Solutions Manual to

Exercises in

Mathematical Tools

For Economics

Exercises for Chapter 1

1.1

1. (i) $A+B=\begin{pmatrix}4 & -2\\ 4 & 8\end{pmatrix},\ -3A=\begin{pmatrix}-6 & 15\\ -18 & -3\end{pmatrix}.$

$$C(A+B)=\begin{pmatrix}0 & 1\\ 5 & 8\end{pmatrix}\begin{pmatrix}4 & -2\\ 4 & 8\end{pmatrix}=\begin{pmatrix}4 & 8\\ 52 & 54\end{pmatrix}.$$

$$CA=\begin{pmatrix}6 & 1\\ 58 & -17\end{pmatrix},\ CB=\begin{pmatrix}-2 & 7\\ -6 & 71\end{pmatrix},\ CA+CB=\begin{pmatrix}4 & 8\\ 52 & 54\end{pmatrix}.$$

$$(A+B)C=\begin{pmatrix}4 & -2\\ 4 & 8\end{pmatrix}\begin{pmatrix}0 & 1\\ 5 & 8\end{pmatrix}=\begin{pmatrix}-10 & -12\\ 40 & 68\end{pmatrix}.$$

(ii) $AB=\begin{pmatrix}2 & -5\\ 6 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\ -2 & 7\end{pmatrix}=\begin{pmatrix}14 & -29\\ 10 & 25\end{pmatrix}.$

$$BA=\begin{pmatrix}2 & 3\\ -2 & 7\end{pmatrix}\begin{pmatrix}2 & -5\\ 6 & 1\end{pmatrix}=\begin{pmatrix}22 & -7\\ 38 & 17\end{pmatrix}$$

$$ABC=\begin{pmatrix}14 & -29\\ 10 & 25\end{pmatrix}\begin{pmatrix}0 & 1\\ 5 & 8\end{pmatrix}=\begin{pmatrix}-145 & -218\\ 125 & 210\end{pmatrix}$$

$$BCA=\begin{pmatrix}15 & 26\\ 35 & 54\end{pmatrix}\begin{pmatrix}2 & -5\\ 6 & 1\end{pmatrix}=\begin{pmatrix}186 & -49\\ 394 & -121\end{pmatrix}$$

$$CAB=\begin{pmatrix}0 & 1\\ 5 & 8\end{pmatrix}\begin{pmatrix}14 & -29\\ 10 & 25\end{pmatrix}=\begin{pmatrix}10 & 25\\ 150 & 55\end{pmatrix}.$$

Clearly AB and BA are not the same.

(iii) $\mathrm{tr}AB=14+25=39$

$\mathrm{tr}BA=22+17=39$

$\text{tr}ABC = -145 + 210 = 65$

$\text{tr}BCA = 186 - 121 = 65$

$\text{tr}CAB = 10 + 55 = 65.$

2. (i) $AB = \begin{pmatrix} 1 & 3 & 7 \\ 5 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 4 & 2 \\ -2 & 5 & -1 \\ 3 & -1 & 6 \end{pmatrix} = \begin{pmatrix} 16 & 12 & 41 \\ 29 & 12 & 58 \end{pmatrix}.$

As B is 3×3 and A is 2×3, BA does not exist.

(ii) $(AB)' = \begin{pmatrix} 16 & 29 \\ 12 & 12 \\ 41 & 58 \end{pmatrix}.$

$$B'A' = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -1 \\ 2 & -1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 0 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 16 & 29 \\ 12 & 12 \\ 41 & 58 \end{pmatrix}.$$

(iii) $AA' = \begin{pmatrix} 1 & 3 & 7 \\ 5 & 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 0 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 59 & 61 \\ 61 & 89 \end{pmatrix},$

which is clearly symmetric.

3. (i) $AB = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

$$BA = \begin{pmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both AB and BA equal the 3×3 null matrix.

(ii) $AC = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} = A.$

$CA = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = C.$

(iii) $A^2 = AA = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} = \begin{pmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{pmatrix} = A.$

$C^2 = CC = \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} \begin{pmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix} = C.$

(iv) (a) $A'ACB = A'AB$ from ii

$= A'\mathrm{O}$ from i

$= \mathrm{O}.$

(b) $(A-B)^2 = (A-B)(A-B)$

$= A^2 - AB - BA + B^2$

$= A^2 + B^2$ from i.

4. A square matrix *A* is symmetric idempotent if $A' = A$ and $A^2 = A.$

(i) $A' = (\mathbf{i}\,\mathbf{i}')'/n = (\mathbf{i}')'\mathbf{i}'/n = A.$

$B' = (\mathrm{I} - A)' = \mathrm{I} - A' = B.$

$AA = \dfrac{\mathbf{i}\,\mathbf{i}'\mathbf{i}\,\mathbf{i}'}{n^2} = A$ as $\mathbf{i}'\mathbf{i} = n.$

$BB = (\mathrm{I} - A)^2 = \mathrm{I} - 2A + A^2 = B.$

$$\text{tr}A = \text{tr}\frac{i\,i'}{n} = \text{tr}\frac{i'i}{n} = 1.$$

$$\text{tr}B = \text{tr}(\text{I} - A) = \text{tr}I - \text{tr}A = n - 1.$$

(ii) $A\mathbf{x} = \mathbf{i}\ \mathbf{i}'x/n.$

Now $\mathbf{i}'\mathbf{x} = (1\ \ldots\ 1)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 + \ \ldots\ + x_n.$

Hence $A\text{x} = \mathbf{i}\,\overline{x} = \begin{pmatrix} \overline{x} \\ \vdots \\ \overline{x} \end{pmatrix}$ where $\overline{x} = \sum_{i=1}^{n} x_i / n$, so Ax is an $n \times 1$ vector where elements are $\overline{x}$. Now

$\text{B}\mathbf{x} = (\text{I} - \text{A})\,\mathbf{x} = \mathbf{x} - \text{A}\mathbf{x} = \begin{pmatrix} x_1 - \overline{x} \\ \vdots \\ x_n - \overline{x} \end{pmatrix}$, so $\text{B}\mathbf{x}$ is an $n \times 1$ vector where *j*th element is $x_j - \overline{x}$.

1.2

1. $|A| = -6,\ |B| = 18 + 4 = 22,$

$$|AB| = \begin{vmatrix} -2 & -4 \\ 4 & 74 \end{vmatrix} = -148 + 16 = -132.$$

$$|A'| = \begin{vmatrix} -1 & 5 \\ 0 & 6 \end{vmatrix} = -6$$

$$|B'| = \begin{vmatrix} 2 & -1 \\ 4 & 9 \end{vmatrix} = 18 + 4 = 22.$$

2. $|A| = \begin{vmatrix} 4 & 1 & 6 \\ 7 & 2 & 9 \\ 3 & 0 & 8 \end{vmatrix} \underset{r_2' = r_2 - 2r_1}{=} \begin{vmatrix} 4 & 1 & 6 \\ -1 & 0 & -3 \\ 3 & 0 & 8 \end{vmatrix}.$

Expanding using the second column we have

$$|A| = 1(-1)^{1+2}\begin{vmatrix} -1 & -3 \\ 3 & 8 \end{vmatrix} = -1(-8+9) = -1.$$

Similarly,

$$|A'| = \begin{vmatrix} 4 & 7 & 3 \\ 1 & 2 & 0 \\ 6 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 4 & -1 & 3 \\ 1 & 0 & 0 \\ 6 & -3 & 8 \end{vmatrix} = -1.$$

3. $$|A| \underset{r_2' = r_2 - r_3}{=} \begin{vmatrix} 1 & 2 & 3 & 2.5 \\ 0 & 0 & 0 & 7 \\ 1 & 2 & 3 & -3 \\ 3 & -1 & 1 & -2 \end{vmatrix}.$$

Expanding using the second row we have

$$|A| = 7(-1)^{2+4}\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & -1 & 1 \end{vmatrix} = 7\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 3 & -1 & 1 \end{vmatrix} = 0.$$

$$|B| \underset{c_4' = c_4 - 3c_2}{=} \begin{vmatrix} 2 & 1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 2 & 4 & 1 & 0 \end{vmatrix} = 0.$$

1.3

1. $\text{Adj}A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}' = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$ and $|A| = 4 + 2 = 6$. Hence $A^{-1} = \frac{1}{6}\begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}$.

$\text{Adj}B = \begin{pmatrix} 8 & -5 \\ 0 & -2 \end{pmatrix}' = \begin{pmatrix} 8 & 0 \\ -5 & -2 \end{pmatrix}$ and $|B| = -16$. Hence $B^{-1} = -\frac{1}{16}\begin{pmatrix} 8 & 0 \\ -5 & -2 \end{pmatrix}$.

$$AB = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 5 & 8 \end{pmatrix} = \begin{pmatrix} -7 & -8 \\ 16 & 32 \end{pmatrix}.$$

$$\mathrm{Adj}(AB)=\begin{pmatrix}32 & -16\\ 8 & -7\end{pmatrix}'=\begin{pmatrix}32 & 8\\ -16 & -7\end{pmatrix} \text{ and } |AB|=-96 \text{ so } (AB)^{-1}=-\frac{1}{96}\begin{pmatrix}32 & 8\\ -16 & -7\end{pmatrix}.$$

2. $$|A| \underset{r_3'=r_3-\frac{3}{2}r_2}{=} \begin{vmatrix}2 & 1 & 0\\ 6 & 2 & 6\\ -13 & -6 & 0\end{vmatrix}=6(-1)^{2+3}\begin{vmatrix}2 & 1\\ -13 & -6\end{vmatrix}=-6, \text{ so } A \text{ is nonsingular.}$$

$$\mathrm{Adj}A=\begin{pmatrix}\begin{vmatrix}2 & 6\\ -3 & 9\end{vmatrix} & -\begin{vmatrix}6 & 6\\ -4 & 9\end{vmatrix} & \begin{vmatrix}6 & 2\\ -4 & -3\end{vmatrix}\\ -\begin{vmatrix}1 & 0\\ -3 & 9\end{vmatrix} & \begin{vmatrix}2 & 0\\ -4 & 9\end{vmatrix} & -\begin{vmatrix}2 & 1\\ -4 & -3\end{vmatrix}\\ \begin{vmatrix}1 & 0\\ 2 & 6\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 6 & 6\end{vmatrix} & \begin{vmatrix}2 & 1\\ 6 & 2\end{vmatrix}\end{pmatrix}'=\begin{pmatrix}36 & -78 & -10\\ -9 & 18 & 2\\ 6 & -12 & -2\end{pmatrix}'.$$

Hence

$$A^{-1}=-\frac{1}{6}\begin{pmatrix}36 & -9 & 6\\ -78 & 18 & -12\\ -10 & 2 & -2\end{pmatrix}.$$

$$|B| \underset{\substack{r_1'=r_1-r_3\\ r_2'=r_2-r_3}}{=} \begin{vmatrix}0 & -1 & 0\\ 0 & -1 & 1\\ 1 & 4 & 3\end{vmatrix}=1(-1)^{3+1}\begin{vmatrix}-1 & 0\\ -1 & 1\end{vmatrix}=-1, \text{ so } B \text{ is nonsingular.}$$

$$\mathrm{Adj}B=\begin{pmatrix}\begin{vmatrix}3 & 4\\ 4 & 3\end{vmatrix} & -\begin{vmatrix}1 & 4\\ 1 & 3\end{vmatrix} & \begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}\\ -\begin{vmatrix}3 & 3\\ 4 & 3\end{vmatrix} & \begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix}\\ \begin{vmatrix}3 & 3\\ 3 & 4\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 1 & 4\end{vmatrix} & \begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix}\end{pmatrix}'=\begin{pmatrix}-7 & 1 & 1\\ 3 & 0 & -1\\ 3 & -1 & 0\end{pmatrix}'=\begin{pmatrix}-7 & 3 & 3\\ 1 & 0 & -1\\ 1 & -1 & 0\end{pmatrix}.$$

Hence $B^{-1}=\begin{pmatrix}7 & -3 & -3\\ -1 & 0 & 1\\ -1 & 1 & 0\end{pmatrix}.$

3. $(A \vdots I) = \begin{pmatrix} 2 & 4 & 5 & \vdots & 1 & 0 & 0 \\ 0 & 3 & 0 & \vdots & 0 & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 0 & 1 \end{pmatrix}$

$$\underset{r_1 \leftrightarrow r_3}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 3 & 0 & \vdots & 0 & 1 & 0 \\ 2 & 4 & 5 & \vdots & 1 & 0 & 0 \end{pmatrix}$$

$$\underset{r_3 = r_3 - 2r_1}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 3 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & -2 \end{pmatrix}$$

$$\underset{r_2' = \frac{1}{3} r_2}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 4 & 3 & \vdots & 1 & 0 & -2 \end{pmatrix}$$

$$\underset{r_3' = r_3 - 4r_2}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 3 & \vdots & 1 & -\frac{4}{3} & -2 \end{pmatrix}$$

$$\underset{r_1' = r_1 - r_3}{\cong} \begin{pmatrix} 1 & 0 & 0 & \vdots & -\frac{1}{3} & \frac{4}{9} & 5/3 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{3} & -\frac{4}{9} & -\frac{2}{3} \end{pmatrix}$$

$$\underset{r_4' = \frac{1}{3} r_4}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \vdots & 1 & -\frac{4}{9} & -2/3 \end{pmatrix}$$

$$\underset{r_4' = \frac{1}{3} r_4}{\cong} \begin{pmatrix} 1 & 0 & 1 & \vdots & 0 & 0 & 1 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \vdots & \frac{1}{3} & -\frac{4}{9} & -2/3 \end{pmatrix}$$

$$\underset{r_1'=r_1-r_3}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{4}{9} & 5/3 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{1}{3} & -\frac{4}{9} & -\frac{2}{3} \end{array}\right)$$

Hence $A^{-1}=\frac{1}{9}\begin{pmatrix} -3 & 4 & 15 \\ 0 & 3 & 0 \\ 3 & -4 & -6 \end{pmatrix}$.

$$\left(B \vdots \mathrm{I}\right)=\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{c_2'=c_2-2c_1 \\ c_3'=c_2-c_1}}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 2 & -3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{c_2'=-c_2}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -1 \\ 0 & 1 & 1 & 0 & -1 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{c_3'=c_3-c_2}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 2 & -3 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 2 & 3 & -1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{\substack{c_1'=c_1+2c_3 \\ c_2'=c_2+3c_3}}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & -3 \\ 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & 0 & 1 & 2 & 3 & 1 \end{array}\right)$$

$$\underset{c_3'=-c_3}{\cong}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & -7 & 3 \\ 0 & 1 & 0 & 2 & 2 & -1 \\ 0 & 0 & 1 & 2 & 3 & -1 \end{array}\right)$$

Hence $B^{-1}=\begin{pmatrix} -5 & -7 & 3 \\ 2 & 2 & -1 \\ 2 & 3 & -1 \end{pmatrix}$.

4. $|A| \underset{r_2'=r_2-r_3}{=} \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \end{vmatrix}$. Expanding this determinant using the third column we have

$$|A| = (-1)^{3+3}\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2, \text{ so } A \text{ is nonsingular.}$$

$$\text{Adj } A = \begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} -1 & 0 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} \end{pmatrix}' = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -3 \\ -1 & -1 & 5 \end{pmatrix}'.$$

Hence $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -3 & 5 \end{pmatrix}$.

1.4

1. (i) A set of *m* $n \times 1$ vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are linearly dependent if there exists scalars $\lambda_1, \ldots, \lambda_m$ not all of which are zero such that $\lambda_1 \boldsymbol{a}_1 + \ldots + \lambda_m a_m = \mathbf{0}$.

(ii) Consider

$$\lambda_1\boldsymbol{x}_1 + \lambda_2\boldsymbol{x}_2 + \lambda_3\boldsymbol{x}_3 = \mathbf{0}$$

which is the following set of equations in $\lambda_1, \lambda_2,$ and λ_3:

$$\begin{aligned} 2\lambda_1 + 2\lambda_2 + 14\lambda_3 &= 0 \\ -7\lambda_1 + 4\lambda_2 - 27\lambda_3 &= 0 \\ \lambda_1 - \lambda_2 + 3\lambda_3 &= 0 \end{aligned}$$

From the third equation $\lambda_1 = \lambda_2 - 3\lambda_3$ and substituting back into the other two equations gives $\lambda_2 = -2\lambda_3$, so if we take $\lambda_3 = 1,\ \lambda_2 = -2$, and $\lambda_1 = -5$ we get $-5\boldsymbol{x}_1 - 2\boldsymbol{x}_2 + \boldsymbol{x}_3 = \mathbf{0}$.

Thus the vectors are linearly dependent.

(iii) Consider

$$\lambda_1 \boldsymbol{x}_1 + \lambda_2 \boldsymbol{x}_2 + \lambda_3 \boldsymbol{x}_3 = \mathbf{0}$$

which gives

$$\begin{aligned} 9\lambda_1 + 2\lambda_2 - 4\lambda_3 &= 0 \\ -3\lambda_1 &= 0 \\ -\lambda_1 + 2\lambda_2 + 8\lambda_3 &= 0. \end{aligned}$$

The second equation gives $\lambda_1 = 0$ so we are left with $2\lambda_2 = 4\lambda_3$ and $2\lambda_2 = -8\lambda_3$ which are true only if $\lambda_2 = \lambda_3 = 0$. Hence the vectors are linearly dependent.

2. (i)

$$A \underset{\substack{r_2'=r_2-2r_1 \\ r_3'=r_3+r_1 \\ r_4'=r_4-r_1}}{\cong} \begin{pmatrix} 1 & -3 & -2 & 11 \\ 0 & 1 & 11 & -33 \\ 0 & -1 & -5 & 15 \\ 0 & 5 & 1 & -3 \end{pmatrix}$$

$$\underset{\substack{r_3'=r_3+r_2 \\ r_4'=r_4-5r_2}}{\cong} \begin{pmatrix} 1 & -3 & -2 & 11 \\ 0 & 1 & 11 & -33 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & -54 & 162 \end{pmatrix}$$

$$\underset{r_4'=r_4 \, 9r_1}{\cong} \begin{pmatrix} 1 & -3 & -2 & 11 \\ 0 & 1 & 11 & -33 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $r(A) = 3$.

(ii)

$$|B| \underset{\substack{r_2'=r_2-r_1 \\ r_3'=r_3+r_1 \\ r_4'=r_4-r_1}}{=} \begin{vmatrix} 1 & -1 & 2 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{vmatrix} = -1(-1)^{1+2} \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\underset{r_3'=r_3-r_1}{=}\begin{vmatrix}1 & 2 & 0\\ 0 & 1 & 1\\ 0 & -2 & -1\end{vmatrix} = 1(-1)^{1+1}\begin{vmatrix}1 & 1\\ -2 & -1\end{vmatrix} = 1.$$

Hence $r(B) = 4.$

3. Expanding $|A|$ by the third column we have

$$|A| = (-1)^{3+1}\begin{vmatrix}2 & 1-x\\ 1 & 0\end{vmatrix} + (1-x)(-1)^{3+3}\begin{vmatrix}5-x & 2\\ 2 & 1-x\end{vmatrix} = -(1-x) + (1-x)\left[(5-x)(1-x)-4\right]$$
$$= (1-x)\ x\ (x-6).$$

Therefore $|A| = 0$ if $x = 1$, or $x = 0$, or $x = 6$ but is nonzero for all other values of *x*.

So if *x* does not take the value 0, 1, or 6 then $r(A) = 3.$ If <u>*x*</u> takes any of these values consider the minor

$$\begin{vmatrix}5-x & 1\\ 2 & 0\end{vmatrix} = -2$$

for all values of *x*. Thus we have $r(A) = 3$ for all *x* other than 0, 1, 6, $r(A) = 2$ for *x* taking the value 0, 1, or 6.

4. (i)

$$N' = \left(X\left(X'X\right)^{-1}X'\right)' = \left(X'\right)'\left[\left(X'X\right)^{-1}\right]'X' = X\left[\left(X'X\right)'\right]^{-1}X' = X\left(X'X\right)^{-1}X' = N.$$

$$M' = \left(I-N\right)' = I' = N' = M.$$

Hence both *N* and *M* are symmetric.

Consider

$$NN = X(X'X)^{-1}X'X(X'X)^{-1}X' = N$$

and

$$MM = (I - N)(I - N) = I - 2N + N^2 = M.$$

So *M* and *N* are idempotent.

