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Gottlob Frege (1848–1925)

MICHAEL DUMMETT

Friedrich Ludwig Gottlob Frege was born in Wismar in 1848, and died in Bad Kleinen in 1925. His whole adult career was spent, from 1874 to 1918, in the Mathematics Department of Jena University. He devoted almost all his life to work on the borderline between mathematics and philosophy. In his lifetime, that work was little regarded, save by Bertrand Russell and Ludwig Wittgenstein; his death was marked by very few. Yet today he is celebrated as the founder of modern mathematical logic and as the grandfather of analytical philosophy.

Frege's intellectual career was unusual. Most philosophers and mathematicians make contributions to diverse topics within their fields; but Frege set himself to achieve one particular, though extensive, task. From 1879 to 1906 he pursued this single ambitious aim: to set arithmetic upon secure foundations. Virtually all that he wrote during those years was devoted to this project, or to the elaboration of ideas that he developed in the course of trying to carry it through and took as integral to it. The term "arithmetic," as he used it, is to be understood in a broad sense: for him, it comprised, not only number theory (i.e. the theory of the natural numbers), but also analysis, that is, the theory of real and complex numbers. He did not attempt to construct the foundations of geometry. He viewed mathematics in the traditional way, as divided into the theory of quantity, and thus of cardinal numbers and of numbers measuring the magnitudes of quantities (real numbers), and the theory of space (points, lines, and planes). He believed these two parts of mathematics to rest on different foundations. The foundations of arithmetic – of number theory and analysis – are purely logical. But although the truths of geometry are a priori, they rest upon spatial intuition: they are synthetic a priori, in the Kantian trichotomy Frege accepted. Kant was therefore right about geometry; but he was wrong about arithmetic. All appeal to spatial or temporal intuition must be expelled from arithmetic: its concepts must be formulated and its basic principles established without recourse to intuition of any kind.

It was to the task of establishing the purely logical foundations of arithmetic that Frege devoted his whole intellectual endeavor. In carrying out this task, he was led into some purely philosophical investigations; it is for this reason that, although a mathematician, he is now held in such high regard by philosophers of the analytic school. His attempt to provide arithmetic with secure logical foundations was embodied in three books, all of high importance, although he also published a number of articles, spin-offs from his central endeavor. The first of these three books was the short *Begriffsschrift* (*Conceptual Notation*) of 1879: this expounded a new formalization of logic. It was the first work of modern mathematical logic: it contained an axiomatic system of predicate logic of precisely the type that was to become standard. The notation was utterly different from that which would become standard, and was essentially two-dimensional, the two clauses of a conditional being written on different lines; but the notation was essentially isomorphic to that which Peano, Hilbert, and Russell later made standard. An English translation of the book and of some related articles was published by T. W. Bynum in 1972. The *Begriffsschrift* received six reviews, including a lengthy one by Ernst Schröder; but none of the reviewers understood Frege's intention or his achievement. The second book was also short, though not quite so short as *Begriffsschrift*. It was called *Die Grundlagen der Arithmetik* (*The Foundations of Arithmetic*) and appeared in 1884. In order that it should be accessible to as many as possible, the book was written without the use of logical symbols. Frege first surveys a range of rival theories on the status of arithmetical propositions and the nature of (cardinal) number, and demolishes them with trenchant arguments; this part of the book is largely philosophical in character. Then, having to his satisfaction left no space for any other theory but his own, he proceeds to sketch a purely logical derivation of the fundamental laws of arithmetic. It is to be doubted whether any other philosophical treatise of comparable length has, since Plato, ever manifested such brilliance as Frege's *Grundlagen*. An English translation by J. L. Austin was published in 1950, and a critical edition by C. Thiel in 1986. This time Frege's book received five reviews, again none of them adequate to their subject matter. One was by Georg Cantor, who unhappily does not seem to have tried to understand the work, with which he might have been expected to be in large sympathy.

Frege had thought that he was on the verge of success in constructing definitive foundations for arithmetic. He had thought that his *Grundlagen* would make this plain to the world of philosophers and mathematicians. He became intensely depressed by his failure to have conveyed to that world the magnitude of his achievement. At the same time, he became aware of deficiencies in the philosophical basis on which, in *Grundlagen*, he had rested his arguments and which underpinned his formal logic. There followed five years during which he published nothing, but engaged in a thoroughgoing revision of his philosophy of logic and of his formal logic. The outcome of this revision he expounded in a lecture, *Function und Begriff* (*Function and Concept*), given in 1891. He then set about a complete formal exposition of his foundations of arithmetic in his third great book, *Grundgesetze der Arithmetik* (*Basic Laws of Arithmetic*). This was utterly unlike *Grundlagen*. The first volume came out in 1893, the second volume not until ten years later, in 1903. The book is incomplete; a third volume must have been planned, but it never appeared. In the first volume of *Grundgesetze*, there is no argument, only exposition. Frege began by explaining his formal logical system, and expounding, without giving any argument for or justification of them, the philosophical, or, more exactly, semantic notions that underpinned it. He provides what is in effect a precise semantic theory for the formal language used in the book. This makes up Part I, of which an English translation by M. Furth was published in 1964.

