CHAPTER ONE

MODELS, SYSTEMS, AND DYNAMICS

We must learn to think in terms of systems. We must learn that in complex systems we cannot do only one thing. Whether we want it to or not, any step we take will affect many other things. We must learn to cope with side effects. We must understand that the effects of our decisions may turn up in places we never expected to see them surface. (Dietrich Dörner, *The Logic of Failure*, p. 198)

1.1 WHAT IS A MODEL?

Our environment is both complex and dynamic. Given this complexity we need a "map" or models to help us to understand what processes and interactions are important and to evaluate the outcomes of interest. The first step in modeling is to clearly define what is the problem or problems of interest. For instance, the problem or question to be answered may be, what will be the population of grizzly bears in a national park next year? Any model that adequately addresses this problem must include hypotheses, or statements, about what influences the bear population. By necessity, such statements cannot be a complete representation of the dynamics of the grizzly bear population. For instance, the accumulation of pesticides and other chemicals in the food chain may have an adverse effect on grizzly bear breeding success rate in the *long run*, but incorporating chemical and pesticide build-up in grizzly bears may not help us to improve our prediction of the grizzly bear population for next year. Thus the *purpose* of the model determines the boundary of the model and what we should or should not include within our "map."

A model can be a highly complex system of equations developed in an iterative process that may take months, or even years, to construct. By contrast, it may be as simple as a single statement that represents an underlying process or relationship that can be used to help resolve a particular research problem. For example, "The population of grizzly bears in Banff National Park next breeding season will equal the current population, plus the number of cubs that survive the current season less the number of juvenile and adult bears that die during the season." This statement can be written out as a mathematical model,

$$x_{t+1} = x_t + b_t - d_t$$

where x_{t+1} is the population of grizzly bears in period $t+1$, x_t is the population of grizzly bears in period t, b_t is the number of cubs successfully reared and d_t is the number of juveniles or adult bears that die.

This model provides an understanding, or an interpretation, of the population dynamics of grizzly bears. The formulation of the model may be derived from watching breeding females raise cubs during the breeding season. If data are available on the current population, the number of cubs successfully raised in the first year of their life and the number of juveniles and adults that die, the model can be tested by comparing its predictions to the number of bears observed in next year's breeding season. If subsequent observations and data match our predictions to an appropriately defined level of significance, then the model has achieved its purpose. However, just because a model is useful does *not* imply that a model is "true." Indeed, no single model can be described as being a correct or true representation of reality as it must, by necessity, be an abstraction.

The specified model of the population dynamics of grizzly bears ignores the possibility of the migration of grizzly bears from other populations to Banff National Park, and from grizzly bears in Banff to populations of bears in other locations. However, if net migration of bears is small compared to the birth or death rates, the model may still be a good predictor of next year's breeding population. If the purpose is to predict next year's breeding population, making the model more realistic (and including net migration) is not necessarily desirable. For instance, if including migration in the model increases the prediction error, or the difference between observed and predicted bear numbers, then it may be preferable to leave out net migration from the model. In other words, if the research problem is simply to predict next year's bear population then a model that achieves this purpose with a lower prediction error is preferred to another model, even if the alternative is more realistic and captures more details of the population dynamics. Thus the judgment of a model is not whether it describes reality well or not, but whether it helps address the research problem for which it was built and whether it does so better than alternative models.

A maxim of modeling, known as Occam's razor, is that the simplest logical model that addresses the research problem is preferred over alternative models. Thus the art of modeling is not to include everything that can be incorporated, but rather to make the model as simple and tractable as possible to help answer the question that was posed. Knowing what to leave in, and what to leave out of a model, requires a good understanding of both the processes being modeled and the purpose of the model. For instance, if the purpose of the model of the population dynamics of bears is to understand the relationships between bears and their prey, then the model given above is useless. If, however, its purpose is to simply predict next year's population the model may be very useful. Consequently the judgment on the usefulness of a model is intricately linked to what problem it tries to address, or the questions for which it was devised to answer.

1.2 MODEL BUILDING

Model building often involves both conjectures and hypotheses based on observations of phenomena, and that may be called induction, as well as the specification of a logical and consistent set of statements that purport to explain the phenomena, and that may be called deduction. Good model building requires both induction and deduction. Theories cannot be developed in a vacuum without an understanding of the phenomena being modeled. Similarly, models based purely on observation run the risk of lacking in rigor and logic where "facts" and observations may support a completely wrong model. In other words, just because observations fail to falsify or refute a model, it does *not* mean that the model is correct. Moreover, correlation between variables that conform to a model's hypotheses does *not* necessarily imply causation. Many variables are correlated with each other, but there is not necessarily an underlying causal relationship between them. For instance, in rich countries the average time spent per week watching television is positively correlated with life expectancy, but this does not imply that watching television *causes* us to live longer. A classic example of how observations can support an incorrect model is provided by Apollonius of Perga (265–190 BC) who was one of the greatest mathematicians of antiquity. He developed a geocentric model of the solar system in which the earth was at the center and all other planets, including the sun, orbited around it. The model was supported by observations over many centuries and was able to predict planetary positions to a surprising degree of accuracy.

The testing or disproving of hypotheses is part of the scientific method whereby propositions or models are formulated and are then tested to see whether they conform to empirical observations. The exception, perhaps, is in mathematics, where "truth" is not determined by experimentation but rather by proof. Thus mathematical truths, that are in the form "If A, then B," are results derived by deduction from the initial axioms or statements or rules. In other words, the proofs or propositions derived from the initial axioms are "true" in a *mathematical* sense whether or not the original axioms were correct or whether or not they conform to reality. An axiomatic approach to modeling can be very useful and can provide fundamental insights, but if we seek an understanding of the world around us then, sooner or later, our models (and axioms!) must connect to reality.

If we employ the scientific method, hypotheses that are found lacking, or can be "disproved" in their current formulation, may be modified, or an entirely new model may be devised to test the hypotheses. Any hypothesis that is "scientific" must be falsifiable in the sense that it can be disproved from empirical observations.

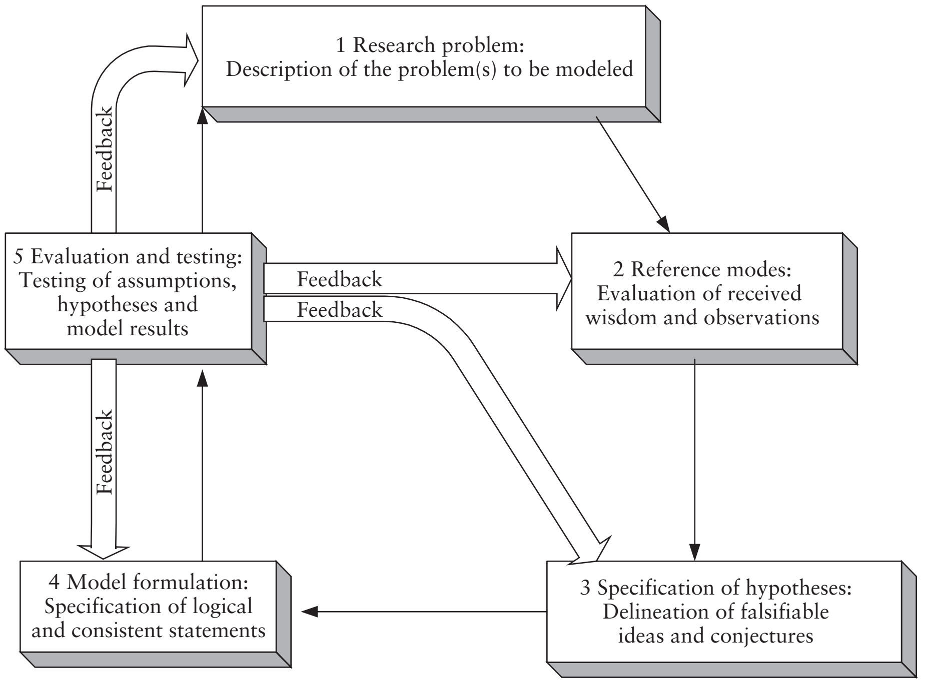

Figure 1.1 The model-building process

Indeed, the falsification process should include the specification in advance of the observations that would falsify the hypothesis. For example, Einstein's theory of relativity (special and general) predicted that light passing through space would be bent when it passed near an object with a massive gravitational field. This prediction was found to be correct in 1919 (14 years after Einstein's special theory was published) when it was observed by British scientists, during a solar eclipse, that distant stars appeared to "move" from a terrestrial perspective as the light they emitted was bent by our sun. Ideally, the falsification of a model should also require that the model being tested make predictions that other models cannot. Sometimes the data or observations may not yet exist to disprove a hypothesis, but provided that such data can be obtained, then the hypothesis is still falsifiable, although it remains untested.

