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Problem Section

Solutions

In vol. 52, nr. 2 of March 1998 we published a solution to problem 335. However, the solution published is not the answer to the question. Your editor should have been more critical.

Problem 334** (N. Schmitz, Münster, Germany)
Does there exist a sequence of identically distributed pairwise uncorrelated random variables (which, of course, fulfils the weak law of large numbers) which does not fulfil the strong law of large numbers?

It was stated that the answer is negative. R. Helmers, D. Gilat, and M. S. Keane referred to theorem 5.1.2 from K. L. Chung (1989), *A course in probability theory*; 2nd edition, Academic Press. However, this does not settle the question. To straighten up things, A. A. Jagers did not suggest a solution, but suggested that the statement and proof of the strong law of large numbers in B. Fristedt & L. Gray (1997), *A modern approach to probability theory*; Birkhäuser, Boston, etc. might be an interesting reference because it avoids a number of unnecessary conditions.

It is well known (namely as a result by Rajchman in 1932) that for the i.i.d. case with existing variance the strong law of large numbers holds true. So, the references mentioned are correct. The complication is that there exist random variables X and Y with finite expectations, infinite variances, and finite expectation EXY. The Cauchy–Schwartz inequality establishes that EXY is finite if the variances of X and Y are finite.

So, the problem has not been solved. In order to avoid unclarities, we give another formulation of the problem. We hope that our readers will come up with a solution.

*Problem 334*** (N. Schmitz, Münster, Germany)
Does there exist a sequence of identically distributed pairwise uncorrelated random variables (with infinite variance; otherwise a result due to Rajchman may be applied) which does not fulfil the strong law of large numbers, or may the theorem of Etemadi be generalized (from pairwise independent) to pairwise uncorrelated random variables?

*Problem 328*** (R. Theodorescu, Ste. Foy, Canada)
Let $m_n(t)$ be the nth moment of a Poisson distribution with parameter $t > 0$. It is known that

$$m_n(t) = \sum_{j=0}^{n} S(n, j)t^j, \tag{1}$$

where $S(n, j)$ is the Stirling number of the second kind; moreover,

$$m_{n+1}(t) = t(m_n(t) + (\mathrm{d}/\mathrm{d}t)m_n(t)).$$

Consider the normalized moments $c_n(t) = m_n(t)/n!$. CONJECTURE: for each $t \geq 1$,

$$c_{n+1}^2(t) > c_n(t)c_{n+2}(t), \quad n \geq 0. \tag{2}$$

Is the logconcavity Conjecture true?

In Statistica Neerlandica, Vol. 51 (1997), (p. 378) A. A. Jagers gave a positive asymptotic answer for $t = 1$ to Problem 328. the proposer contributed some additional comments. For $t \geq 0$, $n \geq 0$,

$$m_{n+1}^2(t) \leq m_n(t)m_{n+2}(t); \tag{3}$$

a property communicated to the proposer by W. Kratz (University of Ulm, Germany). From (3) we obtain

$$(n+1)m_{n+1}^2(t) \leq (n+1)m_n(t)m_{n+2}(t), \quad n \geq 0, \tag{4}$$

whereas Conjecture (2) may be rewritten in the form

$$(n+2)m_{n+1}^2(t) > (n+1)m_n(t)m_{n+2}(t), \quad n \geq 1, \tag{5}$$

Inspection of (4) and (5) explains the difficulty in proving/disproving (2) for each $t \geq 1$.

Another proof of the asymptotic answer is as follows. For $n \geq 0$ there exists t_n such that

$$c_{n+1}^2(t) > c_n(t)c_{n+2}(t) \tag{6}$$

for $t \geq t_n$. This result can be established as follows. From (1) we deduce

$$m_n(t) = t^n + \sum_{k=0}^{n-1} S(n, k)t^k. \tag{7}$$

Hence

$$(n+2)m_{n+1}^2(t) - (n+1)m_n(t)m_{n+2}(t) = t^{2n+2}\{1 + \mathrm{O}(1/t)\} > 0$$

for $t \to \infty$. The Conjecture is not true for $t < 1$ and it is true asymptotically. It still remains open for each $t \geq 1$.

New Problems

Problem 346 (F. W. STEUTEL)

A game is played as follows. Two players A and B take turns in drawing random numbers, independent and uniformly distributed on (0, 1). The person who first draws a number smaller than the last number drawn by his opponent loses the game. Person A draws the first number. Question: What is the probability P that A wins the game?

Problems marked with * are nonelementary, of problems marked with ** no solution is known to the editor; unmarked problems are not necessarily simple. Solutions of the problem in this issue should arrive before January 31, 1999. Problems (preferably with solutions) and solutions (type-written on separate sheets bearing the name of the solver) are welcomed by the column editor.

Contributions are requested to use *electronic mail* and to submit material in *LATEX* format, whenever this is possible, because this speeds up processing the problem section.