There follows in Part II a string of formal derivations, carried out in Frege's far from easily read symbolism, which execute in detail the program sketched in *Grundlagen* for constructing a logical foundation for the theory of cardinal numbers, including the natural numbers, understood as finite cardinals. Part II was concluded in the second volume of 1903, and is followed by a Part III devoted to the real numbers, a topic Frege had never treated in any detail before. Part III is not completed in the second volume; a clear intention to complete it proves that a third volume was contemplated. The first half of Part III is in prose, not symbols; Frege has changed his approach. He attempts in this half to do for real numbers what he had done in *Grundlagen* for cardinal numbers: to review and criticize rival theories so that there would appear no alternative to his own construction of a foundation for analysis, carried out by proof in his formal system in the second half of Part III. Unhappily, he had lost his touch. Whereas in *Grundlagen* nothing is mentioned save what will carry the argument forward, the first half of Part III of *Grundgesetze* reads as if he was merely determined to get his own back on the other theorists who had neglected him, Cantor and Dedekind included. Criticisms are made of features of their work quite irrelevant to the main strand of the argument; errors are pointed out which could easily be rectified, without any indication of how to rectify them. A powerful critique of formalism is included. It is possible to extract Frege's reasons for the strategy he adopts for constructing foundations of analysis; but the whole lacks the brilliance and the exquisite planning of *Grundlagen*.

When he had finished composing the second volume of *Grundgesetze*, Frege must have felt a deep contentment. Still embittered by the neglect of his work, he surely believed that he had attained his life's goal: he had constructed what he thought to be definitive foundations for the theories both of natural numbers and of real numbers. But, while the book was in press, he received the heaviest blow of all, delivered by one of his few admirers, the young Bertrand Russell. Russell wrote in June 1902 to explain the celebrated contradiction he had discovered in the (naive) theory of classes, and to point out that it could be derived in Frege's logical system (see RUSSELL). At first Frege was shattered. Then, as he reflected on the matter, he devised a weakening of his Basic Law V, which governed the abstraction operator used for forming symbols for classes and was responsible for the contradiction; he was confident that this modification would restore consistency to his system. The modification was explained in an Appendix added to Volume II of *Grundgesetze*. The fact was, however, that the modified Basic Law V still allowed the derivation of a contradiction. Whether Frege ever discovered this is uncertain; but what he must have discovered was that, in its presence, none of the proofs he had given of crucial theorems would go through. His wife died in 1905. It took him until August 1906 to convince himself that his logical system could not be patched up; at that point he had to face the fact that the project to which he had devoted his life had failed. *Grundgesetze* remains the only part of Frege's published work of which no full English translation has yet appeared (save for the translation by Furth of Part I, which also contains the Appendix to Volume II, and excerpts in the volume of translations by Geach and Black).

A brief fragment among Frege's literary remains, dated August 1906, asks the question, "What can I regard as the outcome of my work?" In other words, "What remains now that the contradiction has destroyed my logical foundations for arithmetic?" Frege's answer was that what survived of enduring value was his logical system, stripped of the abstraction operator, and the whole structure of philosophical logic which, from 1891 onwards, had underpinned it. He continued to think about these topics, although it was not for years that he published anything more. Eventually, during the war years, he published the first three chapters of what was planned as a comprehensive treatise on logic, although it was never finished. In the very last years of his life, he finally turned again to the foundations of mathematics; reversing his lifelong view, he began a derivation of arithmetic from geometry, but did not carry it very far.

Frege's judgment of 1906 about where the value of his work lay was at first the judgment of those who participated in the revived study of Frege's writings. In the Preface to his *Tractatus Logico-Philosophicus* of 1922, Ludwig Wittgenstein had written of "the great works of Frege and the works of my friend Bertrand Russell," and had referred frequently to Frege in the book. Despite the celebrity of the *Tractatus* and its wide influence, this failed to stimulate more than a very few philosophers to find out anything about Frege. Rudolf Carnap, who had actually heard Frege's lectures, Alonzo Church, and Peter Geach continued to hold him in high regard (see CARNAP); but the majority of philosophers, British, German, and American, went on ignoring him. The revival of interest in him began in a slow way in the 1960s, and gathered momentum in the 1970s; by the end of that decade, he had become required reading for any student of analytic philosophy. But the interest in his philosophy of arithmetic was meager; it was taken for granted that his system's having run foul of Russell's contradiction destroyed all its pretensions to serious consideration. Interest in Frege therefore concentrated on what made him the grandfather of analytic philosophy: his philosophical analysis of language and of thought which underlay his formal logic.