The scientific approach to model building is iterative. It involves a statement of the problem(s) to be addressed, a review of the observed behavior or received wisdom, a formulation of conjectures or statements or equations that purport to explain the processes and relationships, and the subsequent testing and evaluation of the model(s), as illustrated by figure 1.1. The thin black arrows indicate the development, chronology or learning loop of the model-building process that begins first with the research problem and continues through to evaluation and testing. The thick arrows indicate a feedback process that influences all the steps in model building.

The first step in building a model is to establish what is the research problem. The problem must be sufficiently concise and tractable that the model can realistically provide some insight into the question. For example, the problem “What are the costs of climate change?” is so broad that no single model can hope to provide a meaningful answer to the question. This is not to say that the “big” questions should not be asked, but rather that answering such a question requires a research program that will require many models. Indeed, the question regarding the potential consequences of climate change has spawned a huge and multi-disciplinary research program under the auspices of the Intergovernmental Panel on Climate Change (IPCC) that has led to the formulation of many thousands of models. By contrast, the problem “What are the short-term economic costs for Germany from meeting its obligations to reduce its greenhouse gas (GHG) emissions, as specified under the 1997 Kyoto Protocol?” can be investigated (and indeed is currently being investigated) with an appropriate set of economic models.

The second step in modeling is to review the accepted wisdom. This may include a review of the existing theory and evaluation of the results of existing models. This establishes the “reference modes” (Sterman, 2000) or a summary of the fundamentals of what is known. The review should also include an evaluation and assessment of the existing data or observations about the problem or phenomena to be modeled. For example, if the research problem is to predict the future abundance of animal populations, the reference mode should include the history of the population and some measures of its births and deaths. The reference modes, in turn, help shape our initial hypotheses of the relationships, feedbacks, and relative importance of the variables that are to be included in the model.

The third step in the process is to specify conjectures, ideas or a preliminary theory that can be developed into testable hypotheses about the processes for which the model is being built. These hypotheses help dictate the model we ultimately formulate, along with the existing models in the literature. The hypotheses that are to be tested should be sufficiently clear and precise so that they can provide insights into the research problem. The hypotheses to be refuted, and the reference modes, help to formalize the model used to answer the specified research questions. For example, a hypothesis underlying an economic model of climate change could be that reductions in emissions of carbon dioxide reduce real economic growth. Such a hypothesis would require that we build a model that explicitly includes measures of economic activity and carbon dioxide emissions, and their interrelationships.

The fourth, and perhaps hardest, step is to formulate the model. The formal model must be logical, should avoid unnecessary details and be as simple as possible while still being able to help answer the posed research question. What makes a good model is *not* whether it provides an exact description of the phenomena being studied, but whether it can provide real insights and understanding into the research problem. A model should be more than the sum of its parts and should be judged by its ability to provide understanding and insights about the research questions and hypotheses that would otherwise not be possible.

When formulating a model, simplifying assumptions are required about the relationships of the variables under study. For example, we may assume that one variable (such as the price of a good) is unaffected by changes in another variable (such as income). These assumptions, along with the refutable hypotheses, need to be tested if the model is to be of use. In other words, if we assume a certain relationship holds true when formulating a model then for the model to be falsifiable (as it should be!) this assumption should be able to be tested or refuted.

Models may also require us to subsume a set of postulates or assertions that cannot be tested. These assertions presuppose a state of the world, or set of behavior, that cannot be refuted, but may nevertheless be required if the model is to be tractable. For example, we may *assert* that consumers are rational when we are formulating a model of consumer demand that *assumes* that the quantity demanded is a function of the relative price of the good. Without the assertion that consumers are rational (which may or may not be true), it may be difficult to construct a simple model that could, for example, be used to predict future consumption levels of the good. However, the assumption of a functional relationship between the relative price and the consumption of the good in a model, which is used to predict future consumption, must be tested when evaluating the model. Such tests of the model's assumptions are conditional on the assertions or postulates used to formulate the model.

The step that closes the loop in the model-building process is to test and evaluate the model, the results and hypotheses. Testing of the model may involve many different approaches and methods. For example, with econometric or statistical models we can compare our hypotheses with our empirical results. This can be accomplished by tests for misspecification, measurement (and other) errors, influence of different functional forms on the results and whether the assumptions used in estimating the model are valid. In empirical work, care must also be taken to avoid "data mining" in the sense that we select a model that gives the "best" results and levels of significance, but fail to report the many other estimates we discarded to obtain the best model. Such an approach creates a bias in terms of the normal levels of significance we use for testing whether explanatory variables are statistically significant from zero or not.

Empirical models also require tests of robustness to judge their value and should include an analysis of the influence of outliers and influential observations, the effect of the choice of explanatory variables, the selected data series used for the variables and the chosen time period. Further, careful attention should be given to the *economic* significance of the statistical results (McCloskey, 1997). For instance, simulations can be generated from estimated coefficients to help answer "what if?" questions about the effect of changes in the magnitude of one or more of the explanatory variables. Thus, a variable may be statistically significant in the sense that at the 1 percent level of significance we reject the null hypothesis that its estimated coefficient equals zero, but it may have only a small influence on the dependent variable. Conversely, an explanatory variable that may not be statistically significant at the conventional 5 percent level of significance may *potentially* have a very large effect in the sense that a small change in its magnitude could lead to a large change in the dependent variable.

Whatever the form or type of model, "testing" should include a comparison between the results, the initial hypotheses, and the existing literature. Testing of the model also requires that we evaluate competing models or hypotheses that may provide different insights or understanding to the research problem. In other words, the observations may also be consistent with alternative and competing models and not just the model used in the analysis. Moreover, when comparing models that equally fit existing observations, the model that also makes additional and falsifiable predictions is, in general, preferred. The evaluation of the model and competing models should, in turn, stimulate further thinking and inquiry into the original question or problem posed, the accepted or received wisdom and the model that was formulated. Thus, testing and evaluation continue the model-building process and contribute to our understanding of the problems that originally motivated the research.

Parallel to the model-building process is consideration of not only *what* is the research problem, but *who* is the audience for sharing of the insights and results of the model. Too frequently researchers expect that their model and results will "speak for themselves." Unfortunately, even the most brilliant model builder will accomplish little in terms of increasing knowledge and understanding if she fails to present what has been done in a form suitable for the intended audience. If the intended readership is a group of well-trained and knowledgeable researchers then motivating the research problem, describing the model and explaining the results may be sufficient. If, however, the likely audience lacks the training or background to understand the model, or the implications and caveats of the results, then considerable effort is required to explain the model and its implications in a way that is comprehensible to the reader.

1.3 MODEL CHARACTERISTICS

Models can be divided into those that involve optimization, whereby an objective function is optimized over a set of choice or control variables subject to a set of constraints, and models that simulate changes in processes over time. Optimization models are frequently used to answer "what should be"-type questions. For example, what should be the harvest rate in a fishery if we wish to maximize the present value of net profits? Simulation models are often used to answer "what would be" questions such as, what would be the earth's average surface temperature in 2100 if the concentration of carbon dioxide in the atmosphere were to double?