Now

$$NX = X(X'X)^{-1}X'X = X$$

so

$$MX = X - NX = \mathrm{O}.$$

(ii) $r(N) = \mathrm{tr}N = \mathrm{tr}(X'X)^{-1}X'X = \mathrm{tr}I_K = K.$

$r(M) = \mathrm{tr}(I - N) = \mathrm{tr}I_n - \mathrm{tr}N = n - K.$

1.5

1. (i) $$A \otimes B = \begin{pmatrix} -1\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix} & 2\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix} \\ 3\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix} & 4\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -7 & 0 & 14 \\ -8 & -5 & 16 & 10 \\ 0 & 21 & 0 & 28 \\ 24 & 15 & 32 & 20 \end{pmatrix}.$$

$$B \otimes A = \begin{pmatrix} 0\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} & 7\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \\ 8\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} & 5\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -7 & 14 \\ 0 & 0 & 21 & 28 \\ -8 & 16 & -5 & 10 \\ 24 & 32 & 15 & 20 \end{pmatrix}.$$

$$(A \otimes B)' = \begin{pmatrix} 0 & -8 & 0 & 24 \\ -7 & -5 & 21 & 15 \\ 0 & 16 & 0 & 32 \\ 14 & 10 & 28 & 20 \end{pmatrix}.$$

$$A' \otimes B' = \begin{pmatrix} -1\begin{pmatrix} 0 & 8 \\ 7 & 5 \end{pmatrix} & 3\begin{pmatrix} 0 & 8 \\ 7 & 5 \end{pmatrix} \\ 2\begin{pmatrix} 0 & 8 \\ 7 & 5 \end{pmatrix} & 4\begin{pmatrix} 0 & 8 \\ 7 & 5 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & -8 & 0 & 24 \\ -7 & -5 & 21 & 15 \\ 0 & 16 & 0 & 32 \\ 14 & 10 & 28 & 20 \end{pmatrix}.$$

$$\text{tr}(A \otimes D) = 0 - 5 + 0 + 20 = 15$$
$$\text{tr}A \ \text{tr}B = (-1+4)(0+5) = 15.$$

(ii) $ABC = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 16 & 3 \\ 32 & 41 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 54 & 61 \\ 178 & 87 \end{pmatrix},$

so

$$\text{vec}A\text{BC} = \begin{pmatrix} 54 \\ 178 \\ 61 \\ 87 \end{pmatrix}.$$

$$(C' \otimes A)\text{vecB} = \begin{pmatrix} 3\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} & 2\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \\ 4\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} & -1\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} \end{pmatrix}\text{vecB}$$
$$= \begin{pmatrix} -3 & 6 & -2 & 4 \\ 9 & 12 & 6 & 8 \\ -4 & 8 & 1 & -2 \\ 12 & 16 & -3 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 7 \\ 5 \end{pmatrix}$$
$$= \begin{pmatrix} 54 \\ 178 \\ 61 \\ 87 \end{pmatrix}.$$

(iii) $AB = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 7 \\ 8 & 5 \end{pmatrix} = \begin{pmatrix} 16 & 3 \\ 32 & 41 \end{pmatrix}$

so

$$\text{tr}AB = 16 + 41 = 57.$$

$$(\text{vecA}')'\text{vecB} = (-1 \quad 2 \quad 3 \quad 4)\begin{pmatrix} 0 \\ 8 \\ 7 \\ 5 \end{pmatrix} = 57.$$

(iv) $\text{tr}ABC = 54 + 87 = 141.$

C' is 2×2 and vecB is 4×1 so the required identity matrix is 2×2 and we have

$$(\text{vecA}')'(C'\otimes I)\text{vecB} = (\text{vecA}')'\begin{pmatrix} 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}\text{vecB}$$

$$= \begin{pmatrix} -1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 3 & 0 & 2 & 0 \\ 0 & 3 & 0 & 2 \\ 4 & 0 & -1 & 0 \\ 0 & 4 & 0 & -1 \end{pmatrix}\text{vecB}$$

$$= \begin{pmatrix} 9 & 22 & -5 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 8 \\ 7 \\ 5 \end{pmatrix}$$

$$= 176 - 35 = 141.$$

2. Let $\boldsymbol{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. Then $\boldsymbol{ab}' = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}\begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix} = \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_m \\ \vdots & & \vdots \\ a_n b_1 & \cdots & a_n b_m \end{pmatrix}$.

$$\boldsymbol{b}'\otimes\boldsymbol{a} = \left(b_1\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \ldots b_m \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right) = \boldsymbol{ab}'$$

$$\boldsymbol{a}\otimes\boldsymbol{b}' = \begin{pmatrix} a_1\begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix} \\ \vdots \\ a_n\begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix} \end{pmatrix} = \boldsymbol{ab}'.$$

$$\text{vec } \mathbf{ab}' = \begin{pmatrix} a_1 b_1 \\ \vdots \\ a_n b_1 \\ \vdots \\ a_1 b_m \\ \vdots \\ a_n b_n \end{pmatrix} = \text{vec } (\boldsymbol{b}' \otimes \boldsymbol{a}).$$

$$\boldsymbol{b} \otimes \boldsymbol{a} = \begin{pmatrix} b_1 \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \\ \vdots \\ \mathrm{b}_m \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \end{pmatrix} = \text{vec } \boldsymbol{ab}'.$$

Exercises for Chapter 2

2.2

1. (i) In matrix notation we have

$A\mathbf{x} = \mathbf{0}$

where $$A = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$$

$$\underset{\substack{r_2' = r_2 - r_1 \\ r_3' = r_3 - \alpha r_1}}{\cong} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & 1 & 1-2\alpha \end{pmatrix}$$

$$\underset{r_3' = r_3 - 0.5 r_2}{\cong} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & -1 \\ 0 & 0 & 1.5-2\alpha \end{pmatrix}.$$

Thus $r(A) = 3$ for all values of α other than $\alpha = 0.75$, in which case we only have the trivial solution.

(ii). If $\alpha = 0.75$, $r(A) = 2 < 3$ so we have an infinite number of solutions. If this is the case the original system has the same solutions as

$x_1 + 2x_3 = 0$

$2x_2 - x_3 = 0.$

Let $x_3 = \lambda$, any real number. Then $x_2 = \lambda / 2$ and $x_1 = 2\lambda$, so our general solution is

$$\boldsymbol{x}^* = \begin{pmatrix} 2\lambda \\ \lambda/2 \\ \lambda \end{pmatrix}.$$

2. In matrix notation we have

$$A\boldsymbol{x} = 0$$

$$\text{where } A = \begin{pmatrix} 1 & 2 & 2 & -3 \\ 1 & 0 & -2 & 13 \\ 3 & 5 & 4 & 0 \end{pmatrix}.$$

As *A* is 3×4 we must have $r(A) < 4$, the number of variables. Hence if a nontrivial solution exists, an infinite number of solutions exist.

$$A \underset{\substack{r_2' = r_2 - r_1 \\ r_3' = r_3 - 3r_1}}{=} \begin{pmatrix} 1 & 2 & 2 & -3 \\ 0 & -2 & -4 & 16 \\ 0 & -1 & -2 & 9 \end{pmatrix}$$

$$\underset{r_2' = -\frac{1}{2} r_2}{\cong} \begin{pmatrix} 1 & 2 & 2 & -3 \\ 0 & 1 & 2 & -8 \\ 0 & -1 & -2 & 9 \end{pmatrix}$$

$$\underset{r_3' = r_3 + r_2}{\cong} \begin{pmatrix} 1 & 2 & 2 & -3 \\ 0 & 1 & 2 & -8 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hence $r(A) = 3$ and the original system has the same solution as

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - 3x_4 &= 0 \\ x_2 + 2x_3 - 8x_4 &= 0 \\ x_4 &= 0 \end{aligned}$$

$$\Rightarrow x_2 = -2x_3, \; x_1 = 2x_3.$$

Let $x_3 = \lambda$, where λ is any real number. Then $x_2 = -2\lambda$ and $x_1 = 2\lambda$ so

$$\boldsymbol{x}^* = \begin{pmatrix} 2\lambda \\ -2\lambda \\ \lambda \\ 0 \end{pmatrix}$$

is the general solution.

3. Rewrite the system as

$$x+3z+ky=0$$
$$2x-4z+3y=0$$
$$3x-22+ky=0$$

which can be represented in matrix notation as

$$A\boldsymbol{x}=0$$

$$\text{where } A=\begin{pmatrix}1 & 3 & k\\ 2 & -4 & 3\\ 3 & -2 & k\end{pmatrix}$$

$$\underset{\substack{r_2'=r_2-2r_1\\ r_3'=r_3-3r_1}}{\cong}\begin{pmatrix}1 & 3 & k\\ 0 & -10 & 3-2k\\ 0 & -11 & -2k\end{pmatrix}\underset{r_3'=r_3-\frac{11}{10}r_2}{\cong}\begin{pmatrix}1 & 3 & k\\ 0 & -10 & 3-2k\\ 0 & 0 & -3.3+0.2k\end{pmatrix}.$$

If $k\neq 33/2$ then $r(A)=3$ and the only solution is the trivial solution. Thus for $k=33/2$ we have a nontrivial solution and for this case our system has the same solution as

$$x_1+3x_2+33/2x_3=0$$
$$-10x_2-30x_3=0.$$

So $x_2=-3x_3$ and $x_1=-15x_3/2$, and our general solution is

$$\mathbf{x}^*=\begin{pmatrix}-15\lambda/12\\ -3\lambda\\ \lambda\end{pmatrix}$$

for any real number λ.

2.3

1. In matrix notation our system is $A\boldsymbol{x}=\boldsymbol{b}$

where $A=\begin{pmatrix}1 & 1 & 1\\ 2 & 1 & 2\\ 4 & 3 & 4\end{pmatrix}$ and $\boldsymbol{b}=\begin{pmatrix}2\\ 5\\ \mathrm{c}\end{pmatrix}$.

Then $(A\boldsymbol{b})=\begin{pmatrix}1 & 1 & 1 & 2\\ 2 & 1 & 2 & 5\\ 4 & 3 & 4 & \mathrm{c}\end{pmatrix}\underset{\substack{r_2'=r_2-2r_1\\ r_3'=r_3-4r_1}}{\cong}\begin{pmatrix}1 & 1 & 1 & 2\\ 0 & -1 & 0 & 1\\ 0 & -1 & 0 & \mathrm{c}-8\end{pmatrix}\underset{r_2'=-r_2}{\cong}\begin{pmatrix}1 & 1 & 1 & 2\\ 0 & 1 & 0 & -1\\ 0 & -1 & 0 & \mathrm{c}-8\end{pmatrix}$

$$\underset{r_3'=r_3+r_2}{\cong}\begin{pmatrix}1&1&1&2\\0&1&0&-1\\0&0&0&\text{c}-9\end{pmatrix}.$$

If $c\neq 9$ then $r(A)=2$ but $r(A\boldsymbol{b})=3$ and the equations are inconsistent. For $c=9$ our system has the same solution as

$$\begin{aligned}x_1+x_2+x_3&=2\\x_2&=-1.\end{aligned}$$

A particular solution to this system would be $x_3=0,\ x_2=-1,\ x_1=3$. The homogeneous equations are

$$\begin{aligned}x_1+x_2+x_3&=0\\x_2&=0\end{aligned}$$

which have a general solution of the form $x_3=\lambda,\ x_2=0,\ x_1=-\lambda$ for any real number λ. Hence the general solution to our nonhomogeneous equation is

$$\boldsymbol{x}^*=\begin{pmatrix}3-\lambda\\-1\\\lambda\end{pmatrix}.$$

2. In matrix notation we have

$$\begin{pmatrix}1&3&-3\\3&-17&8\\3&4&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}9\\49\\c\end{pmatrix},$$

or $A\boldsymbol{x}=\boldsymbol{b}$ where

$$(A\boldsymbol{b})=\begin{pmatrix}1&3&-2&9\\3&-17&8&49\\3&-4&1&c\end{pmatrix}\underset{\substack{r_2'=r_2-3r_1\\r_3'=r_3-3r_1}}{\cong}\begin{pmatrix}1&3&-2&9\\0&-26&14&22\\0&-13&7&c-27\end{pmatrix}\underset{r_2'=-\frac{1}{2}r_2}{\cong}\begin{pmatrix}1&3&-2&9\\0&13&-7&-11\\0&-13&7&c-27\end{pmatrix}$$

$$\underset{r_3'=r_3+r_2}{\cong}\begin{pmatrix}1&3&-2&9\\0&13&-7&-11\\0&0&0&c-38\end{pmatrix}.$$

Clearly $r(A)=2<3,$ the number of variables. Hence no unique solution exists as this requires that $r(A\boldsymbol{b})=r(A)=3.$

(a) If $c \neq 38$ then $r(A\boldsymbol{b}) \neq r(A)$ and the equations are inconsistent. No solution exists.

(b) If $c = 38$ then $r(A\boldsymbol{b}) = r(A) = 2$ which is less than the number of variables so an infinite number of solutions exist. For this case our equations have the same solution as

$$\begin{aligned} x + 3y - 2z &= 9 \\ 13y - 7z &= -11. \end{aligned}$$

Let $y = 0$, then $z = 11/7$ and $x = 85/7$ so a particular solution to these nonhomogeneous equations is

$$\frac{1}{7}\begin{pmatrix} 85 \\ 0 \\ 11 \end{pmatrix}.$$

The corresponding homogeneous equations have the same solution as

$$\begin{aligned} x + 3y - 2z &= 0 \\ -13y + 7z &= 0. \end{aligned}$$

Let $z = \lambda$, where λ is any real number. Then $y = 7\lambda/13$ and $x = 5\lambda/13$ so the general solution to the homogeneous equations can be written as

$$\frac{\lambda}{13}\begin{pmatrix} 5 \\ 7 \\ 13 \end{pmatrix}.$$

Adding, we have the general solution to the nonhomogeneous equations is

$$\begin{pmatrix} 85/7 + 5\lambda/13 \\ 7\lambda/13 \\ 11/7 + \lambda \end{pmatrix}.$$

3. (i) In matrix notation our system of equations is

$$\begin{pmatrix} 1 & 4 & 17 & 4 \\ 2 & 12 & 46 & 10 \\ 3 & 18 & 69 & 17 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 38 \\ 98 \\ 153 \end{pmatrix}.$$

Consider

$$(A\boldsymbol{b}) = \begin{pmatrix} 1 & 4 & 17 & 4 & 38 \\ 2 & 12 & 46 & 10 & 98 \\ 3 & 18 & 69 & 17 & 153 \end{pmatrix}.$$

If the equations are consistent

$$r(A\boldsymbol{b}) = r(A).$$

But $r(A) \le \min(3,4) < 4,$ where 4 is the number of variables, so the equations will then have an infinite number of solutions.

(ii)

$$(A\boldsymbol{b}) \underset{\substack{r_2'=r_2-2r_1 \\ r_3'=r_3-3r_1}}{\cong} \begin{pmatrix} 1 & 4 & 17 & 4 & 38 \\ 0 & 4 & 12 & 2 & 22 \\ 0 & 6 & 18 & 5 & 39 \end{pmatrix} \underset{r_2'=\frac{1}{2}r_2}{\cong} \begin{pmatrix} 1 & 4 & 17 & 4 & 38 \\ 0 & 2 & 6 & 1 & 11 \\ 0 & 6 & 18 & 5 & 39 \end{pmatrix} \underset{r_3'=r_3-3r_2}{\cong} \begin{pmatrix} 1 & 4 & 17 & 4 & 38 \\ 0 & 2 & 6 & 1 & 11 \\ 0 & 0 & 0 & 2 & 6 \end{pmatrix}.$$

Hence $r(A\boldsymbol{b}) = r(A) = 3$ and the equations are consistent.

$$(A\boldsymbol{b}) \underset{\substack{r_1'=r_1-2r_2 \\ r_3'=\frac{1}{2}r_3}}{\cong} \begin{pmatrix} 1 & 0 & 5 & 2 & 16 \\ 0 & 2 & 6 & 1 & 11 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix} \underset{\substack{r_1'=r_1-2r_3 \\ r_2'=r_2-r_3}}{\cong} \begin{pmatrix} 1 & 0 & 5 & 0 & 10 \\ 0 & 2 & 6 & 0 & 8 \\ 0 & 0 & 0 & 1 & 3 \end{pmatrix}.$$

It follows that the nonhomogeneous equations have the same solution as

$$\begin{aligned} x_1 + 5x_3 &= 10 \\ 2x_2 + 6x_3 &= 8 \\ x_4 &= 3. \end{aligned}$$

Let $x_3 = 0,$ then $x_4 = 4$ and $x_1 = 10,$ so a particular solution to this system is

$$\begin{pmatrix} 10 \\ 4 \\ 0 \\ 3 \end{pmatrix}.$$

The corresponding homogeneous system has the same solution as

$$\begin{aligned} x_1 + 5x_3 &= 0 \\ 2x_2 + 6x_3 &= 0 \\ x_4 &= 0. \end{aligned}$$

Letting $x_3 = \lambda,$ where λ is any real number we have $x_2 = -3\lambda$ and $x_1 = -5\lambda$ so the general solution to these homogeneous equations can be written as

$$\begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \\ 0 \end{pmatrix}.$$

Adding we see that the general solution to the nonhomogeneous equations can be written as

$$\boldsymbol{x}^* = \begin{pmatrix} 10-5\lambda \\ 4-3\lambda \\ \lambda \\ 3 \end{pmatrix}.$$

2.4

1. Writing our system as $\mathrm{A}\mathbf{x} = \mathrm{b}$ we have

(i) $A = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \underset{r_3'=r_3-r_1}{\cong} \begin{pmatrix} 1 & 2 & -2 \\ 0 & 2 & 1 \\ 0 & -2 & 3 \end{pmatrix}.$

Thus $|A| = 1(-1)^{1+1}\begin{vmatrix} 2 & 1 \\ -2 & 3 \end{vmatrix} = 8,$ and the equations have a unique solution. Now

$$A^{-1} = \frac{1}{8}\begin{pmatrix} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} \\ \begin{vmatrix} 2 & -2 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} \end{pmatrix}' = \frac{1}{8}\begin{pmatrix} 2 & 1 & -2 \\ -2 & 3 & 2 \\ 6 & -1 & 2 \end{pmatrix}'$$

so the unique solution is

$$\boldsymbol{x}^* = \frac{1}{8}\begin{pmatrix} 2 & -2 & 6 \\ 1 & 3 & -1 \\ -2 & 2 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 42 \\ 5 \\ 22 \end{pmatrix}.$$

(ii) $$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 2 & 3 \\ 2 & -2 & -2 \end{pmatrix} \underset{r_3'=r_3-2r_1}{\cong} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $|A| = 0$ and no unique solution exists. Moreover the first and the third equations are inconsistent, so no solution exists.

(iii) $$|A| = \begin{pmatrix} 1 & 1 & -2 \\ 2 & 2 & -4 \\ 3 & 3 & -6 \end{pmatrix} \underset{r_2'=r_2-2r_1}{\cong} \begin{pmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 \\ 3 & 2 & -6 \end{pmatrix},$$

so $|A| = 0,$ and no unique solution exists. Consider

$$(A\boldsymbol{b}) = \begin{pmatrix} 1 & 1 & -2 & 3 \\ 2 & 2 & -4 & 6 \\ 3 & 2 & -6 & 9 \end{pmatrix} \underset{r_2 \leftrightarrow r_3}{\cong} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 3 & 2 & -6 & 9 \\ 2 & 2 & -4 & 6 \end{pmatrix} \underset{\substack{r_2'=r_2-3r_1 \\ r_3'=r_3-2r_1}}{\cong} \begin{pmatrix} 1 & 1 & -2 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

thus $r(Ab) = r(A) = 2$ and we have an infinite number of solutions. Our system has the same solution as

$$\begin{aligned} x + y - 22 &= 3 \\ -y &= 0. \end{aligned}$$

Let $z = \lambda,$ any real number. Then the general solution can be written as

$$\boldsymbol{x}^* = \begin{pmatrix} 3+2\lambda \\ 0 \\ \lambda \end{pmatrix}.$$

2. Consider a system of n linear equations in n variables which we write as

$$A\boldsymbol{x} = \boldsymbol{b}.$$

If A is nonsingular then the solution for x_i can be written as

$$\boldsymbol{x}_i^* = \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} / |A|,$$

where the numerator is obtained by replacing the ith column *A* by the vector **b** and taking the determinant.

By Cramer's rule,

$$x_1 = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 4 & 2 & 1 \\ 8 & 0 & 1 \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} -3 & 0 & -3 \\ 4 & 2 & 1 \\ 8 & 0 & 1 \end{vmatrix}}{|A|} = \frac{2\begin{vmatrix} -3 & -3 \\ 8 & 1 \end{vmatrix}}{|A|} = 42/8.$$

$$x_2 = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 0 & 4 & 1 \\ 1 & 8 & 1 \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 0 & 4 & 1 \\ 0 & 7 & 3 \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} 4 & 1 \\ 7 & 3 \end{vmatrix}}{|A|} = 5/8$$

$$x_3 = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 8 \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & -2 & 7 \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} 2 & 4 \\ -2 & 7 \end{vmatrix}}{|A|} = 22/8.$$

Exercises for Chapter 3

3.2

1. (i) The endogenous variables are the variables of interest to us in our economic analysis. The whole purpose of building the model in the first place is to get some insight into what determines the values of these variables. The exogenous variables are variables whose values are taken as final for the purposes of our analysis. There are noneconomic variables, economic variables determined by noneconomic forces, or economic variables determined by economic forces other than those at play in the model. Definitional equations represent relationships between the variables of the model which are true by definition. Behavioral equations purport to tell us something of the behavior of some economic entities. The structural form is the original form of the model that comes to us from economists. The reduced form is obtained by solving for the endogenous variables in terms of the exogenous variables. The solutions are called the equilibrium values of the endogenous variables. A model is complete when the number of linear equations in the model equals the number of endogenous variables and the model has a unique solution.

(ii) Comparative static analysis concerns itself with how the equilibrium values of the endogenous variables change when we change the given values of the exogenous variables. If we write the structural form as

$A\mathbf{x} = \mathbf{b}$

where **x** is a vector containing the endogenous variables and **b** is a vector containing the endogenous variables or linear combinations of the exogenous variables, then the reduced form is given by

$\mathbf{x} = A^{-1}\mathbf{b}$

and all our comparative static results are summarized by

$\Delta\mathbf{x} = A^{-1}\Delta\mathbf{b}$

where $\Delta\mathbf{b}$ is the vector of changes in the values of the exogenous variables. Often in our economic analysis we are not interested in finding the complete reduced form. Instead, all that we are interested in is the equilibrium value of one of the endogenous variables and our comparative static analysis is concerned with how this equilibrium value changes when there are changes in the values of the exogenous variables. In this case we can use Cramer's rule to solve for the equilibrium value of the endogenous variable in question.

2. (i) Isolating the endogenous variables on the left hand side of the equations we have

$$Y - C - I = G$$
$$-bY + C = a$$
$$-dY + I = c$$

or in matrix notation

$$\begin{pmatrix} 1 & -1 & -1 \\ -b & 1 & 0 \\ -d & 0 & 1 \end{pmatrix}\begin{pmatrix} Y \\ C \\ I \end{pmatrix} = \begin{pmatrix} G \\ a \\ c \end{pmatrix},$$

which we write as

$$A\boldsymbol{x} = \boldsymbol{b}.$$

Now we have three equations in three endogenous variables and

$$|A| = \begin{vmatrix} 1 & -1 & -1 \\ 1-b & 0 & -1 \\ -d & 0 & 1 \end{vmatrix} = (-1)(-1)^{1+2}\begin{vmatrix} 1-b & -1 \\ -d & 1 \end{vmatrix} = 1 - b - d,$$

which we assume is nonzero so the model is complete. The reduced form is given by

$$\boldsymbol{x}^* = A^{-1}\boldsymbol{b}$$

$$= \frac{1}{1-b-d}\begin{pmatrix} \begin{vmatrix}1 & 0\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}-b & 0\\ -d & 1\end{vmatrix} & \begin{vmatrix}-b & 1\\ -d & 0\end{vmatrix} \\ -\begin{vmatrix}-1 & -1\\ 0 & 1\end{vmatrix} & \begin{vmatrix}1 & -1\\ -d & 1\end{vmatrix} & -\begin{vmatrix}1 & -1\\ -d & 0\end{vmatrix} \\ \begin{vmatrix}-1 & -1\\ 1 & 0\end{vmatrix} & -\begin{vmatrix}1 & -1\\ -b & 0\end{vmatrix} & \begin{vmatrix}1 & -1\\ -b & 1\end{vmatrix} \end{pmatrix}' \begin{pmatrix}G\\ a\\ c\end{pmatrix}$$

$$= \frac{1}{1-b-d}\begin{pmatrix}1 & b & d\\ 1 & 1-d & d\\ 1 & b & 1-b\end{pmatrix}'\begin{pmatrix}G\\ a\\ c\end{pmatrix} = \frac{1}{1-b-d}\begin{pmatrix}1 & 1 & 1\\ b & 1-d & b\\ d & d & 1-b\end{pmatrix}\begin{pmatrix}G\\ a\\ c\end{pmatrix}.$$

(ii) We go to the first column of A^{-1} and the elements of this column gives us the results. So

$$\Delta Y^* = \frac{1}{1-b-d}$$

$$\Delta C^* = \frac{b}{1-b-d}$$

$$\Delta I^* = \frac{d}{1-b-d}.$$

(iii) Proceding as before we have

$$Y - C - I = G$$

$$-bY + C = a - bT$$

$$-dY + I = c$$

so all that changes is that the vector **b** is now $\begin{pmatrix}G\\ a-bT\\ c\end{pmatrix}$ whereas before it was $\begin{pmatrix}G\\ a\\ c\end{pmatrix}$.