The key to a modern system of predicate logic is of course the quantifier-variable notation for generality, which Frege introduced for the first time in *Begriffsschrift*. He employed only negation, the conditional, and the universal quantifier; he did not use symbols for conjunction, disjunction, or the existential quantifier, but expressed these by means of the three logical constants for which he did have symbols. Frege insisted that his symbolism, unlike that in the Boolean tradition such as Schröder's, could incorporate a formal language: it needed only the addition of suitable nonlogical constants, predicates, etc., to be capable of framing sentences on any subject matter whatever, and of carrying out deductive reasoning concerning it. Frege did not conceive of formulae in his symbolism in the way that Tarski was to do, namely as built up from atomic formulae containing free variables waiting to become bound in the process of forming complex formulae. Rather, he thought of them as built up out of atomic *sentences*. This required that, before attaching a quantifier, there must first be formed, from a suitable sentence, what we should call a predicate, but, for Frege, was a functional expression or expression for a concept. (He eschewed the term "predicate" as too closely associated with the traditional subject–predicate logic.) Such an expression was "incomplete" or "unsaturated": it could not stand on its own, but had gaps in it, being formed from a sentence by omitting one or more occurrences of a singular term. When a quantifier was attached to it, the bound variables governed by it were to be inserted into these gaps, thereby showing to what expression for a concept the quantifier had been attached.

When he wanted to speak of a particular expression for a concept, Frege used the lower-case Greek letter ξ to indicate the gaps in it; but an expression containing ξ was no part of the formal language, but only something to be used metalinguistically to speak about the formal language. Thus from the sentence "Pitt respects Pitt's father" (which represents the way we are meant to understand the colloquial "Pitt respects his father") three different expressions for concepts could be formed: "ξ respects Pitt's father," "Pitt respects ξ's father," and "ξ respects ξ's father." This exemplifies the fact that a sentence can be analyzed in different ways. Frege insists in *Begriffsschrift* that these different analyses have nothing to do with the content of the sentence, but only with our way of looking at it; in other words, our grasp of the content of the sentence does not depend upon our noticing that it is possible to analyze it in one way or another, for example that the proper name "Pitt" occurs twice within it. In one sense, each of the three expressions for concepts occurs within the sentence; but none of the different concepts is part of the content of the sentence. By attaching the universal quantifier to these three expressions for concepts, we obtain respectively "For all *a*, *a* respects Pitt's father," "For all *a*, Pitt respects *a*'s father," and "For all *a*, *a* respects *a*'s father," or, colloquially, "Everyone respects Pitt's father," "Pitt respects everyone's father," and "Everyone respects his own father." And now, in these quantified sentences, Frege says, the expression for the relevant concept *is* part of the content of the sentence.

The process of forming expressions for concepts may likewise be used to form expressions for functions, as we ordinarily conceive them, namely by starting with a complex singular term within which some simpler singular term occurs. And it may also be used to form expressions for relations between two objects, namely by removing from a sentence one or more occurrences of each of two different singular terms. This was of importance for second-order logic, admitting quantification over functions and relations.

Thus Frege's invention of the quantifier-variable notation yielded him several fundamental insights. First, the conception of concept-expressions as incomplete solved the problem of the unity of sentences and the thoughts they express. No glue is needed to make the parts of the sentence adhere to one another. The concept-expression or relation-expression is of its nature incapable of standing alone, but can be present only when its argument-places are filled by singular terms to form a sentence. Or else it is itself the argument of a quantifier, forming a different kind of sentence: it is made to adhere to such terms, or to have a quantifier attached to it, and cannot exist otherwise. Secondly, concept-formation does not consist solely of the psychological abstraction of some common feature from individual objects or of the process of applying Boolean operations to given concepts (conjoining or disjoining them). By the process of omitting singular terms from complete sentences, or, equivalently, of thinking of them as replaceable by other singular terms, we can arrive at expressions for concepts with new boundaries, and so at the concepts thus expressed. Moreover, such expressions were not, in general, actual *parts* of the sentences from which they were formed; they were, rather, patterns exemplified by different sentences. The expression "ξ respects ξ's father" occurs both in "Pitt respects Pitt's father" and in "Fox respects Fox's father"; the two sentences have, not just common words, but a common *pattern*. Thus we should not think of a sentence, or the thought it expresses, as formed out of its component parts, but of the components as attainable by analyzing the sentence; still less should we think of a concept-expression as formed out of its components, but as a result of analyzing a sentence. Because apprehending the possibility of analyzing a sentence in a particular way requires us to see it as manifesting a certain pattern, which is not required for a simple grasp of the sentence's content, and because apprehending this possibility may be essential to recognizing the validity of some deductive inference, there is a creative ingredient in deductive inference. Such inference does not depend only upon a grasp of the contents of the sentences that figure in it; and this explains how deductive inference can lead us to new knowledge, which consideration of its role in mathematics makes evident that it does.

