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Problem Section

Solutions

Problem 319 (A. A. JAGERS, W. C. M. KALLENBERG), revision

Let X and Y be i.i.d. random variables. It is obvious that $X+Y$ and $X-Y$ have the same spread in the sense that $\mathrm{Var}(X+Y)=\mathrm{Var}(X-Y)$. In the L_1-sense this is not the case. From problem 314 it follows that

$$\mathrm{E}\mid X+Y-\mathrm{E}(X+Y)\mid \geq \mathrm{E}\mid X-Y-\mathrm{E}(X-Y)\mid$$

with equality iff $X-\mathrm{E}X$ is symmetric around 0. Here the expectation is used as a measure of location, which is quite natural for spread in the L_2-sense, the variance. In the L_1-sense the median is more natural ($\mathrm{E}\mid Z-a\mid$ is minimized by $a=\mathrm{med}\ Z$). Show that

$$\mathrm{E}\mid X+Y-\mathrm{med}(X+Y)\mid \geq \mathrm{E}\mid X-Y-\mathrm{med}(X-Y)\mid$$

When does the equality sign hold?

A solution was submitted by the proposers.

Solution by THE PROPOSERS

Noting that 0 is the (unique) median of $X-Y$ we get for any median $\mathrm{med}(X+Y)$ of $X+Y$

$$\begin{aligned}
&\mathrm{E}\mid (X+Y)-\mathrm{med}(X+Y)\mid \\
&\quad = \mathrm{E}\mid \{X-\tfrac{1}{2}\mathrm{med}(X+Y)\}+\{Y-\tfrac{1}{2}\mathrm{med}(X+Y)\}\mid \\
&\quad \geq \mathrm{E}\mid \{X-\tfrac{1}{2}\mathrm{med}(X+Y)\}-\{Y-\tfrac{1}{2}\mathrm{med}(X+Y)\}\mid \\
&\quad = \mathrm{E}\mid X-Y\mid = \mathrm{E}\mid (X-Y)-\mathrm{med}(X-Y)\mid
\end{aligned}$$

on account of the result from problem 314, see Statistica Neerlandica, vol. 49 (1995), pages 262–264. The equality sign holds if and only if $X-\frac{1}{2}\mathrm{med}(X+Y)$ is symmetric around 0.

Problem 326 (R. JANS, LOUVAIN, BELGIUM)

In a lottery game, n numbers are randomly drawn from a set of m numbers. On a ticket you may fill out n numbers hoping that they will match the n numbers selected. We want to know the minimum number of tickets one has to fill out in order to be sure that among the tickets filled out there is at least one ticket that has $n-1$ (or $n-2, n-3, \ldots$) numbers correct. Can this problem be solved using calculus of combinations?

A solution was submitted by J. B. Vermetten.

Solution by J. B. VERMETTEN

The number of combinations of n numbers from a set of m numbers is equal to $\binom{m}{n}$. We assume that the tickets have been filled out in a systematic way, so that no two tickets contain the same combination of numbers. In order to solve the problem with the help of calculus of combinations, the set of m numbers is divided into 2 groups, viz. the 1st group with the n numbers selected and the 2nd group with the $m-n$ numbers, which are not selected. The number of combinations of i numbers taken from the first group is $\binom{n}{i}$. The number of combinations of $n-i$ numbers taken from the 2nd group is $\binom{m-n}{n-i}$. Each combination of the 1st group can be combined with each combination of the 2nd group. The two combinations together give a combination of n numbers with i numbers of the 1st group and $n-i$ numbers of the 2nd group. Their number is $\binom{n}{i}\binom{m-n}{n-i}$. To be sure that at least one ticket has a combination with $n-k$ (or more) numbers correct, we must take the total number of combinations with less than $n-k$ correspondances, plus 1. So the solution for the minimum number of tickets is

$$\sum_{i=0}^{n-k-1}\binom{n}{i}\binom{m-n}{n-i}+1$$

If k is small one may summarize all correct combinations, subtract them from the total number of possible combinations, and then again add 1. By substituting j for $n-i$ and cancelling the $+1$ against $j=0$ in the sum, the alternative solution for the minimum number of tickets is

$$\binom{m}{n}-\sum_{j=1}^{k}\binom{n}{j}\binom{m-n}{j}$$

Problem 327 (K. VAN HARN AND F. W. STEUTEL)

For $n \in \mathbb{N}$, let f_n be the gamma probability density with shape parameter $t = n/2$ and scale parameter $\lambda = 1$, so

$$f_n(x) = \frac{1}{\Gamma(n/2)} x^{(n/2)-1} \mathrm{e}^{-x} \quad [x > 0]$$

Define the function: $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) := 2\sum_{n=1}^{\infty}(-1)^{n-1} f_n(x) \quad [x > 0]$$

Show that f is a probability density.