In our answer for (i) we replace a by $a - bT$.

If both exogenous variables increase by unit amounts then

$$\Delta\boldsymbol{b} = \begin{pmatrix}1\\ -b\\ 0\end{pmatrix}$$

and

$$\Delta \boldsymbol{x}^* = \frac{1}{1-b-d}\begin{pmatrix} 1 & 1 & 1 \\ b & 1-d & b \\ b & b & 1-b \end{pmatrix}\begin{pmatrix} 1 \\ -b \\ 0 \end{pmatrix}.$$

That is our results are obtained by adding to the first column of A^{-1}, $-b$ times the second column of A^{-1}. For example

$$\Delta C^* = \frac{b-b(1-d)}{1-b-d} = \frac{bd}{1-b-d}.$$

3. Isolating the endogenous variables on the left hand side we have

$$Q - \beta P = \alpha + \beta T$$
$$Q - bP = a + cR$$

which in matrix notation is

$$\begin{pmatrix} 1 & -\beta \\ 1 & -b \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} \alpha + \beta T \\ a + cR \end{pmatrix}.$$

Using Cramer's rule, the equilibrium values of the endogenous variables are given by

$$Q^* = \frac{\begin{vmatrix} \alpha + \beta T & -\beta \\ a + cR & -b \end{vmatrix}}{\begin{vmatrix} 1 & -\beta \\ 1 & -b \end{vmatrix}} = \frac{-b\left(\alpha + \beta T\right) + \beta\left(a + cR\right)}{\beta - b}$$

and

$$P^* = \frac{\begin{vmatrix} 1 & \alpha + \beta T \\ 1 & a + cR \end{vmatrix}}{\left(\beta - b\right)} = \frac{a + cR - \alpha - \beta T}{\beta - b}.$$

(i) $\Delta \mathrm{T}$

We have

$$\Delta Q^* = -\frac{b\beta\Delta T}{\beta - b} = \frac{(-\text{ve})(+\text{ve})(-\text{ve})(+\text{ve})}{(-\text{ve})-(+\text{ve})} = \frac{+\text{ve}}{-\text{ve}} = -\text{ve}.$$

Hence the increase in *T* leads to a decrease in the equilibrium quantity.

Similarly

$$\Delta P^* = \frac{-\beta\Delta T}{\beta - b} = \frac{(-\text{ve})(-\text{ve})(+\text{ve})}{-\text{ve}} = -\text{ve},$$

so the increase in *T* leads to a decrease in the equilibrium price.

(ii) $\underline{\Delta \text{R}}$

Similarly

$$\Delta Q^* = \frac{\beta c\Delta R}{\beta - b} = \frac{(-\text{ve})(+\text{ve})(-\text{ve})}{-\text{ve}} = -\text{ve},$$

so the decrease in rainfall leads to a decrease in the equilibrium quantity, and

$$\Delta P^* = \frac{c\Delta R}{\beta - b} = \frac{(+\text{ve})(-\text{ve})}{-\text{ve}} = +\text{ve},$$

so the decrease in rainfall leads to an increase in equilibrium price.

4. Substituting the consumption and investments functions into the definitional equation, then isolating the endogenous variables o the left hand side gives

$$Y(1-\beta) - \delta r = \alpha + \gamma + G$$
$$\tau Y + \lambda r = M.$$

Using Cramer's rule, the equilibrium values of our endogenous variables are given by

$$Y^* = \frac{\begin{vmatrix} \alpha+\gamma+G & -\delta \\ M & \lambda \end{vmatrix}}{\begin{vmatrix} 1-\beta & -\delta \\ \tau & \lambda \end{vmatrix}} = \frac{\lambda\left(\alpha+\gamma+G\right)+\delta M}{\lambda(1-\beta)+\delta\tau}$$

$$r^* = \frac{\begin{vmatrix} 1-\beta & \alpha+\gamma+G \\ \tau & M \end{vmatrix}}{\begin{vmatrix} 1-\beta & -\delta \\ \tau & \lambda \end{vmatrix}} = \frac{(1-\beta)M+\tau(\alpha+\gamma+G)}{\lambda(1-\beta)+\delta\tau}.$$

If *G* and *M* change by ΔG and ΔM respectively we have the resultant change in r^* is given by

$$\Delta r^* = \frac{\left(1-\beta\right)\Delta M + \Delta G}{\lambda\left(1-\beta\right)+\delta\tau}.$$

We want Δr^* to be zero which implies that

$$\left(1-\beta\right)\Delta M + \tau\Delta G = 0.$$

So the required decrease in *M* is given by

$$\Delta M = \frac{-\tau\Delta G}{\left(1-\beta\right)}.$$

Exercises for Chapter 4

4.1

(i) $$\begin{pmatrix} x_1 \; x_2 \end{pmatrix}\begin{pmatrix} 1 & 0.5 \\ 0.5 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

(ii) $$\begin{pmatrix} x \; y \end{pmatrix}\begin{pmatrix} 13 & 16 \\ 16 & 17 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

(iii) $$\begin{pmatrix} x_1 \; x_2 \; x_3 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1.5 \\ -1 & 1 & -2 \\ 1.5 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

(iv) $\begin{pmatrix} x_1 \, x_2 \, x_3 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$

(v) $\begin{pmatrix} x_1 \, x_2 \, x_3 \end{pmatrix} \begin{pmatrix} 42 & 1.5 & 4 \\ 1.5 & 2 & 3 \\ 4 & 3 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$

4.2

1. (i) Consider

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 1 = \lambda(\lambda-2)$$

so the matrix has the eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2.$

(ii) Consider

$$|A-\lambda I| = \begin{vmatrix} 5-\lambda & 2 & 1 \\ 2 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 1(-1)^{3+1} \begin{vmatrix} 2 & 1 \\ 1-\lambda & 0 \end{vmatrix} + (1-\lambda)(-1)^{3+3} \begin{vmatrix} 5-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$= \lambda - 1 + (1-\lambda)\left[(5-\lambda)(1-\lambda) - 4\right] = (1-\lambda)\left(5 - 6\lambda + \lambda^2 - 5\right) = (1-\lambda)\lambda(\lambda-6)$$

so the matrix has eigenvalues $\lambda_1 = 1,\ \lambda_2 = 0$ and $\lambda_3 = 6.$

(iii) Consider

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = 1(-1)^{1+2} \begin{vmatrix} 1 & 0 \\ 1 & 1-\lambda \end{vmatrix} + (1-\lambda)(-1)^{2+2} \begin{vmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix}$$

$$= -(1-\lambda) + (1-\lambda)\left[(2-\lambda)(1-\lambda) - 1\right] = (1-\lambda)\left(2 - 3\lambda + \lambda^2 - 2\right)$$

$$= (1-\lambda)\lambda(\lambda-3),$$

so the matrix has eigenvalues $\lambda_1 = 1,\ \lambda_2 = 0$ and $\lambda_3 = 3.$

(iv) Consider

$$|A-\lambda I| = \begin{vmatrix} -3-\lambda & 2 & 0 \\ 2 & -3-\lambda & 0 \\ 0 & 0 & -5-\lambda \end{vmatrix} = -(5+\lambda)(-1)^{3+3} \begin{vmatrix} -3-\lambda & 2 \\ 2 & -3-\lambda \end{vmatrix}$$

$$= -(5+\lambda)\left[(3+\lambda)^2 - 4\right] = -(5+\lambda)\left(\lambda^2 + 6\lambda + 5\right)$$

$$= -(5+\lambda)(\lambda+5)(\lambda+1),$$

so the eigenvalues are $\lambda_1 = \lambda_2 = -5$ and $\lambda_3 = -1.$

2. Consider

$$|A-\lambda I| = \begin{vmatrix} a_1-\lambda & b \\ b & a_2-\lambda \end{vmatrix} = (a_1-\lambda)(a_2-\lambda)-b^2 = a_1a_2-\lambda(a_1+a_2)+\lambda^2-b^2$$

so the equation which defines the eigenvalues is the quadratic equation

$$\lambda^2-\lambda(a_1+a_2)+a_1a_2-b^2=0.$$

This equation has equal roots if and only if

$$(a_1+a_2)^2-4(a_1a_2-b^2)=0.$$

That is $(a_1-a_2)^2-4b^2=0,$ but clearly this is impossible, for $b\neq 0.$

4.4

1. (i) The vectors $\mathbf{x}_1,\ldots,\mathbf{x}_m$ are orthogonal if $\mathbf{x}_i'\mathbf{x}_j=0$ for $i\neq j.$ They are orthonormal if they are orthogonal and $\boldsymbol{x}_i'\boldsymbol{x}_i=1$ for $i=1,\ldots,n.$

(ii) Consider

$$\mathbf{y}'\mathbf{x} = (y_1\ y_2\ y_3)\begin{pmatrix}1\\-1\\3\end{pmatrix} = y_1-y_2+3\,y_3.$$

For **y** and **x** to be orthogonal we want this expression to equal zero. Hence three vectors orthogonal to **x** would be

$$\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\-1\\0\end{pmatrix},\begin{pmatrix}-3\\0\\1\end{pmatrix}.$$

(iii) Consider

$$\mathbf{y}'\mathbf{x}_1 = \frac{1}{\sqrt{3}}(y_1\ y_2\ y_3)\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{\sqrt{3}}(y_1+y_2+y_3).$$

For y to be orthogonal to $\boldsymbol{x}_1$ we want this expression to be zero. Hence

$$\begin{pmatrix}-1\\1\\0\end{pmatrix}\text{and}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

would be orthogonal to $\boldsymbol{x}_1$. Normalizing these vectors we have that

$$x_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} \text{ and } x_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

meet the requirement.

2. (i) Consider

$$|A-\lambda I| = \begin{vmatrix}2-\lambda & 4\\4 & 2-\lambda\end{vmatrix} = (2-\lambda)^2 - 16 = 4 - 4\lambda + \lambda^2 - 16 = \lambda^2 - 4\lambda - 12,$$

so the matrix has two eigenvalues $\lambda_1 = 6$ and $\lambda_2 = -2$.

Eigenvector for $\lambda_1 = 6$

Consider

$$(A-\lambda_1 I)x = \begin{pmatrix}-4 & 4\\4 & -4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Both the equations give

$$x_1 = x_2.$$

We want our eigenvector to be normalized, that is

$$x_1^2 + x_2^2 = 1$$

so

$$2x_2^2 = 1$$

and we can take

$$x_2 = \frac{1}{\sqrt{2}}.$$

Thus an eigenvector corresponding to λ_1 that meets the requirements is

$$x_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}.$$

Eigenvector for $\lambda_2 = -2$

Consider

$$(A-\lambda_2 I)x = \begin{pmatrix}4 & 4\\4 & 4\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

Both these equations give

$$\boldsymbol{x}_1 = -\boldsymbol{x}_2$$

so a normalized vector corresponding to λ_2 would be

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Clearly $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are orthonormal.

(ii) Consider

$$|A - \lambda \mathrm{I}| = \begin{vmatrix} 4-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 4 = \lambda(\lambda - 5),$$

so the matrix has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$.

Eigenvector for $\lambda_1 = 0$

Consider

$$(A - \lambda_1 I)\boldsymbol{x} = A\boldsymbol{x} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both these equations give $x_2 = -2x_1$. We want our eigenvector to be normalized, so we require that $5x_1^2 = 1$ and we can take $x_1 = 1/\sqrt{5}$.

Our eigenvector associated with λ_1 would then be

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 5$

Consider

$$(A - \lambda_2 I)x = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both equations give

$$x_1 = 2x_2$$

so the required eigenvector is

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(iii) From exercise 1. (iii) of 4.2 we know the eigenvalues for this matrix are $\lambda_1 = 1,\ \lambda_2 = 0,$ and $\lambda_3 = 3.$

Eigenvector for $\lambda_1 = 1$

Consider

$$\left(A - \lambda_1 I\right)\boldsymbol{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which render the equations

$$x_1 + x_2 + x_3 = 0$$
$$x_1 = 0,$$

so $x_3 = -x_2$. For the vector to be normalized we want

$$x_1^2 + x_2^2 + x_3^2 = 1$$

which implies that $2x_2^2 = 1$ so take $x_2 = 1/\sqrt{2}.$

Our eigenvector associated with λ_1 would then be

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Eigenvector for $\lambda_2 = 0$

Consider

$$\left(A - \lambda_2 I\right)\boldsymbol{x} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$2x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 = 0$$
$$x_1 + x_3 = 0.$$

Thus $x_1 = -x_3 = -x_2$. The normalized requirement implies that $3x_1^2 = 1$ so we can take $x_1 = \frac{1}{\sqrt{3}}$ and the required eigenvector would be

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

$\text{Eigenvector for } \lambda_3 = 3$

Consider

$$(A - \lambda_3 I)\boldsymbol{x} = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

That is

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 &= 0 \\ x_1 - 2x_2 &= 0, \end{aligned}$$

so $x_1 = 2x_2 = 2x_3$. For the vector to be normalized we require $6x_2^2 = 1$ so we take $x_2 = \frac{1}{\sqrt{6}}$ and the eigenvector associated with λ_3 would be

$$\boldsymbol{x}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

4.5

1. (i) Consider

$$\frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix};$$

so the matrix is orthogonal.

(ii) Consider

$$\frac{1}{6}\begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \\ -\sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{3} & -\sqrt{3} \\ \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which establishes the result.

(iii) Consider

$$\frac{1}{30}\begin{pmatrix} 0 & \sqrt{5} & 5 \\ \sqrt{6} & -2\sqrt{5} & 2 \\ -2\sqrt{6} & -\sqrt{5} & 1 \end{pmatrix}\begin{pmatrix} 0 & \sqrt{6} & -2\sqrt{6} \\ \sqrt{5} & -2\sqrt{5} & -\sqrt{5} \\ 5 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

so the matrix is orthogonal.

2. Now

$$Q'Q = \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

and

$$\begin{aligned} Q'AQ &= \frac{1}{3}\begin{pmatrix} 1 & -\sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 0 & 0 \\ 3\sqrt{2} & 3 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 3 \end{pmatrix}. \end{aligned}$$

3. From exercise 1 of 4.2 we have that the eigenvalues of this matrix are $\lambda_1 = \lambda_2 = -5$ and $\lambda_3 = -1$.

Consider the following set of equations

$$(A - \lambda_1 I)\boldsymbol{x} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which imply that $x_1 = -x_2$ and x_3 can take any value.

Taking $x_3 = 0$ the normalization condition requires that $x_1^2 + x_2^2 = 1$. That is $2x_1^2 = 1$ so we can take $x_1 = 1/\sqrt{2}$.

This gives the following eigenvector

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Taking $x_3 = 1$ the normalization condition requires that $x_1^2 + x_2^2 + 1 = 1$. So $x_1 = x_2 = 0$. *A* second eigenvector associated with the repeated roots that meet our requirements would be

$$\boldsymbol{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Consider now

$$(A-\lambda_3 I)\boldsymbol{x} = \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which imply that

$$x_1 = x_2,\ x_3 = 0.$$

The normalization condition requires then that

$$2x_1^2 = 1$$

so we take $x_1 = \frac{1}{\sqrt{2}}$ and our eigenvector is

$$\boldsymbol{x}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

A Q that meets our requirement is

$$Q = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

The main diagonal elements of $Q'AQ$ are the eigenvalues –5, –5, and –1.

4. (i) (a) From exercise 1. (i) of 4.2 the eigenvalues of this matrix are $\lambda_1 = 0 \text{ and } \lambda_2 = 2.$

Consider the equations

$$(A-\lambda_1 I)\boldsymbol{x} = A\boldsymbol{x} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which renders $x_1 = x_2$. The normalized condition requires $x_1^2 + x_2^2 = 1$ so we take $x_1 = 1/\sqrt{2}$ and

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as our eigenvector.

The equations

$$(A-\lambda_2 I)\boldsymbol{x} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

give $x_1 = -x_2$ so we take

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

as our eigenvector. Then

$$Q = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The main diagonal elements are the eigenvalues 0 and 2.

(b) From exercise 1. (ii) of 4.2 the eigenvalues of this matrix are $\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = 6$, so first we consider

$$(A - \lambda_1 I)\boldsymbol{x} = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives $x_1 = 0$ and $x_3 = -2x_2$. The normalization requirement is then $5x_2^2 = 1$ so we take $x_2 = \frac{1}{\sqrt{5}}$ and

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

as the eigenvector associated with λ_1.

Next consider

$$(A - \lambda_2 I)\boldsymbol{x} = A\boldsymbol{x} = \begin{pmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$\begin{aligned} 5x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 &= 0 \\ x_1 + x_3 &= 0. \end{aligned}$$

So we have $x_1 = -x_3 = -x_2/2$.

The normalization requires then that $\frac{3}{2}x_2^2 = 1$ so we take $x_2 = \sqrt{\frac{2}{3}}$ and as our eigenvector,

$$\boldsymbol{x}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

Lastly,

$$(A-\lambda_3 I)\boldsymbol{x} = \begin{pmatrix} -1 & 2 & 1 \\ 2 & -5 & 0 \\ 1 & 0 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

which gives

$$x_1 - 5x_2 = 0$$
$$2x_1 - 5x_2 = 0$$

so $x_1 = 5x_3 = 2.5x_2$. Normalization requires that $x_1^2 + x_2^2 + x_3^2 = 1$ so $30x_3^2 = 1$ and we take $x_3 = 1/\sqrt{30}$ and as our eigenvector

$$x_3 = \frac{1}{\sqrt{30}}\begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}.$$

An orthogonal Q that meets our requirements is

$$Q = \frac{1}{\sqrt{30}}\begin{pmatrix} 0 & -\sqrt{5} & 5 \\ \sqrt{6} & 2\sqrt{5} & 2 \\ -2\sqrt{6} & \sqrt{5} & 1 \end{pmatrix}.$$

The main diagonal elements of $Q'AQ$ are the eigenvalues 1, 0, 6.

(c) From exercise 2. (iii) of 4.4 we have that a set of orthonormal vectors for this matrix are

$$\boldsymbol{x}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \boldsymbol{x}_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \boldsymbol{x}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

so

$$Q = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & \sqrt{2} & 2 \\ \sqrt{3} & -\sqrt{2} & 1 \\ -\sqrt{3} & -\sqrt{2} & 1 \end{pmatrix}.$$

The main diagonal elements are the eigenvalues.

(ii) (a) As the eigenvalues are 0 and 2 we know the matrix is positive semidefinite and $\boldsymbol{x}'A\boldsymbol{x} \geq 0$ for all $\boldsymbol{x}$.

(b) As the eigenvalues are 0, 1, and 6 the matrix is positive semidefinite and $\boldsymbol{x}'A\boldsymbol{x} \geq 0$ for all ***x***.

(c) Again the eigenvalues are 0, 1, and 3 so the matrix is positive semidefinite and $\boldsymbol{x}'A\boldsymbol{x} \geq 0$ for all ***x***.

4.6

1. (i) From exercise 1 of 4.5 a set of orthonormal eigenvectors for this matrix is

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \text{ and } \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}.$$

so we consider the transformation

$$\boldsymbol{y} = Q'\boldsymbol{x}$$

where $Q = \left(\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \; \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \right)$.

That is

$$y_1 = \frac{1}{\sqrt{3}}x_1 - \frac{\sqrt{2}x_2}{\sqrt{3}}$$

$$y_2 = x_1 + \frac{1}{\sqrt{2}}x_2.$$

(ii) From exercise 2 of 4.5 a transformation that meets our requirements is

$$\boldsymbol{y} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}\boldsymbol{x}.$$

That is

$$y_1 = \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2$$

$$y_2 = x_3$$

$$y_3 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2.$$

2. In matrix notation the quadratic form can be written as

$$(x\, y)\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

so we consider

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4$$

$$= (\lambda - 3)(\lambda + 1)$$

and equating this determinant to zero gives eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$.

Consider

$$\left(A-\lambda_1 I\right)\boldsymbol{x}=\begin{pmatrix}-2 & 2\\ 2 & -2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=0$$

which gives $x_1 = x_2$ so a normalized eigenvector would be

$$\boldsymbol{x}_1=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}.$$

Consider

$$\left(A-\lambda_2 I\right)=\begin{pmatrix}2 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

so $x_1 = -x_2$ and a normalized eigenvector would be

$$\boldsymbol{x}_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}.$$

The required transformation is

$$\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}.$$

The quadratic form is indefinite as one of the eigenvalues is positive whereas the other is negative.

3. If *A* is an $n\times n$ positive definite matrix then all the eigenvalues of *A* are positive so there exists an orthogonal matrix Q such that

$$Q'AQ=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$$

where $\lambda_1,\ldots,\lambda_n$ are all positive. Let

$$\sqrt{D}=\begin{pmatrix}\sqrt{\lambda_1} & & 0\\ & \ddots & \\ 0 & & \sqrt{\lambda_n}\end{pmatrix}.$$

Then as Q is orthogonal

$$A=Q\sqrt{D}\,\sqrt{D}\,Q'$$

so let

$$P=\sqrt{D}\,Q'.$$

4.7

1. Consider

$$A' = \frac{(\mathbf{i}\,\mathbf{i}')'}{n} = (\mathbf{i}')'\,\frac{\mathbf{i}'}{n} = A$$

and

$AA = \mathbf{i}\,\mathbf{i}'\mathbf{i}\,\mathbf{i}'/n^2 = A$ as $\mathbf{i}'\,\mathbf{i} = n,$ so *A* is symmetric and idempotent.