These ideas were expressed in *Begriffsschrift* and in *Grundlagen*. Part I of *Begriffsschrift* was devoted to sentential logic, and Part II to first-order predicate logic. Although Frege did not have the concept of the completeness of a logical system, he had in fact framed a complete formalization of first-order logic. Part III of *Begriffsschrift* is devoted to second-order predicate logic, involving quantification over concepts, relations, and functions; Frege never saw any reason for regarding the first-order fragment as especially significant. To explain second-order quantification in the same way as first-order quantification, Frege has to admit the notion of an expression for a concept of second level, formed by removing from a sentence one or more occurrences of an expression for a first-level concept, or of a relational or functional expression; these second-level concept-expressions all have different types of incompleteness. In Part III Frege gave his purely logical definition of a sequence; since previously the notion of an infinite sequence had usually been explained in temporal terms, as involving its successive construction step by step, or else a successive diversion of attention from one term to the next, Frege regarded this as an essential contribution to the program of expelling intuition from arithmetic in favor of purely logical notions. It was especially important for number theory, since the natural numbers themselves could be defined as the terms of a finite sequence beginning with 0 and proceeding from each term to its successor. Frege's definition of a sequence was so framed that, when the natural numbers are so defined, the principle of finite induction, sometimes claimed as a method of reasoning peculiar to arithmetic, becomes a direct consequence of the definition. Frege's definition of "sequence" is now generally known as the definition of the ancestral of a relation, namely the relation which the first term of a finite sequence has to the last term when each term but the last stands in the original relation to the next term. (It is named the "ancestral relation" because the relation "ancestor of" is the ancestral of the relation "parent of.")

There are three features of *Grundlagen* of especial interest to philosophy in general. The first is the distinction between the actual (*wirklich*) and the objective. Frege used "actual" to mean "concrete" in the sense in which concrete objects are distinguished from abstract ones; an object is actual if it is capable of affecting the senses, directly or indirectly. But something may be objective even though it is not actual; an example he gives is the Equator. You cannot see or trip over the Equator, but it is not fictitious or subjective; statements about it may be objectively true or false. We can make reference to objects which, though objective, are not actual, and make objectively true statements about them. Frege thus rejected what is now called "nominalism" as based on a fundamental error. This was crucial for his philosophy of arithmetic. He took numbers to be objects, objective but not actual: we can refer to them and make objectively true statements about them.

A principle greatly stressed in *Grundlagen* is what has come to be known as the "context principle": that it is only in the context of a sentence that a word has a meaning. It is noteworthy that the principle is formulated linguistically, as concerning the meanings of words in sentences, rather than, say, as "We can think of anything only in the course of thinking that something holds good of it." The interpretation of the dictum is contentious. At the very least, it is an assertion of the primacy of sentences in the order of explanation of meaning. We must first explain what, in general, constitutes the meaning of a sentence, and then explain the meanings of all small expressions as their contributions to the meanings of sentences in which they occur. When we look at how Frege applies the principle in the book, it appears to have a much stronger significance: namely that, to secure a meaning for an expression or type of expression, it suffices to determine the senses of all sentences in which it occurs. Frege never reiterated the context principle in any subsequent writing, although there is a strong echo of it in Part I of *Grundgesetze*.

The other salient feature is the first clear example of the linguistic turn, giving Frege a strong claim to be the grandfather of analytic philosophy. At a critical point of the book, Frege, having already argued that our notion of number is not derivable from sense-perception or intuition, asks, "How, then, are numbers given to us?" The question is both epistemological and ontological: how are we aware of numbers, and what guarantee is there that such objects as numbers exist? In answering it, Frege simply assumes that it can be equated to "How are meanings conferred on numerical terms?" He appeals immediately to the context principle; in virtue of this, the question reduces to, "What sense attaches to statements containing numerical terms?" A question about what objects exist and how we know of them is thus transformed into a question about the meanings of certain sentences.

However, those of Frege's ideas that most interested analytic philosophers when interest in his work revived were the ones he expounded in his middle period (1891–1906). Frege had no general term for meaning, in the sense in which the meaning of a word or expression comprises everything that a speaker must implicitly know about it in order to understand it. He distinguished three features which, in this sense, may contribute to the meaning of a word or sentence: force, tone, and sense. The force of an utterance is what distinguishes an assertion from a question, and Frege recognized only these two types of force: assertoric and interrogative. In English interrogative force is usually indicated by the inversion of the verb and subject; Frege insisted that the sense of a question inviting the answer "Yes" or "No" will coincide with that of the corresponding assertoric statement. What differentiates them is the significance of the utterance: in one case we ask whether the thought expressed is true, in the other we commit ourselves to its truth. It was important for Frege that only a complete utterance can carry force; a declarative sentence serving as, say, one clause of a disjunctive statement or as the antecedent of a conditional one does not have assertoric force, which is attached only to the statement as a whole. It was essential, Frege thought, not to construe the verb or predicate of a sentence as intrinsically containing the assertoric force within it. Natural languages usually lack any express means of indicating that assertoric force is to be attached to a sentence, but Frege considered this a defect of them. In his formal language he used a symbol for just this purpose, the "judgment-stroke" (often called by others the "assertion sign"). It is a philosophical mistake to speak of "judgments" when all that we are concerned with is their contents; unless we are actually concerned with the act of recognizing them as true, we should speak in this connection of thoughts, rather than of judgments. Frege did not recognize imperatival or other kinds of force, though it may plausibly be argued that an imperative sentence expresses the thought that is true if the command is obeyed or the demand complied with. He simply declared that such a sentence expresses a command, not a thought.

