Problem Section

Solutions

Problem 323 (J. J. A. Moors)

In the television programme "Love at first sight", three boys and three girls participate who are strangers to each other. By asking questions they try and get to know each other better. At the end, each boy indicates the girl of his preference; similarly, each girl chooses one of the boys. Each choice is made without knowledge about the other persons' choices. If a boy and a girl choose each other, a couple is formed; all resulting couples are rewarded. Assume that all (six) choices are made at random and independently.

(a) Give the probability distribution of the number of couples that is formed.

**(b) The expected number of couples appears to equal 1. Is there an intuitive argument for this outcome?

A lot of solutions were submitted: A. ten Cate (part b), H. G. Dehling (part b), A. A. Jagers (parts a and b), K. Poortema (parts a and b), M. H. van Raalte and B. Smid (parts a and b), A. J. Stam (parts a and b), R. Teunter (parts a and b), J. B. Vermetten (parts a and b), and the proposer (parts a and b). With regard to part b, several proposers had very short derivations for showing that the expected number of couples is equal to 1. A true intuitive argument has not been given. Several proposers have given various generalizations. Ten Cate, Dehling, and Jagers solved part b in the more general situation of n boys and girls. A further generalization for part b to n boys and m girls has been given by Van Raalte and Smid, Stam, and Vermetten. Stam has also solved part a for this situation. It is possible to obtain another extension to part b; this has been done by Jagers. We first give a short and direct solution that is similar to the one by the proposer.

Solution by M. H. van Raalte and B. Smid

Let A_i denote the event that there are i couples ($i = 0, 1, 2, 3$) and B_j denotes the event that j different girls were chosen by the boys ($j = 1, 2, 3$). It is not difficult to show that

$$P(B_1) = \frac{3}{27}, \quad P(B_2) = \frac{18}{27}, \quad P(B_3) = \frac{6}{27}$$

By using the fact that

$$P(A_i) = \sum_{j=1}^{3} P(A_i \mid B_j)P(B_j)$$

it is obtained

$$P(A_0) = 0\cdot\frac{1}{9} + \frac{2}{9}\cdot\frac{2}{3} + \frac{8}{27}\cdot\frac{2}{9} = \frac{52}{243}$$

$$P(A_1) = 1\cdot\frac{1}{9} + \frac{5}{9}\cdot\frac{2}{3} + \frac{12}{27}\cdot\frac{2}{9} = \frac{141}{243}$$

$$P(A_2) = 0\cdot\frac{1}{9} + \frac{2}{9}\cdot\frac{2}{3} + \frac{6}{27}\cdot\frac{2}{9} = \frac{48}{243}$$

$$P(A_3) = 0\cdot\frac{1}{9} + 0\cdot\frac{2}{3} + \frac{1}{27}\cdot\frac{2}{9} = \frac{2}{243}$$

A simple calculation shows that the expectation is equal to one.

Solution by A. J. Stam

We generalize the problem to n boys, numbered $1, \ldots, n$ and m girls, numbered $1, \ldots, m$. Let X be the number of couples. Equivalently, X is the number of girls that are part of a couple.

a) The distribution of X. First we take $m \le n$. We apply the inclusion–exclusion principle as given in Feller I, Second Edition, 1957, Chapter IV. Let A_i be the event that girl i is part of a couple. Then

$$P(A_{i1}A_{i2}\ldots A_{ik}) = \frac{n!}{(n-k)!}(nm)^{-k}$$

(First specify the k boys that form couples with girls, $i_1, i_2, \ldots, i_k$, in that order. Then the choices made by these boys and girls have to be the correct ones.) Summation over all $i_1, \ldots, i_k$ with $1 \le i_1 < \ldots < i_k \le m$ gives, in Feller's notation

$$S_k = \binom{m}{k}\frac{n!}{(n-k)!}(nm)^{-k} = k!\binom{m}{k}\binom{n}{k}(nm)^{-k} \qquad (*)$$

This relation also holds for $m > n$ since $\binom{n}{k} = 0$ for $k > n$. It gives the correct value $S_0 = 1$. Then we have

$$\begin{aligned} P(X = r) &= \sum_{k=r}^{m}(-1)^{k-r}\binom{k}{r}S_k \\ &= \sum_{k=r}^{m\wedge n}(-1)^{k-r}k!\binom{k}{r}\binom{m}{k}\binom{n}{k}(nm)^{-k} \end{aligned}$$

a sum for which probably no closed formula is available.
For $m = n = 3$ it gives

$$P(X=0) = \frac{52}{243},\;\; P(X=1) = \frac{141}{243},\;\; P(X=2) = \frac{48}{243},\;\; P(X=3) = \frac{2}{143}.$$

b) The expectation. We have $EX = 1$. This follows from $(*)$, since $EX = S_1$. The following short derivation, without explicit reference to $(*)$, may be given. We have

$$EX = \sum_{i=1}^{m} P(A_i)$$

where $P(A_i) = 1/m$, since the conditional probability that girl i is part of a couple, given that she chose boy j, is $1/m$.
A similar argument shows that we have $EX = 1$ under more general conditions, e.g.