Now

$$B' = (I - A)' = I - A' = B$$

as *A* is symmetric and

$$BB = (I - A)(I - A)) = I - 2A + A^2 = B$$

as *A* is idempotent, so *B* is also symmetric idempotent.

$$AB = A(I - A) = A - A^2 = 0$$

as *A* is idempotent.

As *A* and *B* are symmetric idempotent their ranks are equal to their traces so

$$r(A) = \mathrm{tr}A = \mathrm{tr}\mathrm{i}'i/n = 1$$

and

$$r(B) = \mathrm{tr}B = \mathrm{tr}(I - A) = \mathrm{tr}I - \mathrm{tr}A = n - 1.$$

As the eigenvalues of a symmetric idempotent matrix are 1 or 0 and as the rank of a symmetric matrix is equal to the number of nonzero eigenvalues, *A* has *n*–1 eigenvalues of 0 and 1 eigenvalue of 1 whereas *B* has *n*–1 eigenvalues of 1 and 1 eigenvalue of 0. It follows that both *A* and *B* are positive semi definite matrices (clearly all symmetric idempotent matrices are). Moreover as the determinant of a symmetric matrix is the product of the eigenvalues both matrices have zero determinants.

2. Consider

$$N' = \left(X(X'X)^{-1}X'\right)' = (X')'\left[(X'X)'\right]^{-1}X' = N$$

and

$$NN = X(X'X)^{-1}X'X(X'X)^{-1}X' = N$$

so *N* is symmetric idempotent. Now

$$M' = (I - N)' = I - N$$

as N is symmetric and

$$MM = (I - N)'(I - N) = I - 2N + N^2 = M$$

as N is idempotent, so M is also symmetric idempotent. Also

$$MN = (I - N)N = N - N^2 = 0$$

as N is idempotent. Finally

$$r(N) = \text{tr}N = \text{tr}\left(X'X\right)^{-1} X'X = \text{tr}I_K = K$$

and

$$r(M) = \text{tr}M = \text{tr}I_n - K = n - K.$$

The matrix N has K eigenvalues of 1 and $n–K$ eigenvalues of 0 whereas M has $n–K$ eigenvalues of 1 and K eigenvalues of 0. Both matrices have determinants of 0.

3. (i) From exercise 1. (iii) of 4.2 the eigenvalues of this matrix are 1, 0 and 3, so clearly the trace of the matrix is the sum of the eigenvalues. Now

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0,$$

so the determinant is the product of the eigenvalues. The rank is 2.

(ii) From exercise 1. (ii) of 4.2 the eigenvalues are 1, 0, and 6 so clearly the trace is equal to the sum of the eigenvalues. Now

$$\begin{vmatrix} 5 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 1(-1)^{3+3} \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0$$

so the determinant is the product of the eigenvalues. The rank is 2.

(iii) From exercise 4. (i) of 4.5 the eigenvalues are 1, 0, and 3, so clearly the trace is the sum of the eigenvalues. The determinant is clearly zero, the product of the eigenvalues and the rank is 2.

4.8

1. (i) The leading principal minors are

$$-3, \begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix} = 5, \begin{vmatrix} -3 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & -5 \end{vmatrix} = -5(-1)^{3+3} \begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix} = -25.$$

These alternate in sign, the first being negative. Thus the matrix is negative definite.

(ii) The leading principal minors are

$$-3, \begin{vmatrix} -3 & 1 \\ 1 & -1 \end{vmatrix} = 2, \begin{vmatrix} -3 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{vmatrix} = \begin{vmatrix} 0 & -2 & 6 \\ 1 & -1 & 2 \\ 0 & 2 & -8 \end{vmatrix} = 1(-1)^{2+1} \begin{vmatrix} -2 & 6 \\ 2 & -8 \end{vmatrix} = -4.$$

These alternate in sign, the first being negative. Thus the matrix is negative definite.

2. The principal minors are

$$2,\ 1,\ 1, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1, \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 0.$$

The principal minors are all nonnegative with one being zero. Thus the matrix is positive semidefinite.

3. The first order principal minors are 1, 3, and 8 but a second order principal minor is $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$. This is enough to establish that the matrix is indefinite.

Exercises for Chapter 5

5.2

1. Let $f\left(x_1,\dots,x_n\right) = f(\boldsymbol{x})$ be a differentiable function of many variables. Then the gradient vector of $f(\boldsymbol{x})$ is

$$\nabla f(\boldsymbol{x}) = \begin{pmatrix} f_1(\boldsymbol{x}) \\ \vdots \\ f_n(\boldsymbol{x}) \end{pmatrix}.$$

That is it is the vector of first order partial derivatives of the function.

The Hessian matrix of the function is

$$H\left(\boldsymbol{x}\right) = \begin{pmatrix} f_{11}(\boldsymbol{x}) & \dots & f_{1n}(\boldsymbol{x}) \\ \vdots & & \vdots \\ f_{n1}(\boldsymbol{x}) & \cdots & f_{nn}(\boldsymbol{x}) \end{pmatrix}.$$

That is it is the matrix of all second order partial derivatives of $f(\boldsymbol{x})$.

(i) $y_1 = 3x_1^2 x_2^2,\ y_2 = x_1^3\ x_2$

so $\nabla y(\boldsymbol{x}) = \begin{pmatrix} 3x_1^2 x_2^2 \\ 2x_1^3 x_2 \end{pmatrix}.$

$$y_{11} = 6x_1 x_2^2,\ y_{12} = 6x_1^2 x_2,\ y_{22} = 2x_1^3$$

so the Hessian matrix is

$$H(\boldsymbol{x}) = \begin{pmatrix} 6x_1x_2^2 & 6x_1^2x_2 \\ 6x_1^2x_2 & 2x_1^3 \end{pmatrix}.$$

(ii) $y_1 = 5x_1^4 - 6x_1x_2,\ y_2 = -3x_1^2 + 2x_2$

so $\nabla y(\boldsymbol{x}) = \begin{pmatrix} 5x_1^4 - 6x_1x_2 \\ -3x_1^2 + x_2 \end{pmatrix}.$

$$y_{11} = 20x_1^3 - 6x_2,\ y_{12} = -6x_1,\ y_{22} = 2$$

so

$$H(\boldsymbol{x}) = \begin{pmatrix} 20x_1^3 - 6x_2 & -6x_1 \\ -6x_1 & 2 \end{pmatrix}.$$

(iii) $y_1 = 1/x_2,\ y_2 = -x_1/x_2^2$ so

$$\nabla y(\boldsymbol{x}) = \begin{pmatrix} 1/x_2 \\ -x_1/x_2^2 \end{pmatrix}.$$

$$y_{11} = 0,\ y_{12} = -1/x_2^2,\ y_{22} = 2x_1/x_2^3,$$

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & -1/x_2^2 \\ -1/x_2^2 & 2x_1/x_2^3 \end{pmatrix}.$$

(iv) Write $y = (x_1 - x_2)(x_1 + x_2)^{-1}$. Then

$$y_1 = (x_1 + x_2)^{-1} - (x_1 - x_2)(x_1 + x_2)^{-2} = 2x_2(x_1 + x_2)^{-2},$$
$$y_2 = -(x_1 + x_2)^{-1} - (x_1 - x_2)(x_1 + x_2)^{-2} = -2x_1(x_1 + x_2)^{-2},$$

so

$$\nabla y(\boldsymbol{x}) = \begin{pmatrix} 2x_2(x_1 + x_2)^{-2} \\ -2x_1(x_1 + x_2)^{-2} \end{pmatrix}.$$

$$y_{11} = -4x_2(x_1 + x_2)^{-3},$$
$$y_{12} = 2(x_1 + x_2)^{-2} - 4x_2(x_1 + x_2)^{-3} = 2(x_1 - x_2)(x_1 + x_2)^{-3},$$
$$y_{22} = 4x_1(x_1 + x_2)^{-3},$$

so

$$H(\boldsymbol{x}) = \frac{1}{(x_1 + x_2)^3} \begin{pmatrix} -4x_2 & 2(x_1 - x_2) \\ 2(x_1 - x_2) & 4x_1 \end{pmatrix}.$$

(v) $y_1 = 5\left(x_1^2 + x_2^2\right)^4 2x_1 = 10x_1\left(x_1^2 - 2x_2^2\right)^4,$

$y_2 = 5\left(x_1^2 - 2x_2^2\right)^4(-4x_x) = -20x_2\left(x_1^2 - 2x_2^2\right)^4,$

so

$$\nabla y(\boldsymbol{x}) = \left(x_1^2 - 2x_2^2\right)^4 \begin{pmatrix} 10x_1 \\ -20x_2 \end{pmatrix}.$$

$$\begin{aligned} y_{11} &= 10\left(x_1^2 - 2x_2^2\right)^4 + 40x_1\left(x_1^2 - 2x_2^2\right)^3(2x_1) \\ &= \left(x_1^2 - 2x_2^2\right)^3\left(90x_1^2 - 20x_2^2\right) \\ y_{12} &= 40x_1\left(x_1^2 - 2x_2^2\right)^3(-8x_2) = -320x_1x_2\left(x_1^2 - 2x_2^2\right)^3 \\ y_{22} &= -20\left(x_1^2 - 2x_2^2\right)^4 - 80x_2\left(x_1^2 - 2x_2^2\right)^3(-4x_2) \\ &= \left(x_1^2 - 2x_2^2\right)^3\left(360x_1^2 - 20x_2^2\right), \end{aligned}$$

so

$$H(\boldsymbol{x}) = \left(x_1^2 - 2x_2^2\right)^3 \begin{pmatrix} 90x_1^2 - 20x_2^2 & -320x_1x_2 \\ -320x_1x_2 & 360x_2^2 - 20x_1^2 \end{pmatrix}.$$

(vi) $y_1 = \dfrac{-2x_1^{-3}}{x_1^{-2} + x_2^{-3}},\ y_2 = \dfrac{-3x_2^{-4}}{x_1^{-2} + x_2^{-3}},$

so

$$\nabla y(\boldsymbol{x}) = \frac{1}{x_1^{-2} + x_2^{-3}} \begin{pmatrix} -2x_1^{-3} \\ -3x_2^{-4} \end{pmatrix}.$$

Write $y_1 = -2x_1^{-3}\left(x_1^{-2} - x_2^{-3}\right)^{-1}$ so

$$\begin{aligned} y_{11} &= 6x_1^{-4}\left(x_1^{-2} + x_2^{-3}\right)^{-1} + 2x_1^{-3}\left(x_1^{-2} + x_2^{-3}\right)^{-2}\left(-2x_1^{-3}\right) \\ &= 2x_1^{-4}\left(x_1^{-2} + 3x_2^{-3}\right)\left(x_1^{-2} + x_2^{-3}\right)^{-2} \\ y_{12} &= 2x_1^{-3}\left(x_1^{-2} + x_2^{-3}\right)^{-2}\left(-3x_2^{-4}\right) \\ &= -6x_1^{-3}x_2^{-4}\left(x_1^{-2} + x_2^{-3}\right)^{-2}. \end{aligned}$$

Write $y_2 = -3x_2^{-4}\left(x_1^{-2} + x_2^{-3}\right)^{-1}$ so

$$\begin{aligned} y_{22} &= 12x_2^{-5}\left(x_1^{-2} + x_2^{-3}\right)^{-1} + 3x_2^{-4}\left(x_1^{-2} + x_2^{-3}\right)^{-2}\left(-3x_2^{-4}\right) \\ &= 3x_2^{-5}\left(4x_1^{-2} + x_2^{-3}\right)\left(x_1^{-2} + x_2^{-3}\right)^{-2} \end{aligned}$$

so

$$H(\boldsymbol{x}) = \frac{1}{\left(x_1^{-2} + x_2^{-3}\right)^2} \begin{pmatrix} 2x_1^{-4}\left(x_1^{-2} + 3x_2^{-3}\right) & -6x_1^{-3}x_2^{-4} \\ -6x_1^{-3}x_x^{-4} & 3x_2^{-5}\left(4x_1^{-2} + x_2^{-3}\right) \end{pmatrix}.$$

(vii) $y_1 = e^{2x_1+3x_2},\ y_2 = 3e^{2x_1+3x_2}$

so

$$\nabla y(\boldsymbol{x}) = e^{2x_1+3x_2}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

$$y_{11} = 4e^{2x_1+3x_2},\ y_{12} = 6e^{2x_1+3x_2},\ y_{22} = 9e^{2x_1+3x_2},$$

so

$$H(\boldsymbol{x}) = e^{2x_1+3x_2}\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}.$$

(viii) $y_1 = 6x_1x_2 - 7\sqrt{x_2},\ y_2 = 3x_1^2 - 7x_1/2\sqrt{x_2}$

so

$$\nabla y(\boldsymbol{x}) = \begin{pmatrix} 6x_1x_2 - 7\sqrt{x_2} \\ 3x_1^2 - 7x_1/2\sqrt{x_2} \end{pmatrix}$$

$$y_{11} = 6,\ \ y_{12} = 6x_1 - 7/2\sqrt{x_2},\ \ y_{22} = 7x_1/4x_2^{3/2}$$

so

$$H(\boldsymbol{x}) = \begin{pmatrix} 6 & 6x_1 - 7/2\sqrt{x_2} \\ 6x_1 - 7/2\sqrt{x_2} & 7x_1/4x_2^{3/2} \end{pmatrix}.$$

2. $\dfrac{\partial y}{\partial x_1} = \dfrac{x_1}{x_1^2 + x_2^2},\ \dfrac{\partial y}{\partial x_2} = \dfrac{x_2}{x_1^2 + x_2^2},$

$$\frac{\partial^2 y}{\partial x_1^2} = \left(x_1^2 + x_2^2\right)^{-1} - x_1\left(x_1^2 + x_2^2\right)^{-2} 2x_1 = \frac{x_2^2 - x_1^2}{\left(x_2^2 + x_2^2\right)^2}.$$

By symmetry

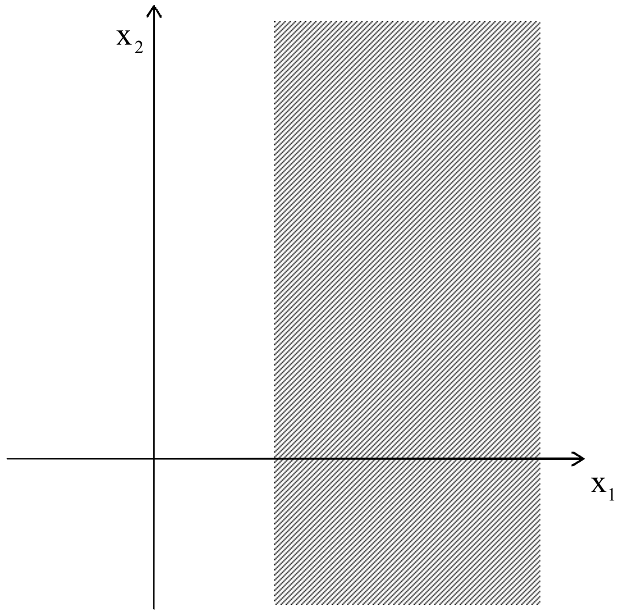
$$\frac{\partial^2 y}{\partial x_2^2} = \frac{x_1^2 - x_2^2}{\left(x_2^2 + x_2^2\right)^2},$$

so clearly,

$$\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_2^2} = 0.$$

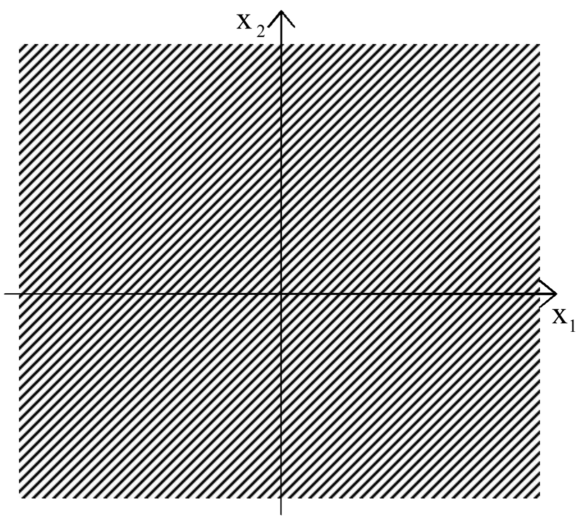
5.3

1. (i) (a) $A \cap B$



$A \cap B$ is convex.

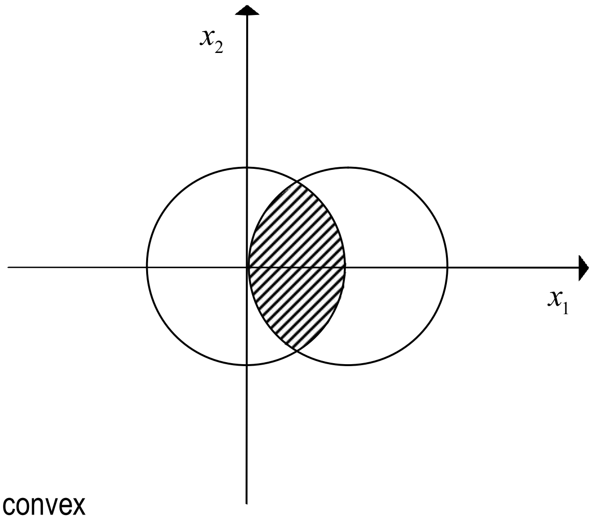
$A \cup B = E^2$



$A \cup B = E^2$ is convex.

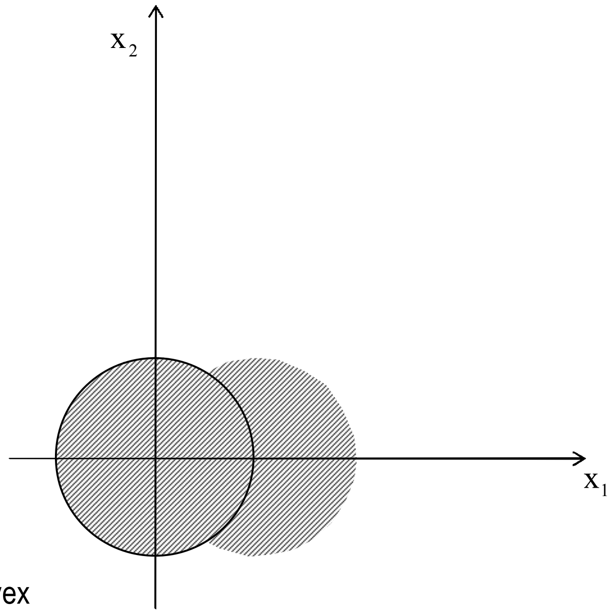
(b)

$A \cap B$



$A \cap B$ is convex

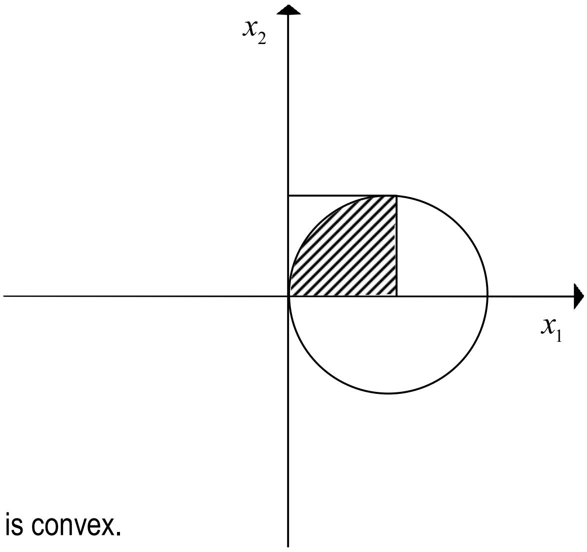
$A \cup B$



$A \cup B$ is not convex

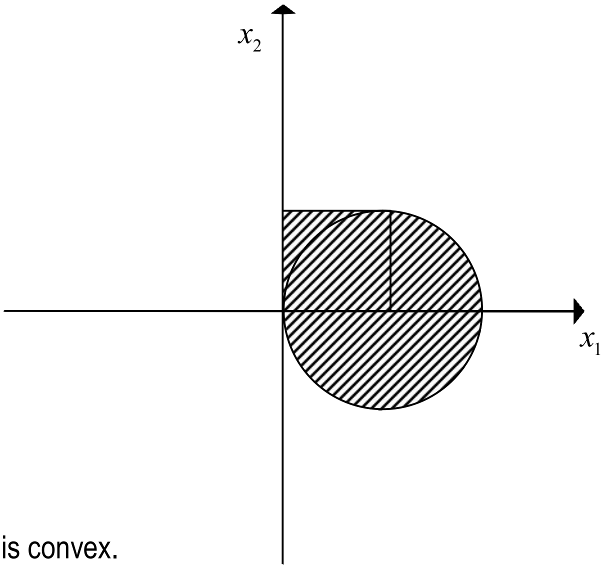
(c)

$A \cap B$



$A \cap B$ is convex.

$A \cup B$



$A \cup B$ is convex.

(ii) (a) Let $\mathbf{u} \in X, \mathbf{v} \in X \Rightarrow \mathbf{c}'\mathbf{u} \le z,$ and $\mathbf{c}'\mathbf{v} \le z$

Consider

$$\begin{aligned} \mathbf{c}'\left(\lambda \mathbf{u} + (1-\lambda)\mathbf{v}\right) &= \lambda \mathbf{c}'\mathbf{u} + (1-\lambda)\mathbf{c}'\mathbf{v} \qquad 0 \le \lambda \le 1 \\ &\le \lambda z + (1-\lambda) z \\ &= z, \end{aligned}$$

so $\lambda \mathbf{u} + (1-\lambda)\mathbf{v} \in X$.

(b) Let $\mathbf{u}, \mathbf{v} \in X \Rightarrow A\mathbf{u} = b$ and $A\mathbf{v} = b$ and consider

$$\begin{aligned} A(\lambda\mathbf{u} + (1-\lambda)\mathbf{v}) &= \lambda A\mathbf{u} + (1-\lambda)A\mathbf{v} \qquad 0 \le \lambda \le 1 \\ &= \lambda b + (1-\lambda)b \\ &= b \end{aligned}$$

so $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in X$.

(c) Similar to (*b*)

(d) Let **u** and $\mathbf{v} \in X \Rightarrow u_i \ge 0,\ v_i \ge 0$ and consider $\lambda\mathbf{u} + (1-\lambda)\mathbf{v}$ for $0 \le \lambda \le 1$. The

*i*th element of this point is

$$\lambda u_i + (1-\lambda)v_i \ge \lambda 0 + (1-\lambda)0 = 0,$$

so this point also belongs to the non negative orthant.