What I have called "tone," and Frege called *Färbung* (coloring) is distinguished from sense in that it cannot affect the truth or falsity of what is said. The English sentences "He has died," "He is deceased," and "He has passed away" do not differ in sense, but only in tone. Likewise, where A and B are sentences, the complex sentences "A and B" and "Not only A but B" do not differ in sense, but in tone: if either is true, the other is true, even if it conveys an inappropriate suggestion. The sense of a whole sentence is the thought that it expresses; the sense of a part of a complex expression, including a sentence, is part of the sense of the whole.

In his middle period, Frege drew a distinction between the significance of an expression and what it signifies, which he had not done in his early period. For the thing signified, he confusingly chose the word "*Bedeutung*," the ordinary German word for "meaning": but the *Bedeutung* of an expression is not part of its meaning, where "meaning" is understood as specified above. It is not necessary, in order to understand a word or phrase, to know its *Bedeutung*, only its sense. Frege's term is conventionally rendered in English either "meaning" or "reference"; neither is happy. The *Bedeutung* of a singular term is the object we use the term to talk about. It is impossible just to know the *Bedeutung* of a singular term, even if that term is logically simple, i.e. it is a proper name in the restricted sense (Frege misleadingly called all singular terms "*Eigennamen*" – proper names). Frege followed Kant in holding that every object of which we are aware is given to us in a particular way; the sense of a singular term embodies the particular way in which its *Bedeutung* is given to us in virtue of our understanding of the term.

But it was not only singular terms which Frege took as having *Bedeutungen*: he ascribed them to every expression that could be a genuine constituent of a sentence, including incomplete ones such as concept-expressions and sentences themselves. He does not argue that any such expression must have a *Bedeutung*; he takes it for granted. The only question he canvasses is what kind of thing the *Bedeutung* of an expression of any given type should be taken to be. This causes much perplexity to those reading Frege for the first time: surely there is nothing to which a concept-expression or a sentence stands as a name stands to the object named. The only way to arrive at an understanding of Frege's notion of *Bedeutung* is to look at the use to which he puts it. That use is governed by four fundamental theses:

1 The *Bedeutung* of a part of a complex expression is not part of the *Bedeutung* of the whole.
2 But the *Bedeutung* of the whole depends uniquely upon the *Bedeutungen* of its parts.
3 If a part lacks *Bedeutung*, the whole lacks *Bedeutung*.
4 The *Bedeutung* of a sentence is its truth-value – its being true or its being false.

Thesis (1) follows from the fact that Sweden is not part of Stockholm, the capital of Sweden; and thesis (3) derives from the consideration that, if there is no such country as Ruritania, then there is no such city as the capital of Ruritania. As for the identification of truth-values as the *Bedeutungen* of sentences in accordance with thesis (4), that follows from Frege's extensionalist logic, despite its failure to fit natural language. According to it, a subsentence of a complex sentence contributes to the truth-value of the whole solely by its own truth-value. Counterexamples from natural language then have to be explained away. Instances are sentences in indirect speech following verbs like "said that" and "believes that"; as is well known, Frege handled these by deeming the sentences following "that" to have a special, indirect sense, whereby their *Bedeutung* became the senses that they would express when in direct speech.

It is plain from these four theses that the *Bedeutung* of an expression constitutes its contribution to the determination of the truth-value of any sentence in which it occurs. This explains why Frege takes it for granted that any expression capable of occurring in a sentence without denying it a truth-value must have a *Bedeutung*. It also makes a Fregean theory of *Bedeutung* for a language equivalent to what we understand as a semantic theory for that language, which is a theory explaining how sentences of the language are determined as true or as false in accordance with their composition. The semantic value of an expression, in such a theory, is precisely that which contributes to the determination of the truth-value of a sentence in which that expression occurs. We may therefore equate the notion of the *Bedeutung* of an expression, as Frege conceived it, with that of its semantic value.

In a conventional semantic theory for a formalized language, the semantic value of an individual constant or other singular term is an element of the domain denoted by the term. The semantic value of a one-place predicate is a class of elements of the domain; the sentence resulting from putting the term in the argument-place of the predicate is true if the element denoted by the term is a member of the class constituting the semantic value of the predicate, false otherwise. Frege did not speak of the domain of quantification; so far as can be determined, he took the individual variables to range over all objects whatever, the *Bedeutung* of any term being such an object. Frege did not take the *Bedeutung* of a concept-expression to be a class. He called the *Bedeutung* of a concept-expression a "concept." This must not be understood in the sense in which we may speak of acquiring a concept or grasping a concept, which has to do with the senses expressed by words. In *Grundlagen*, the word "concept" (*Begriff*) had been used both in this way and in conformity with what was to become Frege's usage in his middle period; but, in that period, he took a concept to stand to a concept-expression as an object stands to a singular term, and thus not at all as the sense of that expression.