Optimization and simulation

Optimization and simulation models share a number of important characteristics and, indeed, sometimes simulations are used to find an "optimum" strategy while optimization models may be used to simulate possible outcomes under alternative specifications of the objective functions and/or constraints.

In environmental and resource economics we often wish to optimize our rate of discharge or depletion or use of an environmental asset. This requires optimizing an objective function subject to a set of constraints. Most economic models optimize over a particular variable whether it be utility, profits, or some other metric subject to constraints. The appropriate metric is determined by the problem addressed by the model. For instance, if we wish to determine the level of harvest of trees that will generate the highest monetary return over time then an objective function that maximizes the discounted net profits is appropriate. By contrast, if we were concerned with the costs of production for a given level of harvest, then an objective function that minimizes the economic costs of production under a harvest constraint would be appropriate. In such problems, the variables whose values are chosen in the optimization program are called control variables and could include, for example, the harvest rate. Variables whose values are determined within the model, but which depend on the values of the control variables, are called state variables. State variables might include, for example, the resource stock. The potential solution is bounded by constraints that may include dynamic constraints that describe the dynamics of the state variables and boundary conditions that specify any constraints on the starting and ending values of variables.

Simulation models provide predicted values of variables of interest based on specified initial values and parameters of the model. In many cases, the parameters and initial conditions for simulation models are obtained from empirical models or observations of the phenomena under study. Simulation models are enormously useful in helping us understand the interactions and processes of systems. The value of simulation models comes from the analysis of the effects of changes in interactions, parameters, and initial values, called sensitivity analysis. To make such comparisons as easy as possible, several software packages are available. The software Vensim (www.vensim.com), Powersim (www.powersim.com) and Stella (www.hps-inc.com) are widely used and are sophisticated enough to build models of highly complex systems.

Endogenous and exogenous variables

Whatever the purpose, the modeler must decide what variables should be determined within the model (be endogenous), and what variables should be determined from outside (be exogenous), but are included in the model. Variables that are neither exogenous nor endogenous to the model are excluded variables and are not incorporated in the model-building process. All variables that are *critical* in determining future states of the model should be endogenous, whether or not the variables change slowly or rapidly. At the very least, model results should be tested for their robustness to changes in values of those variables treated as exogenous.

To some extent, the decision as to which variables are endogenous, exogenous or are excluded depends on both the purpose and the time-scale of the model.

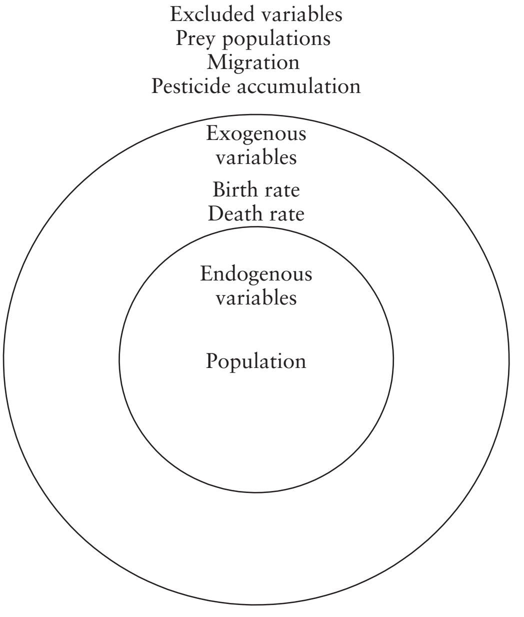

Figure 1.2 Boundaries of a model of the grizzly bear population in Banff National Park

For example, a model designed to predict economic growth over the next year could treat population as an exogenous variable and have little effect on the reliability of the predictions. However, if such a model were used to predict economic growth over 25 years or more it would likely suffer from important deficiencies as economic growth and population growth are co-determined and feed back on each other.

To illustrate the boundaries of models, figure 1.2 shows what variables are excluded, exogenous and endogenous in the model used to predict the bear population in Banff National Park. Outside the model boundary are excluded variables (migration, pesticide accumulation, prey effects). The model includes exogenous variables (birth and death rates) that may be varied by the modeler, but are not determined by the model itself. In the core of the model is the endogenous variable (population) that is determined by the model. The initial and past states of the endogenous variable, in turn, help determine future values of the endogenous variable.

Feedback effects

All complex systems are subject to both positive and negative feedback effects. Positive feedbacks reinforce disturbances to a system and move variables further away from their original state while negative feedbacks tend to return systems

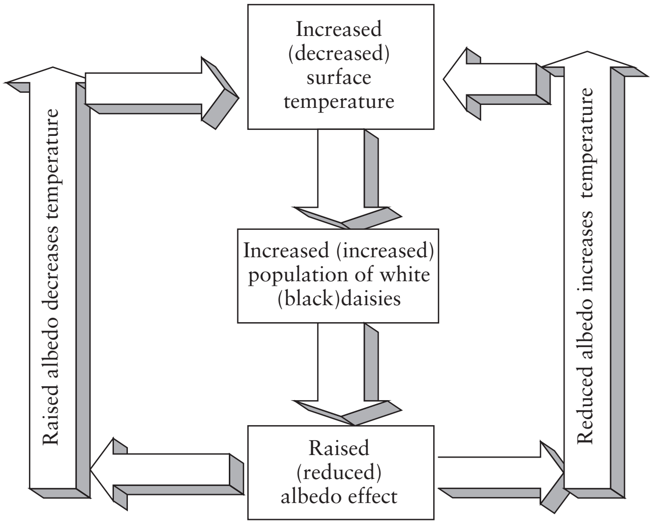

Figure 1.3 Negative feedback effects in Daisyworld

to their former state following disturbance. Negative feedback effects may be illustrated in a simple model of a planet called Daisyworld (Lovelock, 1990). In this world only two plants exist, white and black daisies. White daisies do better at higher temperatures than black daisies, but also have a greater albedo effect and reflect more of the solar radiation reaching the planet's surface. Shocks to the system are provided by changes in solar radiation that affect the planet's surface temperature and the relative abundance of white and black daisies. In turn, the abundance of white and black daisies determines the amount of solar radiation reflected back into space which feeds back to determine the planet's surface temperature and the relative abundance of white and black daisies. This system is presented in figure 1.3.

Both positive and negative feedbacks are important in environmental systems. For instance, the earth's climate includes many different positive and negative feedback effects that contribute to keeping our planet's average surface temperature close to 14 degrees Celsius. These feedbacks are illustrated in figure 1.4. One negative feedback comes from a rising surface temperature that raises the amount of water vapor in the atmosphere that, in turn, increases cloud cover that increases the amount of solar radiation reflected back into space and helps to reduce surface temperature. A positive feedback comes from a rising temperature that increases the melting of the permafrost and wetlands in northern latitudes that, in turn, releases methane (a greenhouse gas) and increases the concentration of greenhouse gases in the atmosphere. An increase in greenhouse gas concentrations increases the ability of the atmosphere to retain heat radiating from the surface and eventually raises surface temperatures.

