, as the planet Venus is the reference of 'The Morning Star') and the *sense* of an expression. ('The Evening Star' differs in sense from 'The Morning Star' though it too, as astronomers discovered, refers to Venus.) The most puzzling and controversial application of Frege's distinction between sense and reference was his theory that it was not only individual words that had reference, but also whole sentences. The reference of a sentence was its truth-value (that is, the True, or the False).

The climax of Frege's career as a philosopher should have been the publication of the volumes of *Grundgesetze der Arithmetik* (*The Basic Laws of Arithmetic*),[3] in which he set out to present in formal manner the logicist construction of arithmetic on the basis of pure logic. This work was intended to execute the task which had been sketched in the earlier books on the philosophy of mathematics: it was to enunciate a set of axioms which would be recognizably truths of logic, to propound a set of undoubtedly sound rules of inference, and then to present, one by one, derivations by these rules from these axioms of the standard truths of arithmetic, in an expanded version of the symbolism of *Concept Script*. However, no publisher would print the manuscript as a whole; Pohle of Jena, who had published 'Function and Concept' as a pamphlet, was willing to publish it in two volumes, the publication of the second instalment being conditional on the success of the first. Late in 1893

3. I shall refer to this work by its German title, since it has never been fully translated into English.

the first volume appeared; the publication of the second was delayed until 1903.

Grundgesetze follows in the main the lines of *The Foundations of Arithmetic*. However, much more emphasis is placed on the notion of class, which is now regarded as essential to the definition of the notion of number. The cardinal numbers are, in effect, defined as classes of equivalent classes, that is, classes with the same number of members; thus the number two is the class of pairs, and the number three is the class of trios. Despite appearances, this definition is not circular, because we can say what is meant by two classes having the same number of members without making use of the notion of number. Two classes are equivalent to each other if they can be mapped one-to-one onto each other. We can define the number zero in purely logical terms as the class of all classes equivalent to the class of objects which are not identical with themselves. We can define the number one as the class of all classes equivalent to the class whose only member is zero. In order to pass from definitions of zero and one to the definition of the other natural numbers, Frege makes use of the definitions of 'successor' and of other mathematical relations within the number series which he had developed in *Concept Script*. A treatment of negative, fractional, irrational and complex numbers was postponed until the second volume.

Frege's magnificent project aborted before it was completed. The first volume was received in general with the chill silence which had greeted his earlier works. As a result of this, publication of the second volume was held up for a decade and it had eventually to be published at the author's own expense. Publication of the first volume did, however, lead to Frege's promotion to a senior professorship at Jena and to a substantial research grant from the foundation set up by the Zeiss camera company. It also led to a fruitful controversy with the Italian logician Giuseppe Peano, who modified his own newly published axiomatization of arithmetic to take account of Frege's criticisms. Through Peano, Frege's work was brought to the notice of the first of his English readers, Bertrand Russell, who was at that time a young Fellow of Trinity College, Cambridge.

Frege occupied much of the time between the appearance of the two volumes of *Grundgesetze* in publishing increasingly bitter and sarcastic attacks on the scholars who had misunderstood his own publications. The most fruitful of these was his hostile review of the *Philosophie der Arithmetik* by the German philosopher Edmund Husserl; this was taken in good part by Husserl, who was converted by it from the psychologism which he had earlier defended, and he joined Frege as one of its severest critics.

While the second volume was in press, in 1902, Frege received a letter from Russell pointing out that the fifth of the initial axioms of *Grundgesetze* made the whole system inconsistent. This axiom states in effect that if every *F* is a *G*, and every *G* is an *F*, then the class of *F*s is identical with the class of *G*s, and vice versa: it was the axiom which, in Frege's words, allowed 'the transition from a concept to its extension', the transition which was essential if it was to be established that numbers were logical objects. Frege's system, with this axiom, permitted the formation of the class of all classes that are not members of themselves. But the formation of such a class, Russell pointed out, leads to paradox: if it is a member of itself then it is not a member of itself; if it is not a member of itself, then it is a member of itself. A system which leads to such a paradox cannot be logically sound.

With good reason, Frege was utterly downcast by this discovery, though he strove to patch his system by weakening the guilty axiom. The paradox and its attempted solution were described in an appendix to the second volume of *Grundgesetze* when it appeared in 1903. Frege's revised system, in its turn, proved inconsistent, though Frege continued to believe in it for some years yet. After his retirement from Jena in 1918 he seems at last to have given up his belief that arithmetic was derivable from logic, and to have returned to the view of Kant that arithmetic, like geometry, is synthetic a priori.

In the last years of his life, between 1918 and his death, Frege attempted to write a full treatise of philosophical logic. All that was completed was a series of articles (*Logical Investigations*, 1919–23) in which he returned to the relationship between logic and philosophical psychology, or the philosophy of mind, and discussed the nature of thought and inference.

Much of what Frege wrote on philosophical logic in his last years remained unpublished at his death. Frege and his wife Margaret had several children, all of whom died young; some time before her death in 1905 they adopted a son, Alfred, who became an engineer. When Frege made his will in January 1925 he left his unpublished papers to Alfred, with the following note:

> Dear Alfred,
> Do not despise the pieces I have written. Even if all is not gold, there is gold in them. I believe there are things here which will one day be prized much more highly than they are now. Take care that nothing gets lost.
> Your loving father
> It is a large part of myself which I bequeath to you herewith.[4]

Six months later Frege died, unaware that he would come to be regarded as the founder of the most influential philosophical movement of the twentieth century. His death was barely noticed by the learned world.

4. Most of Frege's posthumous papers were published in German in 1969 and in English in 1979.