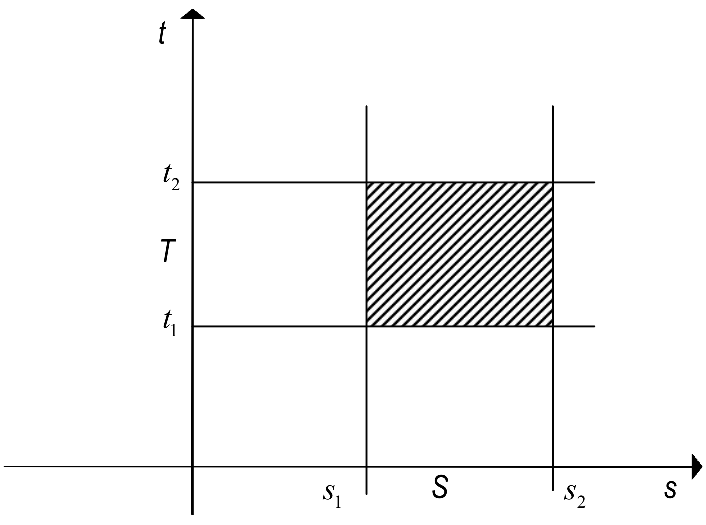
2. Suppose *S* had two points say **u** and **v**. Then all points $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in S$ for $0 \le \lambda \le 1$. But this contradicts S having a finite number of elements. So S has no or one point only.

3. Suppose $(s_1, t_1) \in S \times T \Rightarrow s_1 \in S$ and $t_1 \in T$ and $(s_2, t_2) \in S \times T \Rightarrow s_2 \in S$ and $t_2 \in \mathrm{T}$.
Consider the point

$$\lambda(s_1, t_1) + (1-\lambda)(s_2, t_2) = (\lambda s_1, (1-\lambda)s_2, \lambda t_1 + (1-\lambda)t_2) \quad 0 \le \lambda \le 1$$

As S is convex $\lambda s_1 + (1-\lambda)s_2 \in S$ and as *T* is convex $\lambda t_1 + (1-\lambda)t_2 \in \mathrm{T}$.

Graph of $S \times T$



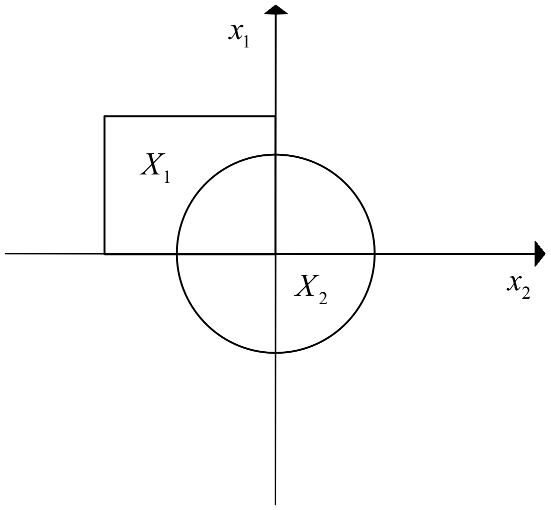
4. Let $\mathbf{u},\mathbf{v} \in X_1 \cap X_2 \Rightarrow \mathbf{u},\mathbf{v} \in X_1$ and $\mathbf{u},\mathbf{v} \in X_2$

$$\Rightarrow \lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in X_1 \text{ and } \lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in X_2$$

$$\Rightarrow \lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in X_1 \cap X_2$$

so $X_1 \cap X_2$ is convex.

$X_1 \cup X_2$ need not be convex as the following counter example shows:



$$X_1 = \{\mathbf{x} \in E^2 / x_1 \le 0,\ x_1 \ge -2,\ x_2 \ge 0,\ x_2 \le 2\}$$
$$X_2 = \{\mathbf{x} \in E^2 / x_1^2 + x_2^2 \le 1\}.$$

5. Let $y = f(x_1,\dots,x_n) = f(\mathbf{x})$ be a function of many variables. This function is homogeneous of degree r if

$$f(\lambda x_1,\dots,\lambda x_n) = \lambda^r f(x_1,\dots,x_n).$$

Euler's theorem

If $y = f(\mathbf{x})$ is homogeneous of degree r and differentiable then

$$x_1\frac{\partial f}{\partial x_1} + \ldots + x_n\frac{\partial f}{\partial x_n} = rf(\mathbf{x}).$$

(i) Consider

$$f(\lambda x,\lambda y)=(\lambda x)^{\frac{1}{2}}(\lambda y)^{\frac{1}{2}}+3(\lambda x)^{2}(\lambda y)^{-1}=\lambda f(x,y),$$

and the function is homogeneous of degree 1. Now

$$f_x=\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}+6xy^{-1},\ f_y=\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}-3x^2y^{-2}$$

so

$$xf_x+yf_y=\frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}+6xy^{-1}+\frac{1}{2}x^{\frac{1}{2}}y^{\frac{1}{2}}-3x^2y^{-1}=f(x,y)$$

and Euler's theorem holds.

(ii) Consider

$$f(\lambda x,\lambda y)=(\lambda x)^{\frac{3}{4}}(\lambda y)^{\frac{1}{4}}+6\lambda x=\lambda f(x,y)$$

so the function is homogeneous of degree 1. Now

$$f_x=\frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}}+6,\ f_y=\frac{1}{4}x^{\frac{3}{4}}y^{-\frac{3}{4}}$$

so

$$xf_x+yf_y=\frac{3}{4}x^{\frac{3}{4}}y^{\frac{1}{4}}+6x+\frac{1}{4}x^{\frac{3}{4}}y^{\frac{1}{4}}=f(x,y),$$

and Euler's theorem holds.

(iii) $$f(\lambda x,\lambda y)=\frac{(\lambda x)^2-(\lambda y)^2}{(\lambda x)^2+(\lambda y)^2}+3=f(x,y)$$

so the function is homogeneous of degree 0.

$$f_x=2x\left(x^2+y^2\right)^{-1}-x^2\left(x^2+y^2\right)^{-2}2x=2xy^2\left(x^2+y^2\right)^{-2}$$

$$f_y=-2y\left(x^2+y^2\right)^{-1}+y^2\left(x^2+y^2\right)2y=-2yx^2\left(x^2+y^2\right)^{-2}.$$

Thus

$$xf_x+yf_y=\left(2x^2y^2-2y^2x^2\right)\left(x^2+y^2\right)^{-2}=0,$$

so Euler's theorem holds.

(iv) $$f(\lambda x,\lambda y)=3(\lambda x)^5(\lambda y)+2(\lambda x)^2(\lambda y)^4-3(\lambda x)^3(\lambda y)^3=\lambda^6 f(x,y)$$

so the function is homogeneous of degree 6.

$$f_x=15x^4y+4xy^4-9x^2y^3,\ f_y=3x^5+8x^2y^3-9x^3y^2,$$

so

$$xf_x + yf_y = 15x^5y + 4x^2y^4 - 9x^3y^3 + 3yx^5 + 8x^2y^4 - 9x^3y^3 = 6f(x,y)$$

and Euler's theorem holds.

6. (i) For the Cobb-Douglas function

$$Q(\lambda K, \lambda L) = A(\lambda K)^{\alpha}(\lambda L)^{\beta} = \lambda^{\alpha+\beta}Q(K,L)$$

so the function is homogeneous of degree $\alpha + \beta$.

For the CES production function

$$Q(\lambda K, \lambda L) = A\left(a_1(\lambda K)^{\rho} + a_2(\lambda L)^{\rho}\right)^{\frac{1}{\rho}}$$

$$= Q(K,L)\left(\lambda^{\rho}\right)^{\frac{1}{\rho}} = \lambda Q(K,L),$$

so the function is homogeneous of degree 1.

(ii) For the Cobb-Douglas function

$$Q_K = \alpha AK^{\alpha-1}L^{\beta},\ Q_L = \beta AK^{\alpha}L^{\beta-1}$$

so

$$KQ_K + LQ_K = \alpha AK^{\alpha}L^{\beta} + \beta AK^{\alpha}L^{\beta} = (\alpha+\beta)Q(K,L),$$

and Euler's theorem holds.

For the CES production function

$$Q_K = \frac{1}{\rho}A\left(a_1K^{\rho} + a_2L^{\rho}\right)^{\frac{1}{\rho}-1}a_1\rho K^{\rho-1},\ Q_L = \frac{1}{\rho}A\left(a_1K^{\rho} + a_2L^{\rho}\right)^{\frac{1}{\rho}-1}a_2\rho L^{\rho-1}$$

so

$$KQ_K + LQ_L = A\left(a_1K^{\rho} + a_2L^{\rho}\right)^{\frac{1}{\rho}-1}\left(a_1K^{\rho} + a_2L^{\rho}\right) = Q(K,L)$$

and Euler's theorem holds.

(iii) For the Cobb-Douglas function

$$Q_K(\lambda K, \lambda L) = \alpha A(\lambda K)^{\alpha-1}(\lambda L)^{\beta}$$

$$= \lambda^{\alpha+\beta-1}Q_K(K,L)$$

so the marginal product Q_K is homogeneous of degree $\alpha+\beta-1$. Similarly Q_L is homogeneous of degree $\alpha+\beta-1$.

For the CES production function

$$Q_K(\lambda K, \lambda L) = A\left(a_1(\lambda K)^{\rho} + a_2(\lambda L)^{\rho}\right)^{\frac{1-\rho}{\rho}}a_1(\lambda K)^{\rho-1} = Q_K(K,L)\left(\lambda^{\rho}\right)^{\frac{1-\rho}{\rho}}\lambda^{\rho-1}$$

$$= \lambda^{o}Q_K(K,L),$$

so the marginal product Q_K is homogeneous of degree 0.

7. As $f_1(x_1, x_2)$ is homogeneous of degree *r*–1 we have by Euler's theorem that

$$x_1 f_{11} + x_2 f_{12} = (r-1) f_1$$

so

$$x_1^2 f_{11} + x_1 x_2 f_{12} = (r-1) x_1 f_1. \ (1)$$

Similarly as $f_2(x_1, x_2)$ is homogeneous of degree *r*–1 we have

$$x_2 x_1 f_{12} + x_2^2 f_{22} = (r-1) x_2 f_2 \ (2)$$

Adding (1) and (2) gives

$$x_1^2 f_{11} + 2x_1 x_2 f_{12} + x_2^2 f_{22} = (r-1)(x_1 f_1 + x_2 f_2) = r(r-1) f,$$

by Euler's theorem.

8. (i) A set $X \subset R^n$ is a convex set if $\mathbf{u}, \mathbf{v} \in X \Rightarrow \lambda \mathbf{u} + (1-\lambda)\mathbf{v} \in X,\ 0 \le \lambda \le 1.$

A function *f*(***x***) is convex if $\lambda f(\mathbf{u}) + (1-\lambda) f(\mathbf{v}) \ge f(\lambda \mathbf{u} + (1-\lambda)\mathbf{v})$ for all points **u**, **v** in the domain of the function.

$$f_1 = x_2,\ f_2 = x_1,\ f_{11} = 0,\ f_{12} = 1,\ f_{22} = 0$$

so the Hessian matrix is

$$H(\boldsymbol{x}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Principal minors are 0, 0, –1, so the function is neither convex nor concave.

(ii) $f_x = 4x^3 + 6xy^2,\ f_y = 4y^3 + 6x^2 y$

$f_{xx} = 12x^2 + 6y^2,\ f_{xy} = 12xy,\ f_{yy} = 12y^2 + 6x^2$

so the Hessian matrix is

$$H(x, y) = \begin{pmatrix} 12x^2 + 6x^2 & 12xy \\ 12xy & 12y^2 + 6y^2 \end{pmatrix}.$$

The principal minors are $12x^2 + 6y^2, 12y^2 + 6x^2$ and

$$(12x^2 + 6x^2)(12y^2 + 6x^2) - 144x^2y^2 = 72x^4 + 72y^4 + 36x^2y^2$$

so these principal minors are ≥ 0 on R^2. Thus the Hessian matrix is positive semidefinite and the function is convex.

(iii) $f_x = -6x + 2y + 3, \quad f_y = 2x - 2y - 4, \quad f_{xx} = -6, \; f_{xy} = 2 \;\; f_{yy} = -2$
so the Hessian matrix is

$$H = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}.$$

The principal minors are –6, –2, 8 so the matrix is negative semidefinite and the function is concave.

(iv) $f_x = e^{x+y} + e^{x-y} - \frac{3}{2}, \quad f_y = e^{x+y} - e^{x-y} - \frac{1}{2},$

$f_{xx} = e^{x+y} + e^{x-y}, \quad f_{xy} = e^{x+y} - e^{x-y}, \quad f_{yy} = e^{x+y} + e^{x-y}$

so the Hessian matrix is

$$H(x,y) = \begin{pmatrix} e^{x+y} + e^{x-y} & e^{x+y} - e^{x-y} \\ e^{x+y} - e^{x-y} & e^{x+y} + e^{x-y} \end{pmatrix}.$$

The principal minors of this matrix are $e^{x+y} + e^{x-y}$ and
$\left(e^{x+y} + e^{x-y}\right)^2 - \left(e^{x+y} - e^{x-y}\right)^2 = 4e^{2x}$ which are all greater than zero on R^2 and thus H is positive semidefinite. The function is (strictly) convex.

(v) $f_x = 2 - 2x - 2y, \quad f_y = -1 - 2x - 2y, \quad f_{xx} = -2, \; f_{xy} = -2, \; f_{yy} = -2,$
so the Hessian matrix is

$$H = \begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix}.$$

The first order principal minors are –2 and –2 and the second order principal minor is 0 so the matrix is negative semidefinite and the function is concave.

(vi) $f_x = 1 - e^x - e^{x+y}, \quad f_y = 1 - e^{x+y}, \quad f_{xx} = -e^x - e^{x+y}, \quad f_{xy} = -e^{x+y}, \quad f_{yy} = -e^{x+y}.$
Thus the Hessian matrix is

$$H(x,y) = \begin{pmatrix} -e^x - e^{x+y} & -e^{x+y} \\ -e^{x+y} & -e^{x+y} \end{pmatrix}.$$

The first order principal minors are $-e^x - e^{x+y}$ and $-e^{x+y}$ which are both < 0 on R^2 and the second order principal minor is

$$e^{x+y}\left(e^x + e^{x+y}\right) - e^{2(x+y)} = e^x e^{x+y}$$

which is > 0 on R^2 thus the Hessian matrix is negative definite on R^2 and the function is concave.

9. $f_x = 2x - y - 3x^2, \quad f_y = -2y - x, \quad f_{xx} = 2 - 6x, \quad f_{xy} = -1, \quad f_{yy} = -2$
so the Hessian matrix is

$$H(x,y) = \begin{pmatrix} 2 - 6x & -1 \\ -1 & -2 \end{pmatrix}.$$

For the function to be concave we want this matrix to be negative semidefinite. Thus we require all the first order principal minors to be ≤ 0,

that is, $2-6x \le 0 \Rightarrow x \ge \frac{1}{3}$.

We also require that the second-order principal minor be ≥ 0,

that is, $-4+12x-1 \ge 0 \Rightarrow x \ge 7/12$.

Hence the largest convex domain in E^2 for which the function is concave is

$$D = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle/ x \ge 7/12 \right\}.$$

10. (i) $f_x = -6x+2y+3, \quad f_y = 2x-2y-4, \quad f_{xx} = -6, \quad f_{xy} = 2, \quad f_{yy} = -2$

so the Hessian matrix is

$$H = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}.$$

Leading principal minors are –6 and $|H| = 8$, so the matrix is negative definite on R^2 and the function is strictly concave.

(ii) $f_x = 4x^3+2xy^2-3, \quad f_y = 2x^2y+4y^3-8,$

$f_{xx} = 12x^2+2y^2, \quad f_{xy} = 4xy, \quad f_{yy} = 2x^2+2y^2$

so the Hessian matrix is

$$H(x,y) = \begin{pmatrix} 12x^2+2y^2 & 4xy \\ 4xy & 2x^2+12y^2 \end{pmatrix}.$$

The first order leading principal minor is $12x^2+2y^2$ which is > 0 on the set. The second order leading principal minor is

$$\left(12x^2+2y^2\right)\left(2x^2+12y^2\right)-16x^2y^2 = 24x^4+132x^2y^2+24y^4$$

which is also > 0 on the set. Hence the Hessian matrix is positive definite on the set and the function is strictly convex on the set.

(iii) $f_x = \frac{1}{4}x^{-\frac{3}{4}}y^{\frac{1}{4}}, \quad f_x = \frac{1}{4}x^{\frac{1}{4}}y^{-\frac{3}{4}},$

$f_{xx} = -\frac{3}{16}x^{-\frac{7}{4}}y^{\frac{1}{4}}, \quad f_{xy} = \frac{1}{16}x^{-\frac{3}{4}}y^{-\frac{3}{4}}, \quad f_{yy} = -\frac{3}{16}x^{\frac{1}{4}}y^{-\frac{7}{4}}$

so the Hessian matrix is

$$H(x,y) = \frac{1}{16}\begin{pmatrix} -3x^{-\frac{7}{4}}y^{\frac{1}{4}} & x^{-\frac{3}{4}}y^{-\frac{3}{4}} \\ x^{-\frac{3}{4}}y^{-\frac{3}{4}} & -3x^{\frac{1}{4}}y^{-\frac{7}{4}} \end{pmatrix}.$$

The first order leading principal minor is $-\frac{3}{16}x^{-\frac{5}{4}}y^{\frac{1}{4}}$ which is < 0 on the positive orthant whereas the second order principal minor is

$$\frac{9}{256}x^{-\frac{3}{2}}y^{-\frac{3}{2}}-\frac{1}{256}x^{-\frac{3}{2}}y^{-\frac{3}{2}}=\frac{1}{32}x^{-\frac{3}{2}}y^{-\frac{3}{2}}$$

which is > 0 on the positive orthant so the Hessian matrix is negative definite on the positive orthant and the function is strictly concave on this set.

(iv) $f_x = 3e^x,\quad f_y = 20y^3,\quad f_z = -1/z,$

$$f_{xx} = 3e^x,\quad f_{xy} = 0,\quad f_{xz} = 0,\quad f_{yy} = 60y^2,\quad f_{yz} = 0,\quad f_{22} = 1/z^2,$$

so the Hessian matrix is

$$H(x,y,z)=\begin{pmatrix} 3e^x & 0 & 0 \\ 0 & 60y^2 & 0 \\ 0 & 0 & 1/z^2 \end{pmatrix}.$$

The first-order principal minor is $3e^x$ which is positive, on the positive orthant, the second order principal minor is $180e^x y^2$ which is positive on the positive orthant, and the third order principal minor is $180e^x y^2/z^2$ which is also positive on the set. So the Hessian matrix is positive definite on the positive orthant and the function is strictly convex on that set.

11. $Q_K = aAK^{a-1}L^b,\quad Q_L = bAK^aL^{b-1},$

$$Q_{KK} = a(a-1)AK^{a-2}L^b,\quad Q_{KL} = abAK^{a-1}L^{b-1},\quad Q_{LL} = b(b-1)AK^aL^{b-2}$$

so the Hessian matrix is

$$H(K,L)=\begin{pmatrix} a(a-1)AK^{a-2}L^b & abAK^{a-1}L^{b-1} \\ abAK^{a-1}L^{b-1} & b(b-1)AK^aL^{b-2} \end{pmatrix}$$

(i) For the function to be concave we want the first-order principal minors to be ≤ 0 and the second order principal minor to be ≥ 0. Consider the first order principal minor

$$a(a-1)AK^{-2}L^b$$

which is ≤ 0 on the positive orthant if $Aa(a-1)\le 0$. With $A > 0$ this requires that $a \ge 0$ and $a \le 1$. Similarly we require $b \ge 0$ and $b \le 1$.

The second-order principal minor is

$$\left(AK^{a-1}L^{b-1}\right)^2\left[ab(a-1)(b-1)-a^2b^2\right]=\left(AK^{a-1}L^{b-1}\right)^2\left[ab(1-a-b)\right].$$

With $A > 0$, $a \ge 0, b \ge 0$ for this to be ≥ 0 on the positive orthant implies that $a+b \le 1$. Moreover $a+b \le 1$ with $a \ge 0$ and $b \ge 0$ implies that both a and b are ≤ 1. Hence the function is concave on the positive orthant if $A > 0$, $a \ge 0, b \ge 0$ and $a + b \le 1$.

(ii) For the function to be strictly concave, the first order leading principal minor of the Hessian matrix should be < 0 which requires that
$Aa(a-1)<0$

so with $A > 0$ we need $a > 0$ and $a < 1$. The second order leading principal minor must also be > 0 which requires that $ab\left[1-(a+b)\right]>0$. With $a > 0$ this implies that $b > 0$ and $a + b < 1$ as required.

12. As $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$ are convex functions

$$f(\lambda\mathbf{u}+(1-\lambda)\mathbf{v})\le\lambda f(\mathbf{u})+(1-\lambda)f(\mathbf{v})$$

$$g(\lambda\mathbf{u}+(1-\lambda)\mathbf{v})\le\lambda g(\mathbf{u})+(1-\lambda)g(\mathbf{v}),$$

for $0\le\lambda\le 1$.

Let $h(\mathbf{x})=af(\mathbf{x})+bg(\mathbf{x})$ and consider

$$\begin{aligned} h(\lambda\mathbf{u}+(1-\lambda)\mathbf{v}) &= af(\lambda\mathbf{u}+(1-\lambda)\mathbf{v})+bg(\lambda\mathbf{u}+(1-\lambda)\mathbf{v}) \\ &\le a(\lambda f(\mathbf{u})+(1-\lambda)f(\mathbf{v}))+b(\lambda g(\mathbf{u})+(1-\lambda)g(\mathbf{v})) \\ &= \lambda h(\mathbf{u})+(1-\lambda)h(\mathbf{v}), \end{aligned}$$

so $h(\boldsymbol{x})$ is convex.

13. Let $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ belong to $S^{\le}$ so $f(\boldsymbol{x}_1)\le k$ and $f(\boldsymbol{x}_2)\le k$.
We wish to prove that $\lambda\mathbf{x}_1+(1-\lambda)\mathbf{x}_2\in S^{\le}$ for $0\le\lambda\le 1$. Now as $f(\boldsymbol{x})$ is convex for $0\le\lambda\le 1$,

$$\begin{aligned} f(\lambda\boldsymbol{x}_1+(1-\lambda)\boldsymbol{x}_2) &\le \lambda f(\boldsymbol{x}_1)+(1-\lambda)f(\boldsymbol{x}_2) \\ &\le \lambda k+(1-\lambda)k \\ &= k. \end{aligned}$$

If $f(\boldsymbol{x})$ is a concave function then $S^{\le}=\{\boldsymbol{x}/f(\boldsymbol{x})\ge k\}$ is a convex set.