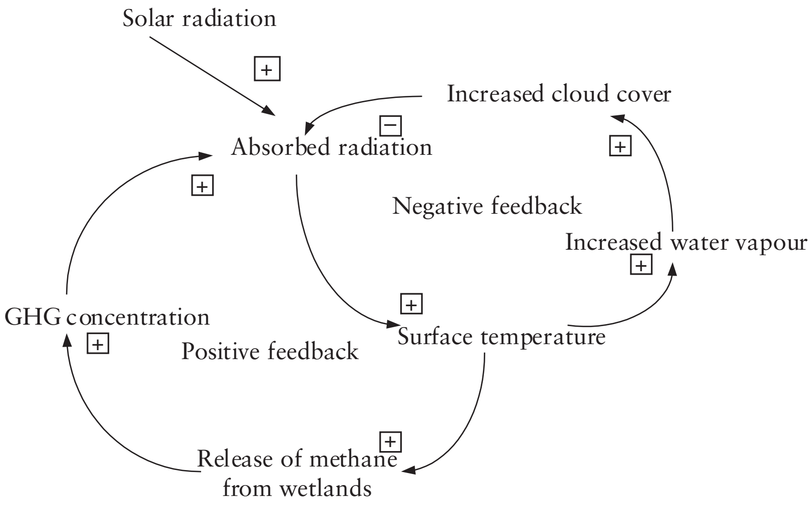

Figure 1.4 Examples of positive and negative feedbacks with climate change

Whatever the model, and whether it be used for optimization or simulation, the fundamental feedbacks of the system should be incorporated. More generally, a failure to incorporate feedback effects into models is likely to result in serious errors in prediction and a failure to understand the important interactions between variables. For example, in a set of models built in the 1970s that were enormously useful in helping people think about the interconnections and dynamics between human activities and environmental outcomes, modelers failed to adequately model the feedbacks between prices, quantity demanded and the supply (proven reserves) for non-renewable resources. In *illustrations* of the possible effects of unlimited economic growth where the demand for resources was assumed to increase exponentially, the model incorrectly predicted that the world's present and known reserves of gold, tin, petroleum, and silver in 1972 would be exhausted by 1990 (Meadows et al., 1974).

Stocks and flows

Common to both optimization and simulation models are stock and flow variables. Stocks, such as the level of capital, can be added to and subtracted from by flows, such as investment and depreciation. In dynamic optimization models, stock and flow relationships are characterized by dynamic constraints that define how a stock changes over time. For example, in an optimization model to maximize the present value of net profits from a fishery, the dynamic constraint that governs the fish stock could be

$$dx/dt = F(x) - h(t)$$

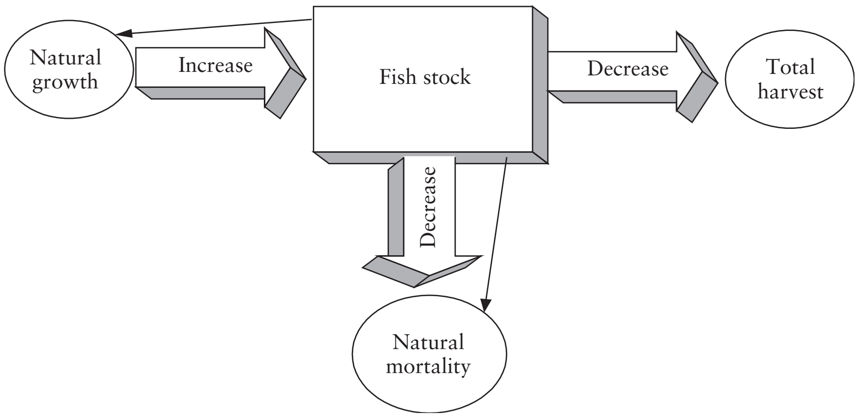

Figure 1.5 Stocks, flows, and feedbacks

where dx/dt is the change in the fish stock with respect to time, $F(x)$ is the natural growth function of the fish stock and h(t) is the harvest per time period. In this case, $F(x)$ is a flow determined by nature and the level of the stock and h(t) is a flow determined by decisions of fishers.

The relationship between stocks and flows can also be visualized in a simulation model, where natural growth is an inflow and natural mortality and the total fishing harvest are outflows represented by large arrows that increase or decrease the stock. A feedback relationship between a flow and a stock is represented by a single-line arrow that indicates the level of the fish stock helps determine both natural growth and natural mortality. A representation of a model in this form in figure 1.5 helps us to understand the relationships, causal connections, and feedbacks in a system.

1.4 MODEL DYNAMICS

The most cursory examination of the world around us reveals that life, our planet, and our universe are continually changing. The fossil record indicates that the earth has suffered from several mass extinctions, and that the earth's biota has changed dramatically in the relatively short period of time that modern humans have been in existence. Thus, researchers who wish to understand environmental challenges, and how to manage natural resources, must recognize that the world is dynamic.

Characteristics of dynamic systems

All natural systems are dynamic in the sense that they change over time, but are able to sustain life despite shocks. For example, the human body is a natural system

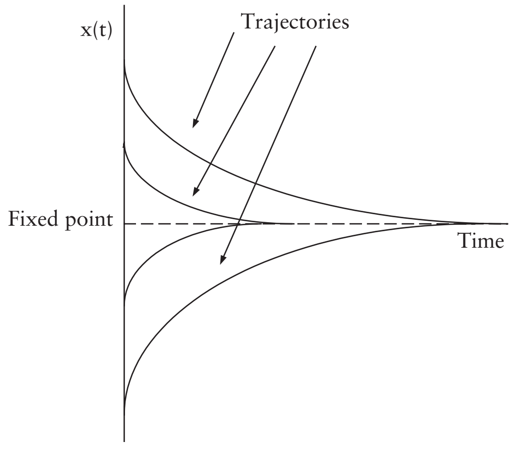

Figure 1.6 Trajectories to a fixed point

whose changes are governed by both underlying processes (such as genetics) and external factors that are partly under our control and predictable (such as our diet) and unpredictable events (such as being struck by lightning). Despite the many changes and shocks that our bodies undergo during our lifetime, they provide us with a blood pressure and a body temperature that vary by surprisingly small amounts despite huge variations and changes in our environment. Such a process that sustains life and that arises from both positive and negative feedbacks is called *homeostasis* and is a common feature in living systems.

Another important feature of dynamic systems is whether they, or variables within the system, tend to converge to a *fixed point* or steady state over time. In other words, is there some point, should it ever be reached, where the variable or system will remain at forever. The existence of fixed points and whether we can ever reach them is of particular importance when managing natural systems. For example, in a fishery we might wish to keep the resource stock within some desirable range and if we are not in this range, we would like to know whether we can arrive at these desirable levels, given sufficient time. This is illustrated in figure 1.6 where the fixed point might represent a desirable level of the resource stock. In this particular example, the fixed point is *globally stable* because whatever the initial value of the variable (be it greater or less than the fixed point) the variable will converge to it over time. The movement or transition of a variable or system from one value to another over time is called a trajectory and is also illustrated in figure 1.6.

A fixed point may or may not be an optimum in the sense that it optimizes a given objective function, but if it is an optimum it provides a point to which we would like the system or variable to converge. Ideally, we would wish for our global optimum (most desirable point) to be a globally stable fixed point in the sense that whatever the initial values of the system the trajectories always converge to the optimum. In reality, dynamic optimization is rarely so straightforward and

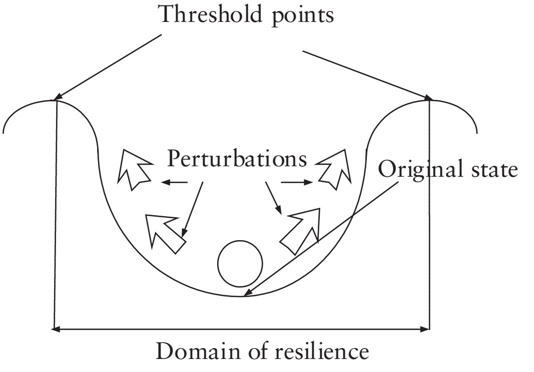

Figure 1.7 Resilience and threshold points

it involves devising a program such that trajectories approach a desired set of values. In some cases, a small change in the trajectories may lead to a radically different (and undesirable) outcome.

Despite the sophistication we can bring to modeling dynamic systems and behavior, our interpretation and prediction of actual systems can be very limited. In part, this arises because system dynamics often arise from both deterministic and stochastic processes and separating the causes, effects, and feedbacks can be very difficult. Fortunately, predicting future values in natural systems is made easier by negative feedbacks. The more able is a system to return to a former state the larger is the magnitude of a shock then the greater is its resilience (Holling, 1973). Unfortunately (for predictive purposes), and no matter how resilient is a system, there is ultimately some threshold point or nonlinearity beyond which the system switches or flips into a fundamentally different state. For example, acid rain over several years may gradually increase the acidity of a fresh-water lake with little apparent effect on the ecosystem, but suddenly at a certain point the environmental system may flip to a fundamentally different state. In the case of acid rain and fresh-water lakes, at a pH threshold point of 5.8 algal mats began to appear along the lake shore disrupting fish breeding and other aspects of the ecosystem.