The problem has been solved by A. A. Jagers and the proposers. The solutions make use of the Laplace transform.

Solution by A. A. Jagers

Let $g : (0, \infty) \to \mathbb{R}$ be given by

$$g(x) = \int_1^\infty \frac{\sqrt{\lambda} e^{-\lambda x}}{\sqrt{\pi x}} \frac{\mathrm{d}\lambda}{\lambda^2} \quad (x > 0)$$

Then g is a probability density being a mixture of gamma probability densities (with shape parameter 1/2 and scale parameter λ). Note that $\int_1^\infty \lambda^{-2} \mathrm{d}\lambda = 1$, indeed. We shall show that $f = g$ by means of Laplace transforms. In fact, for $s > 0$ we have (by dominated convergence)

$$\mathcal{L}(f(x), s) = 2 \sum_{n=1}^\infty (-1)^{n-1} \mathcal{L}(f_n(x), s) = 2 \sum_{n=1}^\infty (-1)^{n-1} (s+1)^{-n/2}$$

$$= \frac{2}{\sqrt{s+1}+1} = \frac{2}{s}(\sqrt{s+1} - 1) = \frac{1}{s} \int_0^s \frac{\mathrm{d}w}{\sqrt{w+1}}$$

$$= \int_0^1 \frac{\mathrm{d}v}{\sqrt{sv+1}} = \int_1^\infty \frac{\lambda^{3/2} \mathrm{d}\lambda}{\sqrt{s+\lambda}}$$

$$= \int_1^\infty \mathcal{L}(e^{-\lambda x}(\pi x)^{-1/2}, s) \lambda^{-3/2} \mathrm{d}\lambda$$

$$= \mathcal{L}(g(x), s), \text{ as required.}$$

*Problem 328*** (R. Theodorescu, Ste. Foy, Canada)

Let $m_n(t)$ be the nth moment of a Poisson distribution with parameter $t > 0$. It is known that

$$m_n(t) = \sum_{j=0}^n S(n,j) t^j$$

where $S(n,j)$ is the Stirling number of the second kind; moreover,

$$m_{n+1}(t) = t(m_n(t) + (\mathrm{d}/\mathrm{d}t) m_n(t))$$

Consider the normalized moments $c_n(t) = m_n(t)/n!$ Conjecture: for each $t > 0$.

$$c_{n+1}^2(t) > c_n(t) c_{n+2}(t), \quad n \geq 1 \ (*)$$

For $n = 0$ $(*)$ holds for $t \geq 1$. Is the logconcavity Conjecture $(*)$ true ?

Unfortunately no solutions were submitted. The problem will be kept open.

New Problems

*Problem 332*** (P. A. Bekker)

Let $X_1,\ X_2,\ X_3$ and X_4 be i.i.d. random vectors in $\mathbb{R}^2$. Let Δ denote the interior of the convex hull of $X_1,\ X_2$ and X_3. The question is to find

$$\sup P\{X_4 \in \Delta\}$$

where the sup is taken over all probability distributions on $\mathbb{R}^2$.

*Problem 333*** (A. G. M. Steerneman)

Assume that (X_1, X_2) follows a bivariate normal distribution:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

Consider the probability distribution of $Q = X_1X_2$. For which values of the parameters is the probability density function of Q unimodal? A p.d.f. f is unimodal if there exists a value x_0 such that f is non-decreasing on $(-\infty, x_0]$ and f is non-increasing on $[x_0, \infty)$.

*Problem 334*** (A. G. M. Steerneman)

Assume that (X_1, X_2) follows a bivariate normal distribution:

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

Define $R = X_1/X_2$. For which value(s) of the parameters is the distribution of R infinitely divisible?

Problems marked with * are nonelementary, of problems marked with ** no solution is known to the editor; unmarked problems are not necessarily simple. Solutions of the problems in this issue should arrive before May 31, 1997. Problems (preferably with solutions) and solutions (type-written on separate sheets bearing the name of the solver) are welcomed by the column editor.

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Correction

Correction to W. K. Fung, On the lack of invariance of some Bayesian outlier models, *Statistica Neerlandica*, **50**, 3, 339–343, 1996

The "S" in the numerator of the right hand side of equation (2) should be replaced by "S_0".