For Frege, a concept must be distinguished from a class, which was for him a particular kind of object. A class is the extension of a concept, comprising those objects that fall under the concept; but the extension of a concept is a derivative notion, only to be so explained. The relation of being a member of a class can be explained only as that of falling under a concept of which the class is the extension; any attempt to explain it in any other way turns the relation into that of part to whole, which is quite different. So we can attain the concept of a class only via that of a concept; and we can characterize any particular class only by citing a concept of which it is the extension.

Since a concept-expression was for Frege incomplete, so its *Bedeutung* cannot be an object of any kind, but must be likewise incomplete, an entity needing an object to saturate it. This is a difficult conception, but is to be thought of by analogy with how we think of functions. We think of an arithmetical function as a principle according to which one number is arrived at, given another: not a *method* of arriving at the value, given the argument, but simply the association of the value to the argument. Its incompleteness consists in the fact that there is nothing to it save this association: its existence consists solely in its linking arguments to values. Likewise, the existence of a concept consists solely in its having certain objects falling under it, and others not falling under it. In facts, according to the doctrines of Frege's middle period, concepts simply are a particular type of function. For he regarded truth-values – truth and falsity – as themselves being objects. So a concept is a function which takes only truth-values as values, mapping an object falling under it on to the value *true*, and one not falling under it on to the value *false*. In the same way, a relation is a function with two arguments, all of whose values are truth-values.

In his early period (1874–85), Frege had not distinguished between significance and thing signified; he had used the one term "content" for both without differentiation. It was a great advance that in his middle period he sharply distinguished them as sense and *Bedeutung*. What we tacitly know in understanding a word or expression is its sense; its sense is the way its *Bedeutung* is given to us. It is not only of an object that it holds good that it must be given to us in a particular way: the same holds good of concepts, relations, and functions. For instance, an arithmetical function may be given to us by means of a particular procedure for computing its value, given its argument or arguments; other procedures might serve to determine the values of just the same function. For this reason, the content of any piece of knowledge that we may have concerning a given expression can never simply consist in our knowing its *Bedeutung*, but must be our knowing its *Bedeutung* as given in a particular way. The *Bedeutung* of an expression is therefore no part of its meaning, where this is what we grasp in understanding the expression: what we grasp is its *sense*. We may indeed grasp more than its sense, namely what was called above its tone. Sense is that part of the meaning of the expression that is relevant to the determination of a sentence containing it as true or as false. But the notions of sense and *Bedeutung* are closely connected: again, the sense of an expression is the way in which its *Bedeutung* is given to us. (The sense is *die Art des Gegebenseins*, usually clumsily translated "the mode of presentation.")

It is not only that each individual speaker must think of the *Bedeutung* of a word as given in some particular way, leaving it possible for different speakers to think of it as given in different ways. For successful communication, the speakers must know that the *Bedeutung* of a word, as each is using it, is the same. To ensure this, it must be a convention of the language that each associates with the word the same sense, that is, the same way of thinking of something as its *Bedeutung*. An imaginary example given by Frege in a letter to Jourdain is that the *Bedeutung* of the name "Afla" might be given as the mountain visible on the northern horizon from such-and-such a place, and that of the name "Ateb" as the mountain visible on the southern horizon from a certain other place. It may prove that the two names have as *Bedeutung* the very same mountain, which was not at first evident; the identity-statement "Afla and Ateb are the same" is informative and reports an empirical discovery. Famously, in his celebrated essay "Über *Sinn* und *Bedeutung*" of 1892, Frege used the example of the names "the Morning Star" and "the Evening Star," which both denote the planet Venus, to illustrate the distinction between sense and *Bedeutung* and so explain how a true statement of identity could be informative (he had used the same example earlier in his lecture "Function and Concept" of 1891).

To know the sense of an expression is, therefore, to know how its *Bedeutung* is determined: not necessarily how *we* can determine it, since we may lack an effective means of doing so, but how, as it were, reality determines it in accordance with the sense we have given it. The sense of a part is part of the sense of the whole; the sense of any given expression is part of the sense of any more complex expression of which the given expression is part. So a grasp of the sense of an expression involves knowing how it may be put together with other expressions to form a complex expression – ultimately, a sentence – and how the sense of the complex is determined from the senses of its parts. To grasp the sense of a concept-expression is to apprehend a particular way of thinking of something incomplete as its *Bedeutung*, something that associates each arbitrary object with a truth-value: a concept that carries each object into the value *true* or the value *false* according as it falls under the concept or not. In general, we grasp the sense of a whole sentence by grasping the sense of each expression composing it, which is its contribution to the sense of the sentence as a whole; and to do this is to have a particular conception of the *Bedeutung* of each constituent, together with a grasp of how these *Bedeutungen* combine to yield the *Bedeutung* of each phrase and ultimately of the sentence itself. But the *Bedeutung* of a sentence is a truth-value; its sense Frege terms a *thought*. In the case of a sentence, the distinction between sense and *Bedeutung* is that between a thought and its truth-value. Thus to grasp a thought is to apprehend how it is determined – by reality, though not necessarily by us – as true or as false. And to grasp a sense that goes to compose a thought by being the sense of a constituent of a sentence that expresses that thought is to understand how the contribution to determining the truth-value of the thought that is made by that constituent is itself determined. In the words of *Grundgesetze*, Part I, the thought expressed by a sentence is the thought that the condition for its truth is fulfilled. This was an expression of what has become the most popular form of a theory of meaning, a truth-conditional theory: truth is the central notion of such a theory, and meaning is to be explained in terms of it.