This system behavior can be visualized in figure 1.7 where movements of the ball represent perturbations to a system and the low point in the "bowl" indicates the system's original state. Provided that the perturbations are not too large the system has a tendency to return to its original state. If, however, the system receives a large shock and is pushed "over the side" the process may become irreversible and the system may never return to its original state.

Discrete time models

Various techniques and approaches have been used to help model the dynamics of the environment and natural resources. Difference equations are used in modeling systems where change occurs at discrete points in time. Difference equations suppose that future values of variables of a system are a function of the current and possibly past values. A first-order difference equation, given below, supposes that the next period value is only a function of the current period value.

$$x_{t+1} = f(x_t)$$

where $f(x_t)$ may be either a linear or nonlinear function.

Difference equations can be used to model both linear and nonlinear behavior. They may also generate fixed points or steady-states (x^*) where x_t is unchanged for all time, i.e.,

$$x^* = f(x_t) \ \forall \ t$$

If the system converges to a fixed point, whatever the initial value of x_t, it is stable or convergent. For example, a system modeled by the difference equation, where a and b are constants,

$$x_{t+1} = a + bx_t$$

will converge to its fixed point of $a/(1-b)$, provided that $|b| < 1$. The fixed point is found by setting $x_{t+1} = x_t$ and then solving for x_t in terms of a and b. If $b < 0$ then the values of x_t will oscillate between positive and negative values. If $b > 1$ then the values of x_t become increasingly large as time progresses and there exists no fixed point or equilibrium. The solution to a difference equation is consistent with the original equation, but contains no lagged values. For this particular difference equation the solution is

$$x_t = a/(1-b) + b^t(x_0 - a/(1-b)).$$

The solution allows us to predict x_t at any time period provided we know the initial value (x_0) and the parameter values (a and b).

Difference equations can also be used to model seemingly very complex system behavior. A commonly used model of the population dynamics of some animal populations is logistic growth,

$$x_{t+1} = a\, x_t(1 - x_t)$$

where a is a constant. Logistic growth characterizes a population that has a low rate of increase when its population level is small and when it is large, and has its highest rates of growth at intermediate levels of the stock. Thus at low population levels a positive feedback exists between the population and growth in the population, but at a high population a negative feedback exists such that further increases in population reduce population growth. Such behavior is

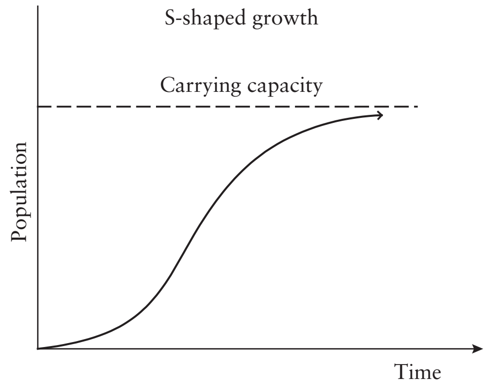

Figure 1.8 S-shaped growth

called density dependent growth. Logistic growth is sometimes referred to as sigmoidal or s-shaped growth, as shown in figure 1.8 because of the shape that it resembles when the total population is plotted against time and begins at a very low level. Because of negative feedback effects the population eventually reaches a carrying capacity beyond which the population cannot be sustained by the environment.

Chaos

To help understand the potential behavior of dynamic but deterministic systems, consider the trajectories or values of x_t over time in a logistic model. Provided that $a < 1$ then x_t converges to the fixed point 0 (population becomes extinct) because with each period of time x_t becomes successively smaller. In this case, the parameter a is at a level that extinction of the population is irreversible, whatever the initial population.

If a is greater than 1 but less than 3 then whatever the initial value of x_t the population will converge to the same fixed point or carrying capacity, for a given value of a. As we progressively increase a above 3 then the trajectory (set of points that represent the level of the population at different periods in time) of x_t starts to move towards not one, but two points called *attractors* and will go back and forth between the points. At increasingly higher values of a the number of attractors for the trajectory also rises such that the number of attractors doubles from 2 to 4 when $a \approx 3.45$ and doubles again to 8 points at $a \approx 3.54$, and keeps on doubling at slightly higher values of a. This switch in the qualitative behavior of a system is called a *bifurcation* and, in this case, is called *period doubling* to indicate that a small change in a parameter in the system doubles the number of attractors. As the number of attractors doubles, the time that it takes the system to return to a given attractor also doubles. Thus it takes twice as long to

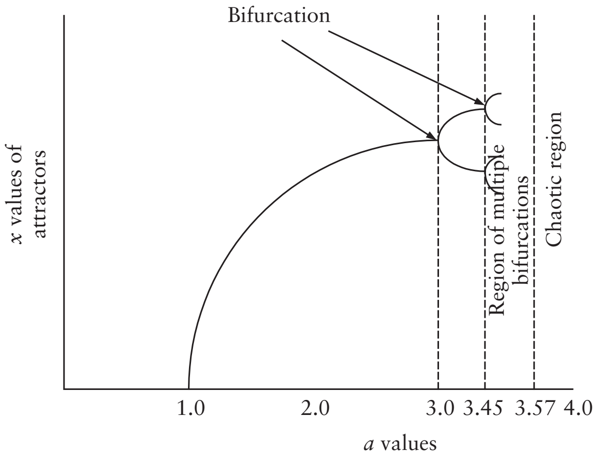

Figure 1.9 Bifurcation to chaos

return to a given attractor when there are four attractors than when there are just two attractors.

For values of a greater than 3.57 and equal to or less than 4, the system exhibits chaos and, depending on the initial value of x_t, the attractor (the points to which the system moves towards over time) may have an infinite number of values. The pattern of attractors for different values of a is illustrated in figure 1.9. Although the system is deterministic such that future values are completely determined, the system is highly sensitive to the initial value of x_t and the parameter a. Moreover, chaos can generate very complex dynamics *without* random shocks or stochastic events and if variables and states of the world are measured imprecisely, we can never predict their long-term values.

In reality, many systems are subject to both deterministic processes and stochastic events. For example, a population that is chaotic (and therefore deterministic) may also be subject to random shocks, such as changes in climate, that also influence its future state. Separating out the effects of shocks from the outcomes of deterministic processes or distinguishing between chaotic systems (which are deterministic) and systems that are not chaotic, but subject to stochastic fluctuations or events, is extremely difficult.

Continuous time models

Another way to model dynamics is to assume that change occurs continuously rather than at discrete points in time. The continuous time analog to difference equations are differential equations that can be written as

$$dx/dt = f(x,t)$$

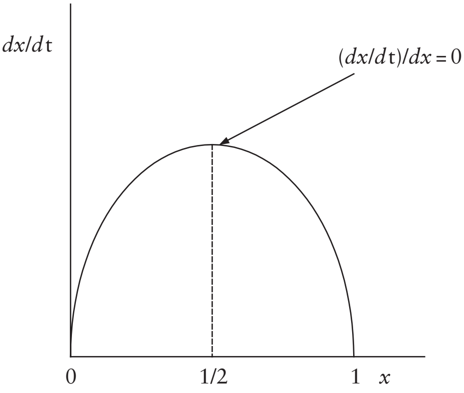

Figure 1.10 Logistic growth curve

where f(x,t) can be a linear or nonlinear function. For comparison, the differential equation and continuous time analog to the difference equation for logistic growth is:

$$dx/dt = ax\ (1 - x)$$

In the case where the differential equation is not a function of time, such as with logistic growth, the equation is said to be autonomous. The population with logistic growth has three fixed points (when $dx/dt = 0$); one when $x = 0$, a second when $x = 1/2$, and a third when $x = 1$. The first case is when the population is extinct, the second case is when the growth in the population is maximized or the point where $(dx/dt)/dx = 0$ and the third point is when the population is at its carrying capacity. The representation of the relationship between dx/dt and x is given in figure 1.10.