5.4

1. Consider a nonlinear equation

$$F(y,x_1,\ldots,x_n)=0. \tag{5.1}$$

The implicit function theorem addresses the question when it is possible to solve this equation for y as a differentiable function of $x_1,\ldots,x_n$.

In nonlinear economic models equilibrium conditions often give rise to an equation like (5.1). In this equation the y is the endogenous variable and $x_1,\ldots,x_n$ are the exogeneous variables. Before any comparative static analysis can be conducted we need to know the condition required to ensure that we can solve the equilibrium equation for *y* as a differentiable function of $x_1,\ldots,x_n$.

2. (i) Clearly the point satisfies the equation and the function on the left hand side has continuous partial derivatives. Consider

$$Fy = 4x^2 - 14xy = -10$$

at the point in question, so an implicit function exists in the neighbourhood of this point.

Differentiating both sides of the equation with respect to *x*, remembering that *y* can now be regarded as a function of *x* gives

$$3x^2 + 8xy + 4x^2\frac{dy}{dx} - 7y^2 - 14xy\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx}\left(4x^2 - 14xy\right) = 7y^2 - 3x^2 + 8xy$$

and

$$\frac{dy}{dx} = \frac{7y^2 - 3x^2 + 8xy}{4x^2 - 14xy}.$$

(ii) The point satisfies the equation and the function on the left hand side has continuous partial derivatives. Moreover

$$Fy = 8x + 4y^3 = 1$$

at the point, so an implicit function exists in the neighbourhood of this point. Differentiating both sides of the equation with respect to *x*, remembering *y* can be treated now as a function of *x* gives

$$2x + 8y + 8x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx}\left(4y^3 + 8x\right) = -\left(2x + 8y\right)$$

and

$$\frac{dy}{dx} = -\frac{2x + 8y}{4y^3 + 8x}.$$

3. The point satisfies the equation and the function of the equation has continuous partial derivatives. Consider

$$Fy = 2x_2 + 2y = 2$$

at the point. So such an implicit function exists. Differentiating both sides of the equation with respect to x_1 and x_2 gives

$$1 + 3x_2 + 2x_2\frac{\partial y}{\partial x_1} + 2y\frac{\partial y}{\partial x_1} = 0$$

so

$$\frac{\partial y}{\partial x_1} 2\left(x_2 + y\right) = -\left(1 + 3x_2\right)$$

and

$$\frac{\partial y}{\partial x_1} = -\frac{1 + 3x_2}{2\left(x_2 + y\right)} = -2$$

at the point in question.

Similarly

$$3x_1 + 2y + 2x_2 \frac{\partial y}{\partial x_2} + 2x_2 + 2y \frac{\partial y}{\partial x_2} = 0$$

giving

$$\frac{\partial y}{\partial x_2} = -\frac{3x_1 + 2y + 2x_2}{2\left(x_2 + y\right)} = -\frac{5}{2}$$

at the point.

4. Write the equilibrium condition as

$$Y - C\left(Y\right) - I\left(Y\right) + M\left(Y\right) - G - X = 0.$$

We assume the functions have continuous derivatives. Consider

$$F_Y = 1 - C' - I' + M'.$$

For a solution to exist we require

$$I - C' - I' + M' \neq 0.$$

Differentiating our equilibrium condition with respect to *G*, regarding *Y* as a function of *G* and *X* gives

$$\frac{\partial Y}{\partial G} - C' \frac{\partial Y}{\partial G} - I' \frac{\partial Y}{\partial G} + M' \frac{\partial Y}{\partial G} - 1 = 0$$

so

$$\frac{\partial Y}{\partial G} = \frac{1}{1 - C' - I' + M'}.$$

5. The point satisfies the equations and the functions have continuous partial derivatives. The Jacobian determinant is

$$|J| = \begin{vmatrix} 5y_1^4 & 1 \\ 2y_2 & 3y_2^2 \end{vmatrix} = 15y_1^4 y_2^2 - 2y_1 = -2$$

at the point so the implicit functions exist.

Differentiating both sides of the equations with respect to *x*, treating y_1 and y_2 as functions of *x*, we have

$$5y_1^4 \frac{dy_1}{dx} + \frac{dy_2}{dx} = 1$$

$$2y_1 \frac{dy_1}{dx} + 3y_2^2\, dy_2/dx = 2x.$$

By Cramer's rule

$$\frac{dy}{dx_1} = \frac{\begin{vmatrix} 1 & 1 \\ 2x & 3y_2^2 \end{vmatrix}}{\begin{vmatrix} 5y_1^4 & 1 \\ 2y_1 & 3y_2^2 \end{vmatrix}} = \frac{\left(3y_2^2 - 2x\right)}{\left(15y_1^4 y_2^2 - 2y_1\right)}$$

and

$$\frac{dy}{dx_2} = \frac{\begin{vmatrix} 5y_1^4 & 1 \\ 2y_1 & 3y_2^2 \end{vmatrix}}{\left(15y_1^4 y_2^2 - 2y_1\right)} = \left(15y_1^4 y_2^2 - 2y_1\right) \Big/ \left(15y_1^4 y_2^2 - 2y_1\right).$$

At the point in question

$$\frac{dy}{dx_1} = 1 \text{ and } \frac{dy}{dx_2} = 1.$$

6. (i) Write our equilibrium equations as

$$F^1\left(Y, r, G, M\right) = Y - C\left(Y\right) - I\left(r\right) - G = 0$$

$$F^2\left(Y, r, G, M\right) = L\left(r, Y\right) - M = 0.$$

If we assume that all functions have continuous derivatives, then it is possible to solve for *Y* and *r* as functions of *G* and *M* at points where

$$\left|J\right| = \begin{vmatrix} F_Y^1 & F_r^1 \\ F_Y^2 & F_r^2 \end{vmatrix} = \begin{vmatrix} 1 - C' & -I' \\ L_Y & L_r \end{vmatrix}$$

is nonzero, that is, the condition we require is

$$L_r\left(1 - C'\right) + I'L_Y \neq 0.$$

(ii) Differentiating both sides of equilibrium equations with respect to *M*, remembering that now we regard *Y* and *r* as functions of *G* and *M* we have

$$\frac{\partial Y}{\partial M} - C'\frac{\partial Y}{\partial M} - I'\frac{\partial r}{\partial M} = 0$$

$$L_r \frac{\partial r}{\partial M} + L_Y \frac{\partial Y}{\partial M} = 1$$

which in matrix notation is

$$\begin{pmatrix} 1-C' & -I' \\ L_Y & L_r \end{pmatrix} \begin{pmatrix} \dfrac{\partial Y}{\partial M} \\ \dfrac{\partial r}{\partial M} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Using Cramer's rule we have

$$\frac{\partial Y}{\partial M} = \frac{\begin{vmatrix} 0 & -I' \\ 1 & L_r \end{vmatrix}}{\begin{vmatrix} 1-C' & -I' \\ L_Y & L_r \end{vmatrix}} = \frac{I'}{L_r\left(L-C'\right)+I'L_Y}.$$

and

$$\frac{\partial r}{\partial M} = \frac{\begin{vmatrix} 1-C' & 0 \\ L_Y & 1 \end{vmatrix}}{\left(L_r\left(I-C'\right)+I'L_Y\right)} = \frac{1-C'}{L_r\left(1-C'\right)+I'L_Y}.$$

Using the apriori information we have on the signs of derivatives we obtain

$$\frac{\partial Y}{\partial M} = \frac{-\text{ve}}{\left(-\text{ve}\right)\left(+\text{ve}\right)+\left(-\text{ve}\right)\left(+\text{ve}\right)} = \frac{-\text{ve}}{-\text{ve}} = +\text{ve}.$$

So *M* and equilibrium *Y* move in the same direction,

$$\frac{\partial r}{\partial M} = \frac{+\text{ve}}{-\text{ve}} = -\text{ve}.$$

So *M* and equilibrium *r* move in opposite directions.

5.5

1. (i) For a function *f*(*x*) of a single variable the Taylor's approximation of order three is

$$f\left(x\right) \approx f\left(x_0\right) + f'\left(x_0\right)\left(x-x_0\right) + \frac{f''\left(x_0\right)}{2!}\left(x-x_0\right)^2 + \frac{f'''\left(x_0\right)}{3!}\left(x-x_0\right)^3$$

$$= f\left(x_0\right) + f'\left(x_0\right)dx + \frac{f''\left(x_0\right)}{2!}dx^2 + \frac{f'''\left(x_0\right)}{3!}dx^3$$

where

$$f\left(x_0\right) = x_0^{\frac{1}{2}}, \quad f'\left(x_0\right) = \frac{1}{2}x_0^{-\frac{1}{2}}, \quad f''\left(x_0\right) = -\frac{1}{4}x_0^{-\frac{3}{2}}, \quad f'''\left(x_0\right) = \frac{3}{8}x_0^{-\frac{5}{2}}.$$

Evaluating our approximation at $x_0 = 4$ and $dx = 0.05$ we have

$$f(4.05) = \sqrt{4.05} \approx 2 + \frac{1}{4}\left(0.05\right) - \frac{1}{64}\left(0.05\right)^2 + \frac{1}{512}\left(0.05\right)^3$$

$$= 2 + 0.0125 - 0.000039 + 0.0000002$$

$$\approx 2.012461.$$

(ii) (a) $f(x) = (x+1)^{\frac{1}{2}}$ $\qquad f(0) = 1$

$f'(x) = \frac{1}{2}(x+1)^{-\frac{1}{2}}$ $\qquad f'(0) = \frac{1}{2}$

$f''(x) = -\frac{1}{4}(x+1)^{-\frac{3}{2}}$ $\qquad f''(0) = -\frac{1}{4}$

$f'''(x) = \frac{3}{8}(x+1)^{-\frac{5}{2}}$ $\qquad f'''(0) = \frac{3}{8}$

so

$$(x+1)^{\frac{1}{2}} \approx 1 + \frac{1}{2}dx - \frac{1}{8}dx^2 + \frac{1}{16}dx^3.$$

For $dx = 0.2$ we have

$$(1.2)^{\frac{1}{2}} \approx 1 + \frac{1}{2}(0.2) - \frac{1}{8}(0.2)^2 + \frac{1}{16}(0.2)^3 = 1 + 0.1 - 0.005 + 0.0005 = 1.0955$$

(b) $f(x) = f'(x) = f''(x) = \mathrm{f}'''(\mathrm{x}) = \mathrm{e}^x$

so all these render 1 at $x = 0$, thus

$$e^x \approx 1 + dx + \frac{1}{2}dx^2 + \frac{1}{3!}dx^3$$

and

$$e^{0.2} \approx 1 + 0.2 + \frac{1}{2}(0.2)^2 + \frac{1}{6}(0.2)^3 = 1 + 0.2 + 0.02 + 0.00133 = 1.22133$$

(c) $f(x) = \log x$ $\qquad f(1) = 0$

$f'(x) = \frac{1}{x}$ $\qquad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$ $\qquad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(1) = 2$

so

$$\log x \approx 0 + dx - \frac{1}{2}dx^2 + \frac{1}{3}dx^3.$$

For dx=0.2 we have

$$\log 1.2 \approx 0.2 - \frac{1}{2}(0.2)^2 + \frac{1}{3}(0.2)^3 = 0.2 - 0.02 + 0.00266 = 0.18266.$$

2. (i) $f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)' d\boldsymbol{x} + \frac{1}{2} d\boldsymbol{x}' H(\boldsymbol{x}_0) d\boldsymbol{x},$

where $\nabla f(\boldsymbol{x})$ is the gradient vector of $f(\boldsymbol{x})$, $H(\boldsymbol{x})$ is the Hessian matrix of $f(\boldsymbol{x})$.

(ii) For this Cobb-Douglas function

$$f(1,1) = 1, \quad f_1 = \frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{3}{4}}, \quad f_2 = \frac{3}{4} x_1^{\frac{1}{4}} x_2^{-\frac{1}{4}},$$

so

$$\nabla f(1,1) = \begin{pmatrix} \frac{1}{4} \\ \frac{3}{4} \end{pmatrix}.$$

Moreover

$$f_{11} = -\frac{3}{16} x_1^{-\frac{7}{4}} x_2^{\frac{3}{4}}, \quad f_{12} = \frac{3}{16} x_1^{-\frac{3}{4}} x_x^{-\frac{1}{4}}, \quad f_{22} = -\frac{3}{16} x_1^{\frac{1}{4}} x_2^{-\frac{5}{4}},$$

so

$$H(1,1) = \frac{3}{16} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Thus

$$f(x_1, x_2) \approx 1 + \frac{1}{4} dx_1 + \frac{3}{4} dx_2 - \frac{3}{32} dx_1{}^2 + \frac{3}{16} dx_1 dx_2 - \frac{3}{32} dx_2{}^2.$$

In moving from the point $x_1^{\mathrm{o}} = 1,\ x_2^{\mathrm{o}} = 1$ to the point $x_1 = 1.1$ and $x_2 = 0.9$ $dx_1 = 0.1$ and $dx_2 = -0.1$. Thus we have

$$(1.1)^{\frac{1}{4}} (0.9)^{\frac{3}{4}} \approx 1 + \frac{1}{4}(0.1) + \frac{3}{4}(-0.1) - \frac{3}{32}(0.1)^2 + \frac{3}{16}(0.1)(-0.1)^2 - \frac{3}{32}(-0.1)^2$$
$$= 0.94625.$$

(iii) $dy = f_1 dx_1 + f_2 dx_2 = \frac{1}{4} x_1^{-\frac{3}{4}} x_2^{\frac{3}{4}} dx_1 + \frac{3}{4} x_1^{\frac{1}{4}} x_2^{-\frac{1}{4}} dx_2$

At the point $x_1^{\mathrm{o}} = 1,\ x_2^{\mathrm{o}} = 1$ with $dx_1 = 0.1$ and $dx_2 = -0.1$ we have

$$dy = \frac{1}{4}(0.1) + \frac{3}{4}(-0.1) = -0.05.$$

3. The total derivative is defined as

$$\frac{dy}{dx_1} = \frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2} \frac{dx_2}{dx_1}.$$

The two concepts differ when x_2 is itself a function of x_1. If this is the case dy/dx_1 is an approximation for the rate of change in *y* when x_1 changes by a small amount, taking into account as it does the direct effect of this change given by $\partial y/\partial x_1$ and the indirect effect through x_2 given by $\frac{\partial y}{\partial x_1}\frac{dx_2}{dx_1}$.

$$\frac{\partial y}{\partial x_1} = 3x_1^2, \quad \frac{\partial y}{\partial x_2} = 14x_2, \quad \frac{dx_2}{dx_1} = \frac{1}{x_1} \quad \text{so}$$
$$\frac{dy}{dx_1} = 3x_1^2 + 14x_2/x_1.$$

4. We know that
$$dy = f_1 dx_1 + f_2 dx_2.$$

Taking the differential of both sides remembering that f_1 and f_2 are functions of x_1 and x_2 we have
$$\begin{aligned} d^2y = d(dy) &= (f_{11}dx_1 + f_{12}dx_2)dx_1 + (f_{21}dx_1 + f_{22}dx_2)dx_2 \\ &= f_{11}(dx_1)^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \\ &= d\boldsymbol{x}'H(\boldsymbol{x})d\boldsymbol{x}. \end{aligned}$$

Exercises for Chapter 6

6.1

1. (i) Critical points where
$$f_1 = 3x_1^2 + 9x_2 = 0$$
$$f_2 = 3x_2^2 + 9x_1 = 0.$$

From the second equation we have $x_1 = x_2^2/3$. Substituting in the first equation gives
$$\frac{x_2^4}{3} + 9x_2 = 0 \Rightarrow x_2(x_2^3 + 27) = 0$$

so $x_2 = 0$ or $x_2 = -3$. The equation then has two critical points
$$\boldsymbol{x}_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \boldsymbol{x}_2^* = \begin{pmatrix} 3 \\ -3 \end{pmatrix}.$$

Now
$$f_{11} = 6x_1, \quad f_{12} = 0, \quad f_{22} = -6x_2$$

so the Hessian matrix is
$$H(x) = \begin{pmatrix} 6x_1 & 9 \\ 9 & -6x_2 \end{pmatrix}.$$

and

$$H\left(x_1^*\right)=\begin{pmatrix}0 & 9\\ 9 & 0\end{pmatrix}.$$

The first order principal minors of this matrix are 0, 0 and the second order principal minor is -81, so $\boldsymbol{x}_1^*$ is a saddle point. Also

$$H\left(\boldsymbol{x}_2^*\right)=\begin{pmatrix}18 & 9\\ 9 & -18\end{pmatrix}.$$

The first order leading principal minors of this matrix is 18 and the second order leading principal minor is 233 so the critical point is strict local minimum.

(ii) Critical points where

$$f_{\mathrm{x}}=-3x^2+y+1=0$$
$$f_{\mathrm{y}}=x+2y=0$$

so $x=-2y$ and

$$-12y^2+y+1=0\Rightarrow(3y-1)(4y+1)=0\Rightarrow y=\tfrac{1}{3}\ \text{or}\ y=-\tfrac{1}{4}.$$

The function then has two critical points

$$\boldsymbol{x}_1^*=\begin{pmatrix}-\frac{2}{3}\\ \frac{1}{3}\end{pmatrix},\quad \boldsymbol{x}_2^*=\begin{pmatrix}\frac{1}{2}\\ -\frac{1}{4}\end{pmatrix}$$

Now

$$f_{xx}=-6x,\quad f_{xy}=1,\quad f_{yy}=2$$

so the Hessian matrix is

$$H(\boldsymbol{x})=\begin{pmatrix}-6x & 1\\ 1 & 2\end{pmatrix}.$$

Now

$$H\left(\boldsymbol{x}_1^*\right)=\begin{pmatrix}4 & 1\\ 1 & 2\end{pmatrix}$$

whose leading principal minors are 4 and 7 so $H\left(\boldsymbol{x}_1^*\right)$ is positive definite and $\boldsymbol{x}_1^*$ is a strict local minimum. Also

$$H\left(\boldsymbol{x}_2^*\right)=\begin{pmatrix}-3 & 1\\ 1 & 2\end{pmatrix},$$

whose first order principal minors are -3 and 2 so $H\left(\boldsymbol{x}_2^*\right)$ is indefinite and the point is a saddle point.

(iii) Critical points where

$$f_x = 4x^3 + 2x - 6y = 0$$

$$f_y = -6x + 6y = 0$$

thus $x = y$ and

$$4x^3 - 4x = 0 \Rightarrow x\left(x^2 - 1\right) = 0$$

so $x = 0$ or $x = 1$ or $x = -1$. The function then has three critical points

$$\boldsymbol{x}_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \boldsymbol{x}_2^* = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ \boldsymbol{x}_3^* = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

The Hessian matrix is

$$H(\boldsymbol{x}) = \begin{pmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{pmatrix}$$

so

$$H\left(\boldsymbol{x}_1^*\right) = \begin{pmatrix} 2 & -6 \\ -6 & 6 \end{pmatrix}.$$

First order principal minors are 2, 6, whereas $\left|H\left(\boldsymbol{x}_1^*\right)\right| = -24$ so this matrix is indefinite and $\boldsymbol{x}_1^*$ is a saddle point. Also

$$H\left(\boldsymbol{x}_2^*\right) = \begin{pmatrix} 14 & -6 \\ -6 & 6 \end{pmatrix},$$

whose leading principal minors are 14 and 48 so this matrix is positive definite and $\boldsymbol{x}_2^*$ is a strict local minimum. Similarly x_3^* is a strict local minimum.

(iv) Critical points where

$$f_1 = 2x_1 - 3x_2 = 0$$
$$f_2 = 6x_2 - 3x_1 + 4x_3 = 0$$
$$f_3 = 4x_2 + 12x_3 = 0,$$

giving $x_2 = -3x_3,\ x_1 = \frac{3}{2}x_2 = \frac{14}{3}x_3$ which is only true if $x_1 = x_2 = x_3 = 0$. Thus the function has one critical point

$$\boldsymbol{x}^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The Hessian matrix is

$$H = \begin{pmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 0 & 4 & 12 \end{pmatrix}.$$

The leading principal minors of this matrix are

$$2, \quad \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 3, \quad |H| = \begin{vmatrix} 2 & -3 & 0 \\ -3 & 6 & 4 \\ 9 & -14 & 0 \end{vmatrix} = 4(-1)^{2+2}\begin{vmatrix} 2 & -3 \\ 9 & -14 \end{vmatrix} = 4$$

so the matrix is positive definite and the critical point is a strict local minimum.

(v) Critical points where

$$f_1 = x_3 + 2x_1 = 0$$
$$f_2 = -1 + x_3 + 2x_2 = 0$$
$$f_3 = x_1 + x_2 + 6x_3 = 0,$$

so $x_3 = -2x_1$, $x_2 = 11x_1$, and $x_1 = 1/20$. Thus the function has one critical point

$$\boldsymbol{x}^* = \frac{1}{20}\begin{pmatrix} 1 \\ 11 \\ -2 \end{pmatrix}.$$

The Hessian matrix is

$$H = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

which has leading principal minors

$$2, \quad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4, \quad |H| = \begin{vmatrix} 0 & -2 & -11 \\ 0 & 2 & 1 \\ 1 & 1 & 6 \end{vmatrix} = 1(-1)^{3+1}\begin{vmatrix} -2 & -11 \\ 2 & 1 \end{vmatrix} = 20.$$

All the leading principal minors are positive so *H* is positive definite and $\boldsymbol{x}^*$ is a strict local minimum.

(vi) (d) The first order conditions are

$$f_x = x + 6y + 6 = 0$$
$$f_y = 6x + 2y - 3z + 17 = 0$$
$$f_z = -3y + 8x - 2 = 0$$

and the first of these equations gives $x=-(3+3y)$ so substituting into the other equations gives

$$-16y-3z=1$$
$$-3y+8z=2,$$

Using Cramer's rule to solve these equations gives

$$y=\frac{\begin{vmatrix}1 & -3\\ 2 & 8\end{vmatrix}}{\begin{vmatrix}-16 & -3\\ -3 & 8\end{vmatrix}}=-14/137$$

$$z=\frac{\begin{vmatrix}-16 & 1\\ -3 & 2\end{vmatrix}}{137}=29/137$$

$$x=-3+42/137=-369/137,$$

and this is the one critical point the function has.