Frege held that anyone who makes a judgment knows implicitly what truth and falsity are. We can express a thought without asserting or judging it to be true, which we do whenever we utter a sentence whose sense it is but to which assertoric force is not attached (e.g. when we ask whether it is true). When we judge the thought to be true, we "advance from the thought to the truth-value." But this advance is not a further thought, to the effect that the original thought is true; by prefacing the sentence expressing the thought with the words "It is true that", we do not confer assertoric force on it, but merely express the very same thought as before. That is why Frege says, in one of his posthumously published writings, that the word "true" seems to make the impossible possible. Frege held the notion of truth to be indefinable: he rejected the correspondence theory of truth, and any other such theory that professes to say what truth is.

Frege was vehemently opposed to psychologistic explanations of concepts, that is, of the senses of linguistic expressions. He opposed explanations in terms of the inner mental operations by which we acquire such concepts. The sense of any expression had to be explained objectively, not subjectively, in terms of the conditions for the truth of sentences containing the expression. A thought, for Frege, is not one of the contents of the mind, as is a sense-impression or a mental image. These are subjective and incommunicable; but it is of the essence of thoughts to be communicable. Different people can grasp the very same thought; it cannot therefore be a content of any of their minds. This rejection of psychologism was of the greatest importance: it rescued the philosophy of thought and of language from explanations given in terms of private psychological processes. Frege's alternative explanation was neither so popular nor so successful. He recognized no intermediate category between the subjective and the wholly objective. He took thoughts and their component senses to constitute a "third realm": like the physical universe, its inhabitants are objective, but, unlike it, they are not in time or space or perceived by the senses. But it is only through our grasp of the inhabitants of the third realm that mere sense-impressions are converted into perceptions, and so we become aware of the external world. We can grasp thoughts and express them: but we human beings can grasp them only as expressed in language or in symbolism.

Frege's attitude to language was ambivalent. He viewed natural language as full of defects: only when it was conducted by means of a purified language, such as his logical symbolism, could deductive reasoning be confidently relied on. So some of the time he inveighs against language, declaring that philosophy must struggle against it and that his real concern is with thoughts and not with the means of their expression. Yet a great deal of his discussions are concerned precisely with language and its workings. His philosophical logic is not a theory of thought, independent of language: it is a systematic theory of meaning, applicable directly to a language purified of the defects of our everyday speech, but indirectly to natural language. The power of his theory of meaning rests upon the capacity of predicate logic – the logic he first invented – to analyze the structure of a great range of sentences and of the thoughts they express. Although many of his ideas were not found acceptable by later analytic philosophers, his theories were seen as a better model of what philosophy should aim at, in framing its basic theories of meaning and of thought, than anything supplied by any other philosopher; and his discussions of problems within that realm a better place to start from than any other.

In recent years there has been a great revival of interest in Frege's philosophy of mathematics, the late George Boolos being one of those to have contributed greatly to it. The comparison between Frege's *Die Grundlagen der Arithmetik* and Richard Dedekind's *Was sind und was sollen die Zahlen?*, two books which approach the same subject matter very differently, is extremely fruitful. Dedekind is concerned to characterize the abstract structure of the sequence of natural numbers; having done so, he arrives at that specific sequence by an operation of psychological abstraction, a quite illegitimate device much favored by mathematicians and philosophers of the time. He acknowledges the use of the natural numbers to give the cardinality of finite classes, but only as a minor corollary. For Frege, by contrast, that use is central. It was for him the primary application of the natural numbers, and must therefore figure in their definition. "It is applicability alone," he wrote in Part III of *Grundgesetze*, "that raises arithmetic from the rank of a game to that of a science." He strongly opposed appeal, such as that made by J. S. Mill, to empirical notions having to do with one or other particular type of application, in defining the natural numbers or the real numbers. But he thought it essential that, in defining them, the general principle underlying all their applications should be made central to their definition. Hence natural numbers were to be presented as finite cardinals: the operator in terms of which all numerical terms were to be framed was "the number of *x*'s such that . . . *x* . . . ," where of course the gap was to be filled by an expression for a concept of first level.

Frege's aim was to show that arithmetical were derivable from purely logical principles. A description of physical space as non-Euclidean is intelligible; so Euclidean geometry is not analytically true. By contrast, any attempt to describe a world in which the truths of arithmetic fail is incoherent. Since Frege characterizes logical notions as those which are topic-neutral, applying to things of every kind, arithmetical notions are already logical ones. But, like Russell's "axiom of infinity," a proposition may be expressed in logical terms without its truth being guaranteed by logic. It therefore remains to be shown that what we take to be the fundamental truths of number theory are derivable from purely logical principles.