As with difference equations, a system of differential equations can be specified to represent the behavior of several and interacting variables over time. Various methods can be used to generate solutions to systems modeled by differential equations. Their solution must be consistent with the original equation, but must not contain any derivative term. Whether or not a system has fixed points and whether the system converges to a fixed point, and from which values, is a fundamental question. Such a question is of particular importance in optimization models where we may be concerned with reaching a target population level (such as a fishery stock) that maximizes our chosen objective function (such as the present value of net profits).

Like difference equations, differential equations can be used to model a range of dynamic behavior. For example, variables in a system may exhibit exponential growth or decay such that the rate of change in the variable over time is proportional to the size of the variable, i.e.,

$$dx/dt = (a - b)x.$$

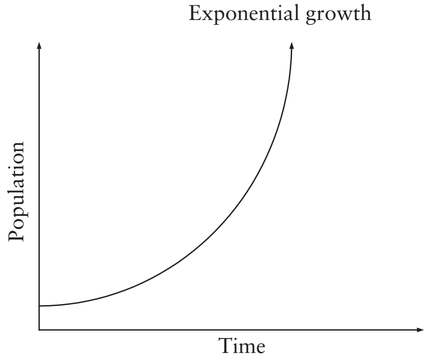

Figure 1.11 Exponential growth

In the case of a population, we can define a as the birth rate and b as the death rate and $(a - b)$ as the net growth rate. If the net growth rate is positive (negative) then the population will continuously grow (decay) over time. The solution to this differential equation can be found by integrating both sides of the equation where the lower and upper limits of integration are 0 and t and is given by,

$$x(\mathrm{t}) = x_o e^{(a - b)\mathrm{t}}$$

where $x(\mathrm{t})$ is the value of the population in time t, and x_o is the initial value of the population. In this system, future values of the variable or population are completely determined by the net growth rate (parameters a and b) and the initial starting value. The system has only one fixed point (point where $dx/d\mathrm{t} = 0$) when $x(\mathrm{t}) = 0$. The dynamics of the system for positive net growth are illustrated in figure 1.11. Although some variables may exhibit exponential growth over periods of time, no natural system can have exponential growth in the long run as ultimately energy, space, or other constraints must place a finite limit on the size of the variable or system.

1.5 DYNAMIC OPTIMIZATION

Dynamic optimization is an important method of analysis in environmental and resource economics. For discrete time problems the method called *dynamic programming*, pioneered by the American mathematician Richard Bellman in the 1950s, is often employed. For continuous time problems, economists frequently use a method called *optimal control* first developed by the Russian mathematician L. S. Pontryagin, and his colleagues, about fifty years ago. To be understood properly, both optimal control and dynamic programming require intensive study. Fortunately, the principles and intuition of both methods can be readily understood and applied in environmental and resource economics.

For both methods, the optimization problem must be properly specified. This requires an understanding of what variable(s) are under the control or decision of the person making the optimizing decisions. Such variables are called *control variables* in optimal control and *decision variables* in dynamic programming. The choice of these variables determines the values of *state variables* that are determined within the dynamic optimization model. The constraints to the problem include both *dynamic constraints* that represent how the state variables change over time and *boundary conditions* that specify the initial or starting values of the state variables, and possibly their value at the end of the program. Whether either approach yields a maximum or not also depends on so-called *sufficiency conditions*. For our purposes, this can be satisfied if the objective functional is differentiable and strictly concave in the control variable, no direct constraints are imposed on the value of the control variable, and the functions that govern how the state variables change over time – the *transition equation* in dynamic programming or the *dynamic constraint* in optimal control — are both differentiable and concave.

Dynamic programming

Dynamic programming is an algorithm that allows us to solve optimization problems that can be written as a multi-stage decision process where information about "the state of the world" is completely summarized in the current value of the state variable(s).

The algorithm is derived from the *principle of optimality* that allows us to solve a set of smaller problems for each decision stage, such that the value of the state variable in the next period depends only on the value of the state variable in the current period and the decision in the current period.

If the objective function satisfies certain sufficiency conditions and is also the sum of the net benefit or stage returns at each stage or point where a decision is made, we can define *Bellman's functional recurrence equation* to solve a discrete dynamic optimization problem. Starting with Bellman's functional recurrence equation for the last stage or final period, the algorithm obliges us to work backwards systematically to the initial period. The initial value of the state variable(s) is then used to solve the problem for all values of the decision variables and state variables at every period in the program. To illustrate, take the following problem,

$$\text{Max} \sum_{t=1}^{T} f_t(s(t), d(t)) \tag{1}$$

Subject to:

$$s(t+1) = g_t\,(s(t), d(t)) \tag{2}$$

$$s(1) = s_1,\ s(T+1) = s_{T+1} \tag{3}$$

where T is the final period in the program, $f_t(s(t),d(t))$ is the *net benefit* or *stage return function* which depends on the state variable at time t, $s(t)$, and the decision variable at time t, $d(t)$. The function $g_t(s(t),d(t))$ is the *transition* or *transformation function* at time t and determines the value of the state variable in the following stage or time period. An initial value of the state variable (s_1) is *always* required to obtain a solution, but this does not necessarily apply for its final value (s_{T+1}). The functional recurrence equation for this problem is,

$$V_t(s(t)) = \max_{d(t)} [f_t(s(t),d(t)) + V_{t+1}(s(t+1))] \tag{4}$$

where from (2), $V_{t+1}(s(t+1)) = V_{t+1}(g_t\ (s(t),d(t)))$.

In general, $V_{T+1}(s(T+1)) = 0$, as it is beyond the final stage or period of the program, T. The method of solution is to first express the problem in the form of the functional recurrence equation for the final stage or time period (T in the problem above) and use the value of the state variable at $T+1$ to obtain an expression for $V_T(s(T))$ solely in terms of $s(T)$. Next, we write the functional recurrence equation for the next to last stage or penultimate period ($T-1$), substitute $V_T(s(T))$ that we found previously into the expression for $V_{T-1}(s(T-1))$ and use the transition equation to substitute out $s(T)$ for $s(T-1)$ and $d(T-1)$. We then use the first-order condition $(\partial V_{T-1}(s(T))/(\partial d(T-1)) = 0$ at time $T-1$ to obtain an expression for $d^*(T-1)$ in terms of $s(T-1)$ and then substitute it into $V_{T-1}(s(T-1))$ so that the equation is solely in terms of $s(T-1)$. This backward recursion continues until we reach the first stage (or $t = 1$ in the problem above) ensuring that for each stage or time period, t, $V_t(s(t))$ has as its argument only $s(t)$. Using the initial condition, or initial value for the state variable, we can then determine $d^*(1)$ and then $s^*(2)$ and so on until $d^*(T)$ and $s^*(T)$, thus offering a full solution to the problem.

To illustrate the approach we can specify a simple two-period "cake eating" problem where a person receives a "cake" at the start of the first period ($t = 1$), but which must be consumed by the end of the program ($t = 3$). The objective is to maximize utility over time by consuming the cake where utility in each period equals the square root of the amount of the cake consumed, i.e.,

$$\text{Max U} = x_1^{1/2} + x_2^{1/2} \tag{5}$$

Subject to:

$$a_1 = 1 \tag{6}$$

$$a_2 = a_1 - x_1 \tag{7}$$

$$a_3 = 0 \tag{8}$$

where x_i is the amount of the cake consumed in period i and a_i is the amount of cake remaining at period i. For this problem, the sufficiency conditions are satisfied, thus, the approach yields a maximum. The functional recurrence equation in this case is,

$$V_t(a_t) = \max_{x(t)} [x_t^{1/2} + V_{t+1}(a_{t+1})] \quad (9)$$

Subject to:

$$a_{t+1} = a_t - x_t \quad (10)$$

where expression (9) or $V_t(a_t)$ is the *return function* and is the maximum value for (5) at time t, given the amount of cake left to be consumed (a_t). Expression (10) is the *transition equation* that determines the value of the next period's state variable.

