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Problem Section

Solutions

In Statistica Neerlandica vol. 50, no. 3, we published a solution to problem 323. Your problem section editor would like to mention that also M. N. Deshpande (India) has given a solution. The solution that has been submitted by A. ten Cate also gives a further generalization for part b to n boys and m girls. In the previous issue, I have stated that he solved the case for n boys and n girls. Sorry!
In Statistica Neerlandica vol. 50, no. 2 we have discussed problem 316. Another solution than the one by the proposer has been submitted by M. S. Keane. It applies a similar idea, but it is more simple.

Problem 316 (V. Mammitzsch, Germany)

Is the distribution function of $f(X + Y) - X$ continuous, if X and Y are independent and have continuous distribution functions?

Solution by M. S. Keane

Let $\{X_n\}$ and $\{Y_n\}$ be independent sequences of i.i.d. random variables each taking the two values 0 and 1 with probability 1/2. Define

$$X = \sum_{n=1}^{\infty} X_n/4^n$$

$$Y = \sum_{n=1}^{\infty} 2Y_n/4^n$$

Then $X + Y$ has a uniform distribution on [0, 1]. It can be shown that X has a continuous distribution function. Moreover, if $X + Y = \alpha$ and if α is not a rational with denominator 4^n for some n, then the value of X is uniquely determined by examining the expansion of α in base 4 in a simple fashion. Let us denote this value by $f(\alpha)$; the value of f on rationals is immaterial. Clearly the distribution of $f(X + Y) - X$ is just the Dirac measure at zero.

We again return to the problem section of Statistica Neerlandica, vol. 50, no. 2, i.e. to problem 320.

*Problem 320*** (R. D. Gill)

Let F and G be continuous distribution functions. Their QQ-plot is the graph of the quantiles $(x_p, y_p) = (F^{-1}(p), G^{-1}(p))$, $0 < p < 1$, i.e. the graph of the function $y = G^{-1}(F(x))$. The monotone function transforms the distribution of $X \sim F$ into

$Y = F^{-1}(G(X))$. If the QQ-plot is convex one says that Y is more skewed to the right than X. Suppose now X and Y are gamma distributed with parameters (α, r) and (β, r). Show that the gamma distribution with smaller shape parameter is more skew to the right.

REMARK: In his Ph.D. thesis W. R. van Zwet mentions that he managed to prove this after many pages of analysis. The pages are in a box in the archive of the now demolished former location of the Mathematical Center (Boerhaavestraat, Amsterdam), so presumably now lost forever!
(Reference: W. R. VAN ZWET (1964), Convex transformations of random variables; *MC Tract 7, Mathematical Centre*, Amsterdam).

No solutions were submitted at that time. Christian Kleiber has pointed out that there is a well-known textbook that contains a rather detailed proof of this result by W. R. Van Zwet, see pages 219-221 from S. Dharmadhikari and K. Joag-Dev (1988), *Unimodality, Convexity, and Applications*; Academic Press.

Problem 326 (R. JANS, LOUVAIN, BELGIUM)

In a lottery game, n numbers are randomly drawn from a set of m numbers. On a ticket you may fill out n numbers hoping that they will match the n numbers selected. We want to know the minimum number of tickets one has to fill out in order to be sure that among the tickets filled out there is at least one ticket that has $n-1$ or (more?) (or $n-2, n-3, \ldots$) numbers correct. Can this problem be solved using calculus of combinations?

Solution by J. B. VERMETTEN

The number of combinations of n numbers from a set of m numbers is equal to

$$\binom{m}{n}.$$

We assume that the tickets have been filled out in a systematic way, so that no two tickets contain the same combination of numbers.
In order to solve the problem with the help of calculus of combinations, the set of m numbers is divided into 2 groups, viz. the 1st group with the n numbers selected, and the 2nd group with the $m-n$ numbers, which are not selected.
The number of combinations of i numbers taken from the 1st group is

$$\binom{n}{i}$$

The number of combinations of $n-i$ numbers taken from the 2nd group is

$$\binom{m-n}{n-i}.$$

Each combination of the 1st group can be combined with each combination of the 2nd group. The two combinations together give a combination of n numbers with i numbers of the 1st group and $n-i$ numbers of the 2nd group. Their number is

$$\binom{n}{i}\binom{m-n}{n-i}.$$

To be sure that at least one ticket has a combination with $n-k$ (or more) numbers correct, we must take the total number of combinations with less than $n-k$ correspondances, plus 1. So the solution for the minimum number of tickets is

$$\sum_{i=0}^{n-k-1}\binom{n}{i}\binom{m-n}{n-i}+1$$

By substituting j for $n-i$ cancelling the $+1$ against $j=0$ in the sum, the alternative solution for the minimum number of tickets becomes

$$\binom{m}{n}\sum_{j=1}^{k}\binom{n}{j}\binom{m-n}{j}$$

Problem 327 (K. van Harn and F. W. Steutel)

For $n \in \mathbb{N}$, let f_n be the gamma probability density with shape parameter $t = n/2$ and scale parameter $\lambda = 1$, so

$$f_n(x) = \frac{1}{\Gamma(n/2)} x^{(n/2)-1}\, \mathrm{e}^{-x} \quad (x > 0)$$

Define the function $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) := 2\sum_{n=1}^{\infty}(-1)^{n-1} f_n(x) \quad (x > 0)$$

Show that f is a probability density.