Frege endorsed the definition of equicardinality that was becoming generally accepted by mathematicians, in particular Cantor:

> There are just as many *F*s as *G*s iff there is a relation which maps the *F*s one-to-one on to the *G*s.

If there is a cup on every saucer on the table, and every cup on the table is on a saucer, we shall know that there are just as many cups as saucers on the table without necessarily knowing how many of each there are. In *Grundlagen* Frege enunciates a basic principle governing his cardinality operator:

> (*) The number of *F*s = the number of *G*s iff there are just as many *F*s as *G*s,

"just as many as" being interpreted in accordance with the foregoing definition. He decides that the cardinality operator cannot be defined contextually, but requires an explicit definition: the one that he chooses is:

> The number of *F*s = the class of concepts *G* such that there are just as many *F*s as *G*s.

Here Frege appeals to the notion of a class for the first time, although he never again considers classes of concepts rather than of objects. But the appeal is solely for the purpose of framing an explicit definition of the cardinality operator; Frege uses it for nothing else than proving the principle (*) from it: all the theorems he goes on to prove about the natural numbers are derived from (*) alone, without further recourse to the definition of "the number of."

The theory sketched in *Grundlagen* is elaborated and fully formalized in *Grundgesetze*, Part II. *Grundgesetze* makes extensive use of the notion of classes, or, rather, of Frege's generalization of it, that of value-ranges: a class is the extension of a first-level concept, while a value-range is the extension of a first-level function of one argument. The latter notion is for Frege the more fundamental one, since concepts are for him a special kind of function. Frege had convinced himself that the notion of a value-range was a logical one. The Basic Law governing the operator forming terms for value-ranges is Law V:

the value-range of f = the value-range of g iff $f(x) = g(x)$ for every x.

It was of course this law which gave rise to the contradiction. Because of this, interest has centered upon a possible modification of Frege's construction of number theory, in which there are no value-ranges or classes, but the cardinality operator, governed by (*), is treated as primitive. Attention has focused on what is now called "Frege's Theorem," namely the proposition that, using Frege's definitions of "0," "successor," and "natural number," all of Peano's axioms, and hence the whole of second-order Peano arithmetic, can be derived in a second-order system from (*) alone. Opinions vary about how close this result brings us to Frege's goal of proving the truths of number theory to be analytic.

While most attention has been paid to Frege's foundations for number theory, some has been given to his foundation for the theory of real numbers, expounded in the incomplete Part III of *Grundgesetze*. Unlike both Cantor and Dedekind, Frege does not first construct the rational numbers and then define real numbers in terms of them: true to his principle that types of number are distinguished by their applications, and holding that both rationals and irrationals serve to give the magnitude of a quantity, he simply treats rational numbers as a kind of real number, defining the latter directly. While cardinal numbers answer questions of the form "How many . . . ?," real numbers answer those of the form "How much . . . ?" Any such question that can be answered by a rational number can also be answered by an irrational number. There are various quantitative domains – lengths, durations, masses, electric charge, etc.; within each, the magnitude of a quantity is given as the ratio of the given quantity to some chosen unit quantity; these ratios are the same from domain to domain. Thus real numbers are to be defined as ratios of quantities belonging to the same domain; such a definition accords with Frege's general tenet, that the definition of a type of number should incorporate the general principle underlying all its applications.

In the sections of Part III included in Volume II of *Grundgesetze*, Frege is concerned to characterize quantitative domains, and he identifies them as groups of permutations of an underlying set satisfying certain conditions. Unknown to Frege, this work had been partially anticipated by Otto Hölder in an article of 1901. Neither Frege nor Hölder uses explicit group-theoretical terminology. Both of them were concerned with groups with an ordering upon them. Hölder is generally credited with having proved the Archimedean law from the completeness of the ordering, which Frege also proved; but Frege's assumptions are much weaker than Hölder's. Frege assumes only that the ordering is right-invariant and that it is upper semi-linear (the ordering is linear upon the elements greater than any given element); Hölder makes the further assumptions that it is also left-invariant, fully linear, and dense. This preliminary part of Frege's construction of the foundations of analysis contains substantial contributions to group theory, and Part III as a whole presents pregnant ideas about how real numbers should be explained.

Frege's work on the philosophy of mathematics offered an explanation of how deductive reasoning can extend our knowledge, and a conception of the significance of the applications of a theory to its foundations. It also challenges us to say on what our recognition of mathematical truth rests, if not on pure logic or, more generally, on purely conceptual truths. But it offers another challenge not so often recognized.

Frege's attempt in Part I of *Grundgesetze* to justify his introduction of value-ranges was undoubtedly a failure: he was attempting simultaneously to specify the domain of his individual variables and to interpret his primitive symbols over that domain. But he was facing a problem that is usually left untackled: how can we without circularity justify the existence of domains sufficiently large to contain the objects of our fundamental mathematical theories such as number theory and analysis? Until a convincing answer is given to this question, we shall not have a satisfying philosophy of mathematics.