The functional recurrence equation when t = 2 is

$$V_2(a_2) = \max[x_2^{1/2} + V_3(a_3)] \quad (11)$$

Subject to:

$$a_3 = a_2 - x_2 \quad (12)$$

$$a_3 = 0 \quad (13)$$

where $V_3(a_3)$ has the value of zero as it is the value of the return function after the end of the program or optimization period. Combining the constraints (12) and (13) we can obtain an expression for x_2 in terms of a_2 that we can use to rewrite the functional recurrence equation solely in terms of a_2, i.e.,

$$V_2(a_2) = a_2^{1/2} \quad (14)$$

The next step is to write the functional recurrence equation for the previous period, t = 1, i.e.,

$$V_1(a_1) = \max[x_1^{1/2} + V_2(a_2)] \quad (15)$$

Subject to:

$$a_2 = a_1 - x_1 \quad (16)$$

$$a_1 = 1 \quad (17)$$

We can substitute in the previously found return function $V_2(a_2)$ and then use (16) to obtain an expression for (15) solely in terms of a_1 and x_1 by substituting out for a_2, i.e.,

$$V_1(a_1) = \max[x_1^{1/2} + (a_1 - x_1)^{1/2}]$$

The necessary condition for a maximum requires that,

$$\begin{aligned} &\partial V_1(a_1)/\partial x_1 = (1/2)x_1^{-1/2} - 1/2(a_1 - x_1)^{-1/2} = 0 \\ &\Rightarrow x_1 \; = a_1 - x_1 \\ &\Rightarrow x_1^* = (1/2)a_1 \end{aligned} \tag{18}$$

Given that $a_1 = 1$, then $(x_1^*, x_2^*, a_2^*) = (0.5, 0.5, 0.5)$. This represents a complete solution to the "cake eating" problem over two periods.

Optimal control

Optimal control provides a set of necessary conditions to help solve dynamic problems in continuous time. These necessary conditions, sometimes called *the maximum principle*, are used to solve for *optimal paths* or trajectories for the control and state variables. The general form of problem that can be solved using optimal control, without discounting the future and where the end of the program T is fixed, can be represented by (19)–(21).

$$\text{Max V} = \int_{t=0}^{T} f[a(t),x(t),t]\, dt \tag{19}$$

Subject to:

$$da/dt = g[a(t),\, x(t),\, t] \tag{20}$$

$$a(0) = a_0 \tag{21}$$

In this problem, V is called the *objective functional*, x(t) is the control variable and a(t) is the state variable. All of the variables are functions of time. The dynamic constraint is given by (20) and governs how the state variable changes over time. The minimal boundary condition is the initial value of the state variable and is given by (21). In some problems, the terminal value of the state variable may also be specified as another boundary condition.

The method of solution is to write a function called a *Hamiltonian* that consists of the objective functional plus the dynamic constraint multiplied by a *co-state* or *adjoint variable* that is also a function of time, normally defined by the Greek symbol lambda, or λ. The co-state variable can be interpreted as the shadow or imputed price of the state variable at a given instant in time and, in this sense, is analogous to the notion of a Lagrangian multiplier in static optimization.

At the end of the program, denoted by T, it must be the case that $\lambda(T) = 0$ if $a(T) > 0$, otherwise we would not be on an optimal path and we would not be maximizing the objective functional subject to the constraints. To understand this point, consider the situation if $a(T) > 0$ and $\lambda(T) > 0$. In this case the state variable has a positive value (because $\lambda(T) > 0$), yet we have chosen to leave some of it at the end of the program. This must be sub-optimal because we could reduce the amount of the state variable remaining at the end of the program and simultaneously increase the objective functional.

For the problem specified by (19)–(21), the Hamiltonian function is as follows,

$$\mathrm{H}[a(\mathrm{t}), x(\mathrm{t}), \lambda(\mathrm{t}), \mathrm{t}] = f[a(\mathrm{t}), x(\mathrm{t}), \mathrm{t}] + \lambda(\mathrm{t})g[a(\mathrm{t}), x(\mathrm{t}), \mathrm{t}] \tag{22}$$

Provided there are no constraints on the control variable and the objective functional is differentiable in the control variable, the necessary conditions for solving (19)–(21) are listed below.

$$\partial \mathrm{H}/\partial x(\mathrm{t}) = 0 \tag{23}$$

$$d\lambda(\mathrm{t})/d\mathrm{t} = -\partial \mathrm{H}/\partial a(\mathrm{t}) \tag{24}$$

$$da/d\mathrm{t} = g[a(\mathrm{t}), x(\mathrm{t}), \mathrm{t}] \tag{25}$$

$$a(0) = a_0 \tag{26}$$

$$\lambda(T) = 0 \quad \text{if } a(T) > 0 \tag{27}$$

Condition (23) states that an optimal path requires that the partial derivative of the Hamiltonian function with respect to the control variable must equal zero at each point in time. Condition (24) states that the change in the co-state variable with respect to time must equal the negative of the partial derivative of the Hamiltonian function with respect to the state variable. Conditions (25) and (26) recover the dynamic constraint given by (20) and the boundary condition given by (21) in the original problem. Condition (27) is called a *transversality condition* that ensures the trajectories are optimal at terminal time T when the program ends.

Given that the conditions (23)–(27) use variables that are functions of time, finding the optimal paths for the control, state, and co-state variables often involves solving differential equations. Sometimes explicit solutions of these differential equations are impossible. In such cases, the "solution" or optimal paths of the variables may be represented qualitatively in terms of phase diagrams provided that the problem is *autonomous* such that time only appears as a function in the control, state, or co-state variables and not explicitly by itself. Phase diagrams trace out the points where the control and state variables are unchanging with respect to time, i.e., the points where $dx(\mathrm{t})/d\mathrm{t} = da(\mathrm{t})/d\mathrm{t} = 0$. Phase diagrams may also be constructed where explicit solutions are possible as they allow us to visualize and characterize the steady state of the dynamic system and the potential trajectories (if any) to the steady state.

To illustrate the optimal control approach, we can solve the continuous time analog to the "cake eating" problem. In continuous time, the problem can be defined by (28)–(31).

$$\mathrm{Max\ V} \int_{t=0}^{2} x(\mathrm{t})^{1/2}\, d\mathrm{t} \tag{28}$$

Subject to:

$$da/dt = -x(t) \tag{29}$$
$$a(0) = 1 \tag{30}$$
$$a(2) = 0 \tag{31}$$

In this problem, x(t) is the control variable and is the amount of cake eaten at an instant in time and a(t) is the state variable, or the amount of cake remaining at an instant in time. Expression (29) is the dynamic constraint where da/dt is the instantaneous change in the amount of cake remaining with respect to time and equals the negative of the amount of cake consumed at each instant in time. Equation (30) is the initial boundary condition and specifies the initial amount of cake available at $t = 0$. We also have an extra boundary condition, given by constraint (31), that specifies that all the cake must be eaten by the end of the program.