The Hessian matrix is

$$H=\begin{pmatrix}2 & 6 & 0\\ 6 & 2 & -3\\ 0 & -3 & 8\end{pmatrix},$$

which has leading principal minors of 2 and $\begin{vmatrix}2 & 6\\ 6 & 2\end{vmatrix}=-32,$ so *H* is indefinite and the critical point is a saddle point.

2. Critical points where

$$f_x=y^2+3x^2y-y=y\left(y+3x^2-1\right)=0 \qquad (1)$$

$$f_y=2xy+x^3-x=x\left(2y+x^2-1\right)=0. \qquad (2)$$

Clearly $y=0$ and $x=0$ define critical points. When $y=0$ we have from (2) that $x\left(x^2-1\right)=0$ which gives $x=0,\ x=1,$ or $x=-1$. When $x=0$ from (1) we get $y\left(y-1\right)=0$ which gives $y=0$ or $y=1$.

Alternatively critical points are defined by

$$y+3x^2-1=0$$
$$2y+x^2-1=0$$

which yields $5x^2=1$ so $x=\frac{1}{\sqrt{5}}$ or $-\frac{1}{\sqrt{5}}$ with $y=\frac{2}{5}$ in both cases.

All in all we have six critical points

$$\boldsymbol{x}_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix},\ \boldsymbol{x}_2^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ \boldsymbol{x}_3^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ \boldsymbol{x}_4^* = \begin{pmatrix} -1 \\ 0 \end{pmatrix},\ \boldsymbol{x}_5^* = \begin{pmatrix} 1/\sqrt{5} \\ 2/5 \end{pmatrix},\ \boldsymbol{x}_6^* = \begin{pmatrix} -1/\sqrt{5} \\ 2/5 \end{pmatrix}.$$

The Hessian matrix is

$$H(\boldsymbol{x}) = \begin{pmatrix} 6xy & 2y + 3x^2 - 1 \\ 2y + 3x^2 - 1 & 2x \end{pmatrix}.$$

$H(\boldsymbol{x}_1^*) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ which is indefinite so $\boldsymbol{x}_1^*$ is a saddle point.

$H(\boldsymbol{x}_2^*) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which is indefinite so $\boldsymbol{x}_2^*$ is a saddle point.

$H(\boldsymbol{x}_3^*) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ which is indefinite so $\boldsymbol{x}_3^*$ is a saddle point.

$H(\boldsymbol{x}_4^*) = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix}$ which is indefinite so $\boldsymbol{x}_4^*$ is a saddle point.

$$H(\boldsymbol{x}_5^*) = \begin{pmatrix} \frac{12}{5\sqrt{5}} & \frac{2}{5} \\ \frac{2}{5} & \frac{2}{\sqrt{5}} \end{pmatrix}.$$

The leading principal minors of this matrix are $12/5\sqrt{5}$ and 4/5 so $H(\boldsymbol{x}_5^*)$ is positive definite and $\boldsymbol{x}_5^*$ is a strict local minimum.

$$H(\boldsymbol{x}_6^*) = \begin{pmatrix} \frac{-12}{5\sqrt{5}} & \frac{2}{5} \\ \frac{2}{5} & -\frac{2}{\sqrt{5}} \end{pmatrix}.$$

The first leading principal minor is $-12/5\sqrt{5}$ and the second leading principal minor is 4/5 so $H(\boldsymbol{x}_6^*)$ is negative definite and $\boldsymbol{x}_6^*$ is a strict local maximum.

6.2

1. (i) Total profit is given by

$$\Pi = p\left(L^{\frac{1}{2}} + K^{\frac{1}{2}}\right) - wL - rK.$$

(ii) The critical point is obtained from

$$\Pi_L = \frac{1}{2}pL^{-\frac{1}{2}} - w = 0$$

$$\Pi_K = \frac{1}{2}pK^{-\frac{1}{2}} - r = 0,$$

so we have one critical point

$$L^* = p^2/4w^2,\ \ K^* = p^2/4r^2.$$

(iii) For a global maximum we require that the objective function Π is concave on the nonnegative orthant. The Hessian matrix of this function is

$$H(L,K) = -\frac{1}{4}\begin{pmatrix} pL^{-\frac{3}{2}} & 0 \\ 0 & pK^{-\frac{3}{2}} \end{pmatrix}.$$

The first order principal minors of this matrix are

$$-\frac{1}{4}pL^{-\frac{3}{2}} \text{ and } -\frac{1}{4}pK^{-\frac{3}{2}}$$

which are both ≤ 0 on the nonnegative orthant and the second order principal minor is

$$|H| = \frac{1}{16}p^2L^{-\frac{3}{2}}K^{-\frac{3}{2}}$$

which is ≥ 0 in the nonnegative orthant so *H*(*L*,*K*) is negative semidefinite on the domain of our function, and thus Π is concave on its domain. Any local maximum will then be a global maximum.

(iv) $L^*(\lambda p, \lambda w) = (\lambda p)^2 / 4(\lambda w)^2 = L^*(p,w),$ so the demand for labour is homogeneous of degree 0. Similarly for K^*.

(v) $\frac{\partial L^*}{\partial w} = -\frac{p^2}{2w^3} \Rightarrow \Delta L^* \approx -\frac{p^2}{2w^3}\Delta w.$ Similarly $\Delta K^* \approx -\frac{p^2}{2r^3}\Delta r.$

(vi) Substituting L^* and K^* back into the profit function gives the maximum profit function

$$M(p,w,r) = p\left(L^{*\frac{1}{2}} + K^{*\frac{1}{2}}\right) - wL^* - rL^* = \frac{p^2}{4w} + \frac{p^2}{4r}$$

$$\frac{\partial M}{\partial p} = \frac{p}{2w} + \frac{p}{2r} \Rightarrow \Delta M \approx \left(\frac{p}{2w} + \frac{p}{2r} \right) \Delta p$$

$$\frac{\partial M}{\partial w} = -\frac{p^2}{4w^2} \Rightarrow \Delta M \approx -\frac{p^2}{4w^2} \Delta w$$

$$\frac{\partial M}{\partial r} = -\frac{p^2}{4r^2} \Rightarrow \Delta M \approx -\frac{p^2}{4r^2} \Delta r.$$

2. (i) Total profit is given by

$$\Pi = 4pL^{\frac{1}{4}}K^{\frac{1}{2}} - wL - rK.$$

(ii) $\Pi_L = pL^{-\frac{3}{4}}K^{\frac{1}{2}} - w = 0,\ \Pi_K = 2pL^{\frac{1}{4}}K^{-\frac{1}{2}} - r = 0,$

$$\Rightarrow \frac{K}{2L} = \frac{w}{r} \Rightarrow K = \frac{2wL}{r}.$$

Substituting into the first order conditions gives

$$\sqrt{2}pL^{-\frac{1}{4}}w^{\frac{1}{2}}$$

$$\Rightarrow L^* = \frac{4p^4}{r^2w^2},\quad K^* = \frac{8p^4}{r^3w}.$$

(iii) For a global maximum want Π to be concave on its domain, the nonnegative orthant. The Hessian matrix is

$$H(L,K) = \mathrm{p}\begin{pmatrix} -\frac{3}{4}L^{-\frac{7}{4}}K^{\frac{1}{2}} & \frac{1}{2}L^{-\frac{3}{4}}K^{-\frac{1}{2}} \\ \frac{1}{2}L^{-\frac{3}{4}}K^{-\frac{1}{2}} & -L^{\frac{1}{4}}K^{-\frac{3}{2}} \end{pmatrix}.$$

The first order principal minors are

$$-\frac{3}{4}pL^{-\frac{7}{4}}K^{\frac{1}{2}} \text{ and } -L^{\frac{1}{4}}K^{-\frac{3}{2}}$$

which are ≤ 0 on the domain of Π. The second order principal minor is

$$|H| = \frac{1}{2}p^2L^{-\frac{3}{2}}K^{-1}$$

which is ≥ 0 on the domain. Hence the Hessian matrix is negative definite, Π is concave on its domain, and any local maximum is a global maximum.

(iv) $\mathrm{L}^*(\lambda p, \lambda r, \lambda w) = \dfrac{4(\lambda p)^4}{(\lambda r)^2(\lambda w)^2} = L^*(p,r,w)$, so L^* is homogeneous of degree zero.

Similarly for K^*.

(v) $\dfrac{\partial L^*}{\partial w} = -\dfrac{8p^4}{rw^3} \Rightarrow \Delta L^* \approx -\dfrac{8p^4}{rw^3}\Delta w$

$\dfrac{\partial K^*}{\partial r} = -\dfrac{24p^4}{r^4 w} \Rightarrow \Delta K^* \approx -\dfrac{24p^4}{r^4 w}\Delta r.$

(vi) The maximum total profit is formed by substituting L^* and K^* into Π. Thus the firm's profit function is

$$M(p,r,w) = 4pL^{*\frac{1}{4}}K^{*\frac{1}{2}} - wL^* - rK^* = 4p^4/r^2w$$

$$\frac{\partial M}{\partial p} = 16p^3/r^2w \Rightarrow \Delta M \approx 16p^3\Delta p/r^2w$$

$$\frac{\partial M}{\partial w} = -4p^4/r^2w^2 \Rightarrow \Delta M \approx -4p^4\Delta w/r^2w^2$$

$$\frac{\partial M}{\partial r} = -8p^4/r^3w \Rightarrow \Delta M \approx -8p^4\Delta r/r^3w.$$

6.3

1. The Lagrangian function is

$$Z = (x_1+1)(x_2+2) + \lambda(12 - 2x_1 - 4x_2).$$

The first order conditions are

$$Z_\lambda = 12 - 2x_1 - 4x_2 = 0$$
$$Z_1 = x_2 + 2 - 2\lambda = 0$$
$$Z_2 = x_1 + 1 - 4\lambda = 0$$
$$\Rightarrow x_1 = 2x_2 + 3$$
$$\Rightarrow x_2 = \frac{3}{4},\ x_1 = \frac{9}{2},\ \lambda = \frac{11}{8}.$$

The bordered Hessian matrix is

$$\bar{H} = \begin{pmatrix} 0 & 2 & 4 \\ 2 & 0 & 1 \\ 4 & 1 & 0 \end{pmatrix}.$$

$$|\bar{H}| = \begin{vmatrix} -8 & 0 & 4 \\ 2 & 0 & 1 \\ 4 & 1 & 0 \end{vmatrix} = 1(-1)^{3+2}\begin{vmatrix} -8 & 4 \\ 2 & 1 \end{vmatrix} = 16,$$

so we have a local maximum.

2. (i) The Lagrangian function is $Z = x_1x_2 + \lambda(Y - p_1x_1 - p_2x_2)$.

(ii) The first order conditions are

$$Z_\lambda = Y - p_1x_1 - p_2x_2 = 0$$
$$Z_1 = x_2 - \lambda p_1 = 0$$
$$Z_2 = x_1 - \lambda p_2 = 0.$$

(iii) The border Hessian matrix is

$$\bar{H} = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & 0 & 1 \\ p_2 & 1 & 0 \end{pmatrix}$$

$$|\bar{H}| = (-1)^{1+2} p_1 \begin{vmatrix} p_1 & p_2 \\ 1 & 0 \end{vmatrix} + (-1)^{1+3} p_2 \begin{vmatrix} p_1 & p_2 \\ 0 & 1 \end{vmatrix}$$
$$= 2p_1p_2 > 0,$$

so the second order condition holds for maximization.

(iv) The second two of the first order conditions imply that

$$x_2 = \frac{p_1}{p_2}x_1$$

so substituting into the first of these gives

$$x_1^* = Y/2p_1,\ \ x_2^* = Y/2p_2.$$

Also $x_1^*(\lambda p_1, \lambda Y) = \lambda Y / 2\lambda p_1 = x_1^*(p_1, Y)$ so x_1^* is homogeneous of degree 0. Similarly for x_2^*.

Suppose prices and income increase so they are λ times their original values. All that changes in our problem is the constraint which now becomes

$$\lambda p_1x_1 + \lambda p_2 x_2 = \lambda Y.$$

But this is the original constraint so nothing changes in the problem. Thus we would expect this result.

(v) $M(p_1, p_2, Y) = x_1^*x_2^* = Y^2/4p_1p_2$

(a) $\dfrac{\partial M}{\partial p_1} = -\dfrac{Y^2}{4p_1p_2} \Rightarrow \Delta M \approx -\dfrac{Y^2}{4p_1^2p_2}\Delta p_1$

(b) $\dfrac{\partial M}{\partial Y} = \dfrac{Y}{2p_1^2p_2} \Rightarrow \Delta M \approx \dfrac{Y}{2p_1p_2}\Delta Y.$

(vi) From the first order conditions we obtain

$$\lambda^* = x_1^* / p_2 = \frac{Y}{2p_1p_2} = \frac{\partial M}{\partial Y}.$$

3. (i) The problem is,

Minimize $wL + rK$

subject to $4K^{\frac{1}{4}}L^{\frac{1}{4}} = Q_o$,

so the Lagrangian function is

$$Z = wL + rK + \lambda\left(Q_o - 4K^{\frac{1}{4}}L^{\frac{1}{4}}\right).$$

(ii) The first order conditions are

$$Z_\lambda = Q_o - 4K^{\frac{1}{4}}L^{\frac{1}{4}} = 0$$

$$Z_L = w - \lambda K^{\frac{1}{4}}L^{-\frac{3}{4}} = 0$$

$$Z_K = r - \lambda K^{-\frac{3}{4}}L^{\frac{1}{4}} = 0.$$

(iii) The deriviatives needed to form the bordered Hessian matrix are

$$g_L = K^{\frac{1}{4}}L^{-\frac{3}{4}}, \qquad g_K = K^{-\frac{3}{4}}L^{\frac{1}{4}}$$

$$Z_{LL} = -\frac{3}{4}\lambda K^{\frac{1}{4}}L^{-\frac{7}{4}}, \qquad Z_{LK} = -\frac{1}{4}\lambda K^{-\frac{3}{4}}L^{-\frac{3}{4}}$$

$$Z_{KK} = -\frac{3}{4}\lambda K^{-\frac{7}{4}}L^{-\frac{1}{4}},$$

so the bordered Hessian matrix is

$$\bar{H}(L,K,\lambda) = \frac{1}{4}L^{-\frac{7}{4}}K^{-\frac{7}{4}}\begin{pmatrix} 0 & 4K^2L & 4KL^2 \\ 4K^2L & -3\lambda K^2 & -\lambda KL \\ 4KL^2 & -\lambda KL & -3\lambda L^2 \end{pmatrix}$$

.

As our problem is a minimization problem we require $\left|\bar{H}\right| < 0$ at the critical point obtained from the first order conditions.

(iv) From the last two first order conditions we have

$$\frac{w}{r} = \frac{K}{L} \Rightarrow K = \frac{wL}{r}.$$

Substituting into the first of these conditions gives

$$Q^o - 4\left(\frac{w}{r}\right)^{\frac{1}{4}}L^{\frac{1}{2}} = 0 \Rightarrow L^* = \left(\frac{Q_o}{4}\right)^2\left(\frac{r}{w}\right)^{\frac{1}{2}} \Rightarrow K^* = \left(\frac{Q_o}{4}\right)^2\left(\frac{w}{r}\right)^{\frac{1}{2}}.$$

(v) The cost function is

$$M\left(w,r,Q_{o}\right)=wL^{*}+rK^{*}=Q_{o}^{2}\left(rw\right)^{\frac{1}{2}}/8.$$

(vi) $\frac{\partial M}{\partial Q_{o}}=Q_{o}\left(rw\right)^{\frac{1}{2}}/4.$

(vii) From the first order completions

$$\lambda^{*}=rK^{*\frac{3}{4}}L^{*-\frac{1}{4}}=Q_{o}\left(rw\right)^{\frac{1}{2}}/4=\partial M/\partial Q_{o}$$

4. (i) The problem is,

Minimize $p_1x_1+p_2x_2$

subject to $x_1^a x_2^b=u_o$,

and the Lagrangian function associated with this problem is

$$Z=p_1x_1+p_2x_2+\lambda\left(u_o-x_1^a x_2^b\right)$$

The first order conditions are

$$Z_\lambda=u_o-x_1^a x_2^b=0 \quad (1)$$

$$Z_1=p_1-a\lambda x_1^{a-1}x_2^b=0 \quad (2)$$

$$Z_2=p_2-b\lambda x_1^a x_2^{b-1}=0 \quad (3)$$

(ii) $Z_{11}=-a\left(a-1\right)\lambda x_1^{a-2}x_2^b$

$Z_{12}=-ab\lambda x_1^{a-1}x_2^{b-1}$

$Z_{22}=-b\left(b-1\right)\lambda x_1^a x_2^{b-2}$

so the border Hessian matrix is

$$\bar{H}=x_1^{a-2}x_2^{b-2}\begin{pmatrix}0 & ax_1x_2^2 & bx_1^2x_2\\ ax_1x_2^2 & -a\left(a-1\right)\lambda x_2^2 & -ab\lambda x_1x_2\\ bx_1^2x_2 & -ab\lambda x_1x_2 & -b\left(b-1\right)\lambda x_1^2\end{pmatrix}.$$

(iii) As this is a minimization problem we require $\left|\bar{H}\right|<0$ at the critical point obtained from the first conditions.

(iv) From (2) and (3)

$$\frac{ax_2}{bx_1}=\frac{p_1}{p_2}\Rightarrow x_2=\frac{bp_1x_1}{ap_2}.$$

Substituting into (1) gives

$$u_o - x_1^{a+b}\left(\frac{bp_1}{ap_2}\right) = 0.$$

Solving for x_1 and using the fact that $a+b=1$ gives

$$\bar{x}_1 = u_o\left(\frac{ap_2}{bp_1}\right)^b .$$

By symmetry,

$$\bar{x}_2 = u_o\left(\frac{bp_1}{ap_2}\right)^a .$$

(v) Substituting $\bar{x}_1 \,\text{and}\, \bar{x}_2$ into the objective function gives

$$\begin{aligned} M &= p_1 u_o\left(\frac{ap_2}{bp_1}\right)^b + p_2 u_o\left(\frac{bp_1}{ap_2}\right)^a \\ &= p_1^a p_2^b u_o a^{1-a} b^{-b} + p_2^b p_1^a u_o a^{-a} b^{1-b} \\ &= p_1^a p_2^b u_o a^{-a} b^{-b}\left(a+b\right) \\ &= a^{-a} b^{-b} p_1^a p_2^b u_o, \end{aligned}$$

as required.

(iv) $\dfrac{\partial M}{\partial p_1} = a a^{-a} b^{-b} p_1^{a-1} p_2^b u_o,$

thus for small changes

$$\Delta M \approx a^b b^{-p} p_1^{-b} p_2^b u_o \Delta p_1.$$

6.4

The set of feasible solutions is a hyperplane as it is the set of points which satisfy a linear constraint so this set is a closed convex set. Consider the Hessian matrix of the objective function which is

$$H\left(x_1, x_2\right) = \begin{pmatrix} -\dfrac{1}{x_1^3} & 0 \\ 0 & -\dfrac{1}{x_2^3} \end{pmatrix}$$

The first leading principal minor is $-\frac{1}{x_1^3} < 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ and the second leading principal minor is $\frac{1}{x_1^3 x_2^3} > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$ so this matrix is negative definite and the objective function is strictly concave. Thus there will be a unique local maximum which is also a global maximum.

1. (i) The Lagrangian function is

$$Z = \frac{1}{x_1} - \frac{1}{x_2} + \lambda\left(Y - p_1 x_1 - p_2 x_2\right).$$

(ii) The first order conditions are

$$Z_\lambda = Y - p_1 x_1 - p_2 x_2 = 0$$

$$Z_1 = \frac{1}{x_1^2} - \lambda p_1 = 0$$

$$Z_2 = \frac{1}{x_2^2} - \lambda p_2 = 0.$$

(iii) The bordered Hessian matrix of this problem is

$$\bar{H}(x_1, x_2) = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & -\frac{2}{x_1^3} & 0 \\ p_2 & 0 & -2/x_2^3 \end{pmatrix}$$

$$\left|\bar{H}\right| = p_1(-1)^{2+1}\begin{vmatrix} p_1 & p_2 \\ 0 & -2/x_2^3 \end{vmatrix} + p_2(-1)^{3+1}\begin{vmatrix} p_1 & p_2 \\ -2/x_1^3 & 0 \end{vmatrix}$$

$$= \frac{2p_1^2}{x_2^3} + \frac{2p_2^2}{x_1^3} > 0,$$

for all positive x_1 and x_2. Thus the second order condition holds.

(iv) From the first order conditions we have

$$\frac{x_2^2}{x_1^2} = \frac{p_1}{p_2} \Rightarrow \mathrm{x}_2 = \sqrt{\frac{p_1}{p_2}}x_1.$$

Substituting gives

$$Y - p_1 x_1 - p_2\sqrt{\frac{p_1}{p_2}}x_1 = 0$$

$$\Rightarrow x_1^* = \frac{Y}{\left(p_1 + \sqrt{p_1 p_2}\right)}$$

$$x_2^* = \frac{Y}{\left(p_2 + \sqrt{p_1 p_2}\right)}.$$

Now

$$x_1^*\left(\lambda p_1, \lambda p_2, \lambda Y\right) = \lambda Y / \left(\lambda p_1 + \sqrt{\lambda^2 p_1 p_2}\right) = x_1^*\left(p_1, p_{2,} Y\right),$$

so x_1^* is homogeneous of degree zero. Similarly for x_2^*.