The problem was solved by A. A. Jagers and the proposers.

Solution by A. A. Jagers

Let $g : (0, \infty) \to \mathbb{R}$ be given by

$$g(x) = \int_1^{\infty} \frac{\sqrt{\lambda}\, \mathrm{e}^{-\lambda x}}{\sqrt{\pi x}} \frac{\mathrm{d}\lambda}{\lambda^2} \quad (x > 0)$$

Then g is a probability density being a mixture of gamma probability densities (with shape parameter 1/2 and scale parameter λ). Note that g is a completely monotone density. Indeed we have

$$\int_1^\infty \lambda^{-2}\,\mathrm{d}\lambda = 1.$$

We shall show that $f = g$ by means of Laplace transforms. In fact for $s > 0$ we have (by dominated convergence)

$$\begin{aligned}\mathcal{L}(f(x), s) &= 2\sum_{n=1}^{\infty}(-1)^{n-1}\mathcal{L}(f_n(x), s) = 2\sum_{n=1}^{\infty}(-1)^{n-1}(s+1)^{-n/2} \\ &= \frac{2}{\sqrt{s+1}+1} = \frac{2}{s}(\sqrt{s+1}-1) = \frac{1}{s}\int_0^s \frac{\mathrm{d}w}{\sqrt{w+1}} \\ &= \int_0^1 \frac{\mathrm{d}v}{\sqrt{sv+1}} = \int_1^\infty \frac{\lambda^{-3/2}\,\mathrm{d}\lambda}{\sqrt{s+\lambda}} \\ &= \int_1^\infty \mathcal{L}(\mathrm{e}^{-\lambda x}(\pi x)^{-1/2}, s)\lambda^{-3/2}\,\mathrm{d}\lambda = \mathcal{L}(g(x), s), \text{ as required.}\end{aligned}$$

*Problem 328*** (R. THEODORESCU, STE. FOY, CANADA)

Let $m_n(t)$ be the nth moment of a Poisson distribution with parameter $t > 0$. It is known that

$$m_n(t) = \sum_{j=0}^{n} S(n, j)t^j$$

where $S(n, j)$ is the Stirling number of the second kind; moreover,

$$m_{n+1}(t) = t(m_n(t) + (\mathrm{d}/\mathrm{d}t)m_n(t))$$

Consider the normalized moments $c_n(t) = m_n(t)/n!$. CONJECTURE: for each $t > 0$

$$c_{n+1}^2(t) > c_n(t)c_{n+2}(t), \quad n \geq 1 \quad (^*)$$

For $n = 0(^*)$ holds for $t \geq 1$. Is the logconcavity Conjecture $(^*)$ true?

No solutions were submitted. The problem will be kept open until further notice.

New Problems

*Problem 335*** (N. SCHMITZ, MÜNSTER, GERMANY)

Does there exist a sequence of identically distributed pairwise uncorrelated random variables (which, of course, fulfills the weak law of large numbers) which does not fulfill the strong law of large numbers?

Problem 336 (B. Kopociński, Wrocław, Poland)

Let X_1, X_2, X_3 denote the random sample with the probability distribution function F_X, absolutely continuous with density f_X. Let $X_{1,3} \leq X_{2,3} \leq X_{3,3}$ denote the order statistics in the sample. Define

$$U = \frac{X_{2,3} - X_{1,3}}{X_{3,3} - X_{1,3}}$$

Note that the p.d.f. of U does not depend upon the position and scale parameters of the Xs hence it may be used to test the type of distribution of the smallest sample range. Prove that

(a) if F_X is uniform then U has also uniform p.d.f.;
(b) if F_X is normal then U has the truncated Cauchy distribution function.

Problem 337 (F. J. Don)

A given expression consists of n elements related by $n - 1$ nonassociative binary operators. However, the parentheses indicating the order of precedence have disappeared. Let $a_n (n > 1)$ indicate the number of different ways there are to put the $n - 1$ pairs of parentheses back into the expression, and let $a_1 = 1$.

(a) Prove the recursion $a_n = \sum_{j=1}^{n-1} a_j \cdot a_{n-j}$.
(b) Find a closed formula for a_n.

Problems marked with * are nonelementary, of problems marked with ** no solution is known to the editor; unmarked problems are not necessarily simple. Solutions of the problems in this issue should arrive before September 30, 1997. Problems (preferably with solutions) and solutions (type-written on separate sheets bearing the name of the solver) are welcomed by the column editor.