The Hamiltonian for the dynamic problem given by (28)–(31) is,

$$H[x(t), \lambda(t)] = x(t)^{1/2} + \lambda(t)[-x(t)] \tag{32}$$

The necessary conditions that must be satisfied to solve (28)–(31) are given below.

$$\partial H/\partial x(t) = \tfrac{1}{2}x(t)^{-1/2} - \lambda(t) = 0 \tag{33}$$
$$d\lambda(t)/dt = -\partial H/\partial a(t) = 0 \tag{34}$$
$$da/dt = -x(t) \tag{35}$$
$$a(0) = 1 \tag{36}$$
$$a(2) = 0 \tag{37}$$

In this case, we do not specify that $\lambda(T) = 0$ as the transversality condition is superfluous given the boundary condition specified by (37). Simplifying (33), we obtain the following expression for x(t) in terms of λ(t), i.e.,

$$x(t) = \frac{1}{4\lambda(t)^2} \tag{33'}$$

Substituting (33') into (35), or the dynamic constraint, and observing from (34) that $d\lambda(t)/dt = 0$, we can integrate both sides of the resulting expression with respect to time to obtain equation (38), where K is a constant of integration, i.e.,

$$a(t) = K - t/4(\lambda(t))^2 \tag{38}$$

The value of K in (38) can be solved by substituting the initial boundary condition, or (36), into (38) for when $t = 0$, i.e.,

$$1 = K - 0/4(\lambda(0))^2 \quad \Rightarrow K = 1$$

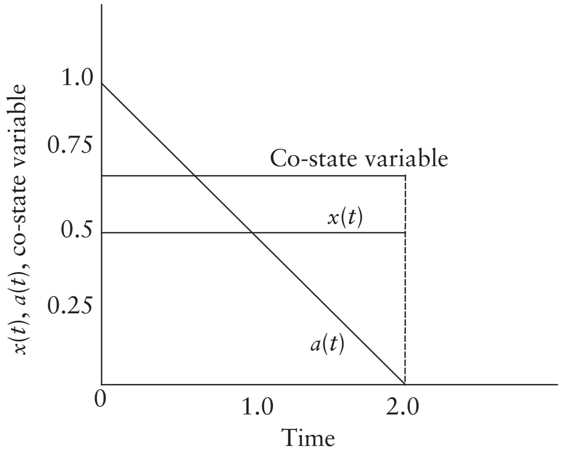

Figure 1.12 Optimal paths in a "cake-eating" problem

Thus, we can rewrite the optimal path for the state variable, a(t), as follows,

$$a(\text{t}) = 1 - \text{t}/4(\lambda(\text{t}))^2 \qquad (38')$$

It now remains to solve for the co-state variable, λ(t). Before we do so, we can characterize the solution by noting from (34) that the co-state variable is unchanging with respect to time, i.e., it is a constant. Similarly, from the expression for x(t) in (33′), expression (34) also implies that the control variable is unchanging with respect to time, i.e., $dx(\text{t})/d\text{t} = 0$. In other words, both the control and co-state variables will be a constant over the program from t = 0 to t = 2.

From the boundary condition at the end of the program, condition (37), we can solve out for the value of the co-state variable at t = 2 using (38′) and thus find the value of λ(2), i.e.,

$$0 = 1 - 2/4(\lambda(2))^2 \ \Rightarrow \lambda(2) = \sqrt{1/2} \qquad (39)$$

If λ(t) has the value of the $\sqrt{1/2}$ at t = 2, it must also have this value at every point in time during the program given condition (34). Substituting the value of λ(t) given in (39) into (33′), and also into (38′), we obtain the optimal paths for the control and state variables, i.e.,

$$x(\text{t}) = 1/2 \qquad (33'')$$
$$a(\text{t}) = 1 - \text{t}/2 \qquad (38'')$$

The optimal paths described by (33″) and (38″) are illustrated in figure 1.12.

In figure 1.12, the area defined by triangle 0-1-2 equals one, as does the rectangular area defined beneath the x(t) line and the horizontal axis, indicating that the amount consumed over the program exactly equals the amount of cake at the beginning of the program. The slope of the a(t) line is da(t)/dt and equals $-1/2$ (or the negative of x(t)) and characterizes the dynamic constraint given by (29) or (35).

If we discount future values and costs, both the Hamiltonian and the necessary conditions need to be modified. In the case of discounting, the objective functional can be specified by (40).

$$\text{Max V} = \int_{t=0}^{\infty} f[a(\text{t}),x(\text{t}),\text{t}]e^{-\delta \text{t}}\, d\text{t} \tag{40}$$

In (40), $\exp^{-\delta t}$ or $e^{-\delta t}$ is the continuous time discount factor, e = base of the natural logarithm and δ is the instantaneous discount rate. If we use the constraints given by (20) and (21) and the objective functional given by (40), the Hamiltonian with discounting is given by (41).

$$\text{H}[a(\text{t}), x(\text{t}), \lambda(\text{t}), \delta, \text{t}] = f[a(\text{t}), x(\text{t}), \text{t}]e^{-\delta \text{t}} + \lambda(\text{t})g[a(\text{t}), x(\text{t}), \text{t}] \tag{41}$$

Expression (41) is defined as the *present-value Hamiltonian*. More commonly, the necessary conditions are defined from the *current-value Hamiltonian* defined as $H = \text{H}\, e^{\delta t}$, i.e.,

$$H = f[a(\text{t}), x(\text{t}), \text{t}] + \mu(\text{t})g[a(\text{t}), x(\text{t}), \text{t}] \tag{42}$$

where $\mu(\text{t}) = e^{\delta t}\lambda(\text{t})$. The only changes to the necessary conditions (now defined in terms of the current-value Hamiltonian) given by (23) to (27) are in terms of the co-state variable. These modified necessary conditions are given by (43)–(47).

$$\partial H/\partial x(\text{t}) = 0 \tag{43}$$

$$d\mu(\text{t})/d\text{t} - \delta\mu(\text{t}) = -\partial H/\partial a(\text{t}) \tag{44}$$

$$da/d\text{t} = g[a(\text{t}), x(\text{t}), \text{t}] \tag{45}$$

$$a(0) = a_0 \tag{46}$$

$$\mu(T)e^{-\delta t} = 0 \text{ if } a(T) > 0 \tag{47}$$

If the program has an infinite time horizon and the problem is autonomous then the transversality condition given by (47) only holds true in the limit as $\text{t} \to \infty$, provided no constraints are imposed on the value of the state variables. For problems where the terminal time T is chosen by the solution to the program, an additional transversality constraint also applies, namely, $H(T)\, e^{-\delta T} = 0$. In other words, the present-value Hamiltonian must be zero at terminal time.

1.6 DYNAMICS AND ENVIRONMENTAL AND RESOURCE ECONOMICS

An understanding of models, model building, dynamics and systems provides a useful starting point for appreciating the research problems and approaches that predominate in environmental, ecological, and resource economics. In models of fisheries, water, forestry, and other natural resources a fundamental question is, how can we do the best we can given our own and nature's constraints? For such problems, dynamic optimization models are widely employed. Depending upon the nature of the problem, several different approaches can be used for their solution. Such problems can be solved by "pen and paper," but software is also available. For models that are linear in both the objective function and constraints powerful algorithms exist for their solution and several different software packages are available, including GAMS (www.gams.com) that can solve very large mathematical programming problems. However, even quite complex optimization problems can be solved using spreadsheet software, such as Excel (Conrad, 1999). For highly nonlinear objective functions, maxima and minima can be solved for using software packages such as MATHEMATICA (www.wolfram.com) or MAPLE (www.maplesoft.com).

A comprehension of environmental values, environmental accounting, economic growth and the environment, and the interconnections in the global commons also requires that we understand the broad dynamics and feedback effects of the systems we wish to understand. Whatever the question or problem, a systematic and scientific approach to modeling provides us with a framework for increasing our understanding of and, ultimately, improving our environment.

FURTHER READING

This chapter provides an introduction to modeling, systems, and dynamics. Given the importance of modeling in economics, surprisingly very few books explain or discuss how to economically model research problems. A wonderful exception is Blaug (1980) who provides a description of the key methodological issues in economics. Sterman (2000), chapter 3, gives an excellent introduction to the building of simulation models.

A plethora of texts exist on the methods of dynamic analysis. A useful introduction that offers questions and answers in mathematical economics is Grafton and Sargent (1996). A textbook on mathematical economics that is comprehensive and comprehensible is Hoy et al. (2001). Three of the best textbooks on dynamic optimization models, with applications to economics, are Shone (1997), Léonard and Van Long (1992) and Chiang (1992). An excellent book on the solution of dynamic optimization models in natural resource economics is Conrad (1999). Several good texts on building and using simulation models exist including Ford (1999) and Deaton and Winebrake (1999). Williams (1997) is a rigorous but highly accessible book on chaos.

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