(1) The choices by boys and by girls are independent.
(2) The probability that boy h chooses girl k is $1/m$.

The choices by the boys need not be independent, e.g. they may choose at random "without replacement". For the choices by the girls no conditions are assumed.
Still more general: Assume (1) and let girl i choose boy j with probability p_{ij} and let boy h choose girl k with probability q_{hk}. Then $EX = \Sigma_{i=1}^{m} P(A_i) = \Sigma_{i=1}^{m}\Sigma_{j=1}^{n} p_{ij}q_{ji}$. When $p_{ij} = p_j$, independent of i, $EX = \Sigma_{j=1}^{n} p_j \Sigma_{i=1}^{m} q_{ji} = \Sigma_{j=1}^{n} p_j = 1$.

*Problem 324*** (T. Sellke)

A sequence is constructed that describes itself. It consists of blocks of more than one or two ones or twos. You start with 1, this is a block of length one. So, the following block has to start with 2 and we have 1 2. The 1 means a block of length one and the 2 means a second block of length two. This gives 1 2 2. The third 2 means that the third block is of length 2, hence we have 1 2 2 1 1. The fourth block is of length one, and this also holds for the fifth block. Hence, we arrive at 1 2 2 1 1 2 1. By continuing this sequence we obtain.

1 2 2 1 1 2 1 2 2 1 2 2 1 1 2 1 1 2 2 . . .

The question is whether the number of 1's is "asymptotically" equal to the number of 2's.

The editor has received a reaction by F. M. Dekking. The sequence of numbers is known as the Prouhet-Thue-Morse sequence. Dekking gives, more or less, a presentation of the state-of-the-art in a technical report entitled "What is the long range order in the Kolakoski sequence?" to appear in Proceedings of the NATO ASI Conference "The Mathematics of Aperiodic Order", August 21–September 1, 1995, Waterloo, Canada, edited by R. V. Moody and J. Patera. It will be published in 1996 by Kluwer. He advises everybody not to go deeply into this problem.

*Problem 325*** (A. G. M. Steerneman)

Let X be a random variable on the non-negative integers.
Let $p_n = P\{X = n\}$, $n = 0, 1, \ldots$ Define $\mu = \mathrm{E}X > 0$. Assume that $p_0 \geq p_1 \geq p_2 \geq \ldots$ For $r = 0, 1, 2, \ldots$ define

$$m_r = \mathrm{E}(X/\mu - 1)^{2r+1}$$

Prove that m_r is nondecreasing in r, or give a counterexample.

No solutions were submitted to this problem. The problem is kept open until further notice.

New Problems

Problem 329 (K. Nevels)

Let $p = 1 - q \in (0, 1)$. For $n = 0, 1, \ldots$ find an expression for

$$u_n = \sum_{k=0}^{\infty} \binom{n+2k}{k} p^{n+k} q^k$$

Is there an elegant probability theoretical interpretion of u_n?

Problem 330 (A. G. M. Steerneman)

Consider Euler's function $\phi(n)$. It is defined as the number of positive integers which are primes relative to n, larger than one and smaller than n. A well-known expression is

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

The product is taken over all prime divisors p of n. Give a probabilistic derivation of this result.

*Problem 331*** (A. G. M. Steerneman)

Let $X_1, \ldots, X_n$ be independently distributed with $X_i \sim N_p(\mu_i, \Sigma)$ where Σ is nonsingular. We assume that $n > p$ and that the matrix $(\mu_1 \ldots \mu_n)$ is of rank 1. Define

$$S = \sum_{i=1}^{n} X_i X_i^t$$

Is it possible to find an explicit expression for $\mathrm{E}S^{-1}$?

Problems marked with $*$ are non-elementary, of problems marked with $**$ no solution is known to the editor; unmarked problems are not necessarily simple. Solutions of the problems in this issue should arrive before January 31, 1997.

Problems (preferably with solutions) and solutions (type-written on separate sheets bearing the name and the address of the solver) are welcomed by the column editor.