(v) $$M\left(p_1, p_2, Y\right) = -\frac{1}{x_1^*} - \frac{1}{x_2^*} = -\left(\sqrt{p_1} + \sqrt{p_2}\right)^2 / Y.$$

(a) $$\frac{\partial M}{\partial p_1} = -p_1^{-\frac{1}{2}}\left(\sqrt{p_1} + \sqrt{p_2}\right)/Y \Rightarrow \Delta M = -p_1^{-\frac{1}{2}}\left(\sqrt{p_1} + \sqrt{p_2}\right)\Delta p_1 / Y$$

(b) $$\frac{\partial M}{\partial Y} = \left(\sqrt{p_1} + \sqrt{p_2}\right)^2 / Y^2 \Rightarrow \Delta M = \left(\sqrt{p_1} + \sqrt{p_2}\right)^2 \Delta Y / Y^2.$$

(vi) From the first order conditions

$$\lambda^* = \frac{1}{p_1 x_1^{*2}} = \frac{\left(\sqrt{p_1} + \sqrt{p_2}\right)^2}{Y^2} = \frac{\partial M}{\partial Y}.$$

Exercises for Chapter 7

7.2

1. Total profit is

$$\Pi = p\left(L^{\frac{1}{2}} + K^{\frac{1}{2}}\right) - wL - rK.$$

Let *M* be the maximum profit function. By the Envelope Theorem

(i) $$\frac{\partial M}{\partial w} = \left.\frac{\partial \Pi}{\partial w}\right|_{L^*} = -L^* = -p^2 / 4w^2$$

(ii) $$\frac{\partial M}{\partial r} = \left.\frac{\partial \Pi}{\partial r}\right|_{K^*} = -K^* = -p^2 / 4r^2$$

(iii) $$\frac{\partial M}{\partial p} = \left.\frac{\partial \Pi}{\partial p}\right|_{L^*, K^*} = -L^{*\frac{1}{2}} + K^{*\frac{1}{2}} = \frac{p}{2w} + \frac{p}{2r}$$

which are the same results we got in exercise 1 of 6.2.

2. Total profit is

$$\Pi = 4pL^{\frac{1}{4}}K^{\frac{1}{2}} - wL - rK.$$

Let *M* be the maximum profit function. By the Envelope Theorem

(i) $$\frac{\partial M}{\partial w} = \left.\frac{\partial \Pi}{\partial w}\right|_{L^*} = -L^* = -\frac{4p^4}{r^2 w^2}$$

(ii) $$\frac{\partial M}{\partial r} = \left.\frac{\partial \Pi}{\partial r}\right|_{K^*} = -K^* = -\frac{8p^4}{r^3 w}$$

(iii) $$\frac{\partial M}{\partial p} = \left.\frac{\partial \Pi}{\partial p}\right|_{L^*,K^*} = 4L^{*\frac{1}{4}}K^{*\frac{1}{2}} = 16p^3 / r^3 w,$$

which are the same results we got in exercise 2 of 6.2.

7.3

1. (i) The Lagrangian function is
$$Z = x_1 x_2 + \lambda\left(Y - p_1 x_1 - p_2 x_2\right).$$

Let *M* be the indirect utility function.

Then by the Envelope Theorem

$$\frac{\partial M}{\partial p_1} = \left.\frac{\partial Z}{\partial p_1}\right|_{\lambda^*, x_1^*} = -\lambda^* x_1^* = -\frac{Y^2}{4p_1^2 p_2},$$

$$\frac{\partial M}{\partial p_2} = \left.\frac{\partial Z}{\partial p_2}\right|_{\lambda^*, x_2^*} = -\lambda^* x_2^* = -\frac{Y^2}{4p_1^2 p_2},$$

$$\frac{\partial M}{\partial Y} = \left.\frac{\partial Z}{\partial Y}\right|_{\lambda^*} = \lambda^* = \frac{Y}{2p_1 p_2},$$

which are the same results we got in exercise 2 of 6.3.

(ii) The Lagrangian function is
$$Z = -\frac{1}{x_1} - \frac{1}{x_2} + \lambda\left(Y - p_1 x_1 - p_2 x_2\right).$$

If *M* is the indirect utility function, then by the Envelope Theorem
$$\frac{\partial M}{\partial p_1} = \left.\frac{\partial Z}{\partial p_1}\right|_{\lambda^*, x_1^*} = -\lambda^* x_1^* = -\frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_1}\,Y},$$

$$\frac{\partial M}{\partial p_2} = \left.\frac{\partial Z}{\partial p_2}\right|_{\lambda^*, x_2^*} = -\lambda^* x_2^* = -\frac{\sqrt{p_1} + \sqrt{p_2}}{\sqrt{p_2} Y},$$

$$\frac{\partial M}{\partial Y} = \left.\frac{\partial Z}{\partial Y}\right|_{\lambda^*} = \lambda^* = \frac{\left(\sqrt{p_1} + \sqrt{p_2}\right)^2}{Y^2},$$

which are the same results as we obtained for exercise 6.4.

2. The Lagrangian function is

$$Z = wL + rK + \lambda\left(Q_o - 4K^{\frac{1}{4}}L^{\frac{1}{4}}\right).$$

If *M* is the cost function then by the Envelope Theorem

$$\frac{\partial M}{\partial w} = \left.\frac{\partial Z}{\partial w}\right|_{L^*} = L^* = \left(\frac{Q_o}{4}\right)^2 \left(\frac{r}{w}\right)^{\frac{1}{2}},$$

$$\frac{\partial M}{\partial r} = \left.\frac{\partial Z}{\partial r}\right|_{K^*} = K^* = \left(\frac{Q_o}{4}\right)^2 \left(\frac{w}{r}\right)^{\frac{1}{2}},$$

$$\frac{\partial M}{\partial Q_o} = \left.\frac{\partial Z}{\partial Q_o}\right|_{\lambda^*} = \lambda^* = \frac{Q_o (rw)^{\frac{1}{2}}}{4}.$$

3. The Lagrangian function is

$$Z = p_1 x_1 + p_2 x_2 + \lambda\left(u_o - x_1^a x_2^b\right).$$

Let *M* be the minimum value function. Then by the Envelope Theorem

$$\frac{\partial M}{\partial p_2} = \left.\frac{\partial Z}{\partial p_2}\right|_{\bar{x}_2^*} = \bar{x}_2^* = u_o \left(\frac{bp_1}{ap_2}\right)^a,$$

$$\frac{\partial M}{\partial u_o} = \left.\frac{\partial Z}{\partial u_o}\right|_{\lambda} = \bar{\lambda} = a^{-a} b^{-b} p_1^a p_2^b.$$

7.4

1. (i) We write the Lagrangian function as

$$Z = U\left(x_1, x_2\right) + \lambda\left(p_2 x_2 + p_2 x_2 - Y\right).$$

Then the first order conditions are

$$Z_1 = U_1 + \lambda p_1 = 0$$
$$Z_2 = U_2 + \lambda p_2 = 0$$
$$Z_\lambda = p_1 x_1 + p_2 x_2 - Y = 0.$$

(ii) The bordered Hessian matrix is

$$\bar{H}\left(x_1, x_2\right) = \begin{pmatrix} 0 & p_1 & p_2 \\ p_1 & U_{11} & U_{12} \\ p_2 & U_{21} & U_{22} \end{pmatrix}$$

and the second order condition is

$$\left|\bar{H}\right| > 0$$

at the critical point obtained from the first order conditions.

(iii) We have

$$p_1 dx_1 + x_1 dp_1 + p_2 dx_2 + x_2 dp_2 - d\mathrm{Y} = 0$$
$$U_{11} dx_1 + U_{12} dx_2 + \lambda dp_1 + p_1 d\lambda = 0$$
$$U_{21} dx_1 + U_{22} dx_2 + \lambda dp_2 + p_2 d\lambda = 0.$$

Isolating the differentials of the exogenous variables on the right hand side we have

$$p_1 dx_1 + p_2 dx_2 = dY - x_1 dp_1 - x_2 dp_2$$
$$U_{11} dx_1 + U_{12} dx_2 + p_1 d\lambda = -\lambda dp_1$$
$$U_{21} dx_1 + U_{22} dx_2 + p_2 d\lambda = -\lambda dp_2$$

or in matrix notation

$$\bar{H}\begin{pmatrix} d\lambda \\ dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} dY - x_1 dp_1 - x_2 dp_2 \\ -\lambda dp_1 \\ -\lambda dp_2 \end{pmatrix}. \qquad (1)$$

(iv) In (1) let $dp_1 = dY = 0$

Then we obtain

$$\bar{H}\begin{pmatrix} d\lambda \\ dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} -x_2 dp_2 \\ 0 \\ -\lambda dp_2 \end{pmatrix}$$

so using Cramer's rule to solve for dx_1 we get

$$dx_1 = \frac{\begin{vmatrix} 0 & -x_2 dp_2 & p_2 \\ p_1 & 0 & U_{12} \\ p_2 & -\lambda dp_2 & U_{22} \end{vmatrix}}{|\bar{H}|}$$

Expanding the determinant in the numerator using the second column gives

$$dx_1 = -x_2 dp_2 \bar{H}_{12} - \lambda dp_2 \frac{\bar{H}_{32}}{|\bar{H}|},$$

where $\bar{H}_{12}$ and $\bar{H}_{32}$ are the cofactors of the (1, 2) and (3, 2) elements of $\bar{H}$.

Dividing both sides of this equation by dp_2, and remembering that the ratio of differentials is a derivative we have

$$\frac{\partial x_1}{\partial p_2} = -x_2 \frac{\bar{H}_{12}}{|\bar{H}|} - \lambda \frac{\bar{H}_{32}}{|\bar{H}|}. \tag{2}$$

(v) Let $dp_1 = dp_2 = 0$ in (1) to obtain

$$\bar{H}\begin{pmatrix} d\lambda \\ dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} dY \\ 0 \\ 0 \end{pmatrix}$$

and using Cramer's rule to solve for dx_1 we have

$$dx_1 = \frac{\begin{vmatrix} 0 & dY & p_2 \\ p_1 & 0 & U_{12} \\ p_2 & 0 & U_{22} \end{vmatrix}}{|\bar{H}| = \frac{dY\bar{H}_{12}}{|\bar{H}|}}.$$

Again dividing through both sides of this equation by *dY* gives

$$\left(\frac{\partial x_1}{\partial Y}\right)_{\substack{\text{Prices} \\ \text{constant}}} = \frac{\bar{H}_{12}}{|\bar{H}|}. \tag{3}$$

(vi) If utility remains constant then

$$dU = U_1 dx_1 + U_2 dx_2 = 0$$

or

$$\frac{U_1}{U_2} dx_1 + dx_2 = 0.$$

But from the first order conditions

$$\frac{U_1}{U_2} = \frac{p_1}{p_2}$$

so holding utility constant requires

$$p_1 dx_1 + p_2 dx_2 = 0.$$

But taking the differential of both sides of the budget constraint gives

$$p_1 dx_1 + x_2 dp_1 + p_2 dx_2 + x_2 dp_2 = dY$$

or

$$p_1 dx_1 + p_2 dx_2 = dY - x_1 dp_1 - x_2 dp_2.$$

Thus holding utility constant requires that

$$dY - x_1 dp_1 - x_2 dp_2 = 0. \qquad (4)$$

(vii) Suppose that Y and p_2 change in such a way that utility remains constant and suppose further p_1 does not change.

Then we must have

$$dY - x_1 dp_1 - x_2 dp_2 = 0$$
$$dp_1 = 0.$$

Substitute these conditions in (1) gives

$$\bar{H}\begin{pmatrix} d\lambda \\ dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\lambda dp_2 \end{pmatrix},$$

and by Cramer's rule

$$dx_1 = \frac{\begin{vmatrix} 0 & 0 & p_2 \\ p_1 & 0 & U_{12} \\ p_2 & -\lambda dp_2 & U_{22} \end{vmatrix}}{|\bar{H}|} = -\frac{dp_2 \bar{H}_{32}}{|\bar{H}|}$$

so

$$\left(\frac{\partial x_1}{\partial p_2}\right)_{\substack{\text{Utility}\\ \text{constant}}} = \frac{-\lambda \bar{H}_{32}}{|\bar{H}|} \qquad . \qquad (5)$$

Substituting (3) and (5) into (2) gives Slutsky's equation

$$\frac{\partial x_1}{\partial p_1} = \left(\frac{\partial x_1}{\partial p_1}\right)_{\substack{\text{Utility}\\ \text{constant}}} - x_2\left(\frac{\partial x_1}{\partial Y}\right)_{\substack{\text{Prices}\\ \text{constant.}}}$$

(vii) The substitution effect of a change in p_2 on good 1 is

$$\left(\frac{\partial x_1}{\partial p_2}\right)_{\substack{\text{Utility}\\ \text{constant}}} = \frac{-\lambda \bar{H}_{32}}{|\bar{H}|}.$$

Clearly the substitution effect of a change in p_1 on good 2 is

$$\left(\frac{\partial x_2}{\partial p_1}\right)_{\substack{\text{Utility}\\ \text{constant}}} = \frac{-\lambda \bar{H}_{23}}{|\bar{H}|}.$$

But as the bordered Hessian matrix $\bar{H}$ is symmetric, $\bar{H}_{23} = \bar{H}_{23}$ and the substitution effects are the same.

2. (i) Multiply both sides of (7.7) by p_2 / x_x we have

$$\left(\frac{\partial x_1}{\partial p_2}\right)\frac{p_2}{x_1} = \left(\frac{\partial x_1}{\partial p_2}\right)_{\substack{\text{Utility}\\ \text{constant}}}\frac{p_2}{x_1} - x_2\left(\frac{\partial x_1}{\partial Y}\right)_{\substack{\text{Prices}\\ \text{constant}}}\frac{p_2}{x_1}.$$

Now we can write

$$x_2\left(\frac{\partial x_1}{\partial Y}\right)_{\substack{\text{Prices}\\ \text{constant}}}\frac{p_2}{x_1} = \frac{p_2 x_2}{Y}\left(\frac{\partial x_1}{\partial Y}\right)_{\substack{\text{Prices}\\ \text{constant}}}\frac{Y}{x_1} = \alpha_2 \eta_1$$

where

$\alpha_2 =$ the proportion of income Y spent on good 2

$\eta_1 =$ income elasticity of demand for good 1.

Thus in terms of elasticities we can write Slutsky's equation (7.7) as

$$e_{12} = \varepsilon_{12} - \alpha_2 \eta_2$$

where e_{12} is the cross elasticity of demand.

(ii) $$\varepsilon_{11}+\varepsilon_{12}=\left(\frac{\partial x_1}{\partial p_1}\right)_{\substack{\text{Utility}\\ \text{constant}}}\frac{p_1}{x_1}+\left(\frac{\partial x_2}{\partial p_1}\right)_{\substack{\text{Utility}\\ \text{constant}}}\frac{p_1}{x_2}$$
$$=\frac{-\lambda}{|\bar{H}|x_1}\left(p_1\bar{H}_{22}+p\bar{H}_{32}\right).$$

But the term in the bracket is obtained by multiplying elements of the first column of $\bar{H}$ by the corresponding cofactors of the second column of $\bar{H}$ so by our theorem this term is zero and thus
$$\varepsilon_{11}+\varepsilon_{12}=0.$$

7.5

1. (i) The Marshallian demand functions are the optimal point of the following problem:
$$\begin{aligned}&\text{max imize} && x_1^{\alpha}x_2^{\beta}\\ &\text{subject to} && p_1x_1+p_2x_2=Y\end{aligned}$$

The Lagrangian function associated with this problem is
$$Z=x_1^{\alpha}x_2^{\beta}+\lambda\left(Y-p_1x_1-p_2x_2\right)$$

and the first order condition are
$$Z_1=\alpha x_1^{\alpha-1}x_2^{\beta}-\lambda p_1=0$$
$$Z_2=\beta x_2^{\beta-1}x^{\alpha}-\lambda p_2=0$$
$$Z_{\lambda}=Y-p_1x_1-p_2x_2=0.$$

From the first two equations we obtain
$$\frac{\alpha x_2}{\beta x_1}=\frac{p_1}{p_2}\Rightarrow x_2=\frac{\beta x_1p_1}{\alpha p_2}.$$

Substituting into the third equation gives
$$x_1^*=\frac{\alpha Y}{(\alpha+\beta)p_1}.$$

From symmetry
$$x_2^*=\frac{\beta Y}{(\alpha+\beta)p_2}.$$

These are the Marshallian demand functions (we assume the second order conditions hold).

(ii) The indirect utility function is the maximum value function
$$V(\boldsymbol{p},Y)=x_1^{*\alpha}x_2^{*\beta}=\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{\mathrm{Y}}{\alpha+\beta}\right)^{\alpha+\beta}.$$

(iii) From the consistency properties

$$V(\mathbf{p}, e) = u,$$

so

$$\left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{\beta}{p_2}\right)^{\beta}\left(\frac{e}{\alpha+\beta}\right)^{\alpha+\beta} = u \;\Rightarrow\; e = (\alpha+\beta)\left[u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}.$$

(iv) By Shephard's Lemma the Hicksian demand function $\bar{x}_1$ is given by $\bar{x}_1 = \dfrac{\partial e}{\partial p_1}$, so

$$\begin{aligned}
\bar{x}_1 &= \left[u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right]^{\frac{1-\alpha-\beta}{\alpha+\beta}} u\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_1}{\alpha}\right)^{\alpha-1} \\
&= \left[u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right]^{\frac{1-\alpha-\beta}{\alpha+\beta}} u\left(\frac{p_2}{\beta}\right)^{\beta}\left(\frac{p_1}{\alpha}\right)^{\alpha}\frac{\alpha}{\mathrm{p}_1} \\
&= \left[u\left(\frac{p_1}{\alpha}\right)^{\alpha}\left(\frac{p_2}{\beta}\right)^{\beta}\right]^{\frac{1}{\alpha+\beta}}\frac{\alpha}{p_1} = u^{\frac{1}{\alpha+\beta}}\left(\frac{\alpha p_2}{\beta p_1}\right)^{\frac{\beta}{\alpha+\beta}}.
\end{aligned}$$

By symmetry

$$\bar{x}_2 = u^{\frac{1}{\alpha+\beta}}\left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

2. (i) The Hicksian demand functions are the optimal point of the following problem:

$$\begin{aligned}
&\text{minimize} && p_1x_1 + p_2x_2 \\
&\text{subject to} && \left(x_1^{\rho} + x_2^{\rho}\right)^{\frac{1}{\rho}} = u.
\end{aligned}$$

The Lagrangian function associated with this problem is

$$L = p_1x_1 + p_2x_2 + \lambda\left(u - \left(x_1^{\rho} + x_2^{\rho}\right)^{\frac{1}{\rho}}\right)$$

and the first order conditions are

$$\begin{aligned}
L_1 &= p_1 - \lambda\left(x_1^{\rho} + x_2^{\rho}\right)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} = 0 \\
L_2 &= p_2 - \lambda\left(x_1^{\rho} + x_2^{\rho}\right)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} = 0 \\
L_{\lambda} &= u - \left(x_1^{\rho} + x_2^{\rho}\right)^{\frac{1}{\rho}} = 0.
\end{aligned}$$

From the first two equations we have

$$\frac{p_1}{p_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} \Rightarrow x_1 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\rho-1}} x_2.$$

Substituting into the third equation we have

$$u = \left[x_2^{\rho} \left(\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{\rho-1}} + 1 \right) \right]^{\frac{1}{\rho}} = \left[\frac{x_2^{\rho}\left(p_1^{\sigma} + p_2^{\sigma}\right)}{p_2^{\sigma}} \right]^{\frac{1}{\rho}} \qquad \text{where } \sigma = \rho/(\rho-1)$$

$$= \frac{x_2\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\rho}}}{p_2^{\frac{1}{\rho-1}}}$$

so

$$\bar{x}_2 = \frac{u p_2^{\frac{1}{\rho-1}}}{\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\rho}}},$$

and

$$\bar{x}_1 = \frac{u p_1^{\frac{1}{\rho-1}}}{\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\rho}}}.$$

(ii) The expenditure function is the minimum value function of this problem that is

$$e(\boldsymbol{p}, u) = p_1\bar{x}_1 + p_2\bar{x}_2 = \frac{u\left(p_1^{\sigma} + p_2^{\sigma}\right)}{\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\rho}}} = u\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\sigma}}.$$

By the consistency properties

$$e(p, V) = Y$$

so

$$V\left(p_1^{\sigma} + p_2^{\sigma}\right)^{\frac{1}{\sigma}} = y \Rightarrow V = y\left(p_1^{\sigma} + p_2^{\sigma}\right)^{-\frac{1}{\sigma}}$$

(iii) By Roy's Identity the Marshallian demand function is given by

$$x_1^* = \frac{-\dfrac{\partial V}{\partial p_1}}{\dfrac{\partial V}{\partial y}}$$

$$= \frac{-\left(-\dfrac{1}{\sigma}\right) y \left(p_1^\sigma + p_2^\sigma\right)^{-(1+\sigma)/\sigma\ \sigma p_1^{\sigma-1}}}{\left(p_1^\sigma + p_2^\sigma\right)^{-\frac{1}{\sigma}}}$$

$$= \frac{y p_1^{\sigma-1}}{p_1^\sigma + p_2^\sigma}.$$

By symmetry

$$x_2^* = \frac{y p_2^{\sigma-1}}{p_1^\sigma + p_2^\sigma}.$$

3. (i) The conditional demand functions for inputs are the optimal point of the following problem

$$\text{minimize} \qquad w_1 x_1 + w_2 x_2$$

$$\text{subject to} \qquad y = \left[x_1^\rho + x_2^\rho\right]^{\frac{1}{\rho}}.$$

The Lagrangian function is

$$Z = w_1 x_1 + w_2 x_2 + \lambda\left(y - \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}}\right),$$

and the first order conditions are

$$Z_1 = w_1 - \frac{\lambda}{\rho}\left(x_1^\rho + x_2^\rho\right)^{\frac{1-\rho}{\rho}} \rho x_1^{\rho-1} = 0$$

$$\Rightarrow w_1 - \lambda\left(x_1^\rho + x_2^\rho\right)^{\frac{1-\rho}{\rho}} x_1^{\rho-1} = 0$$

$$Z_2 = w_2 - \lambda\left(x_1^\rho + x_2^\rho\right)^{\frac{1-\rho}{\rho}} x_2^{\rho-1} = 0$$

$$Z_\lambda = y - \left(x_1^\rho + x_2^\rho\right)^{\frac{1}{\rho}} = 0.$$

The first two conditions give

$$\frac{w_1}{w_2} = \left(\frac{x_1}{x_2}\right)^{\rho-1} \Rightarrow x_1^{\rho-1} = \frac{w_1}{w_2} x_2^{\rho-1} \Rightarrow x_1^\rho = \left(\frac{w_1}{w_2}\right)^\sigma x_2^\rho, \text{ where } \sigma = \frac{\rho}{(\rho-1)}.$$

Substituting into the third condition gives

$$y = \left(\left(\frac{w_1^\sigma + w_2^\sigma}{w_2^\sigma}\right) x_2^\rho\right)^{\frac{1}{\rho}} \Rightarrow \bar{x}_2 = \frac{w_2^{\frac{\sigma}{\rho}} y}{\left(w_1^\sigma + w_2^\sigma\right)^{\frac{1}{\rho}}}.$$

By symmetry

$$\overline{\mathrm{x}}_1 = \frac{w_1^{\frac{\sigma}{\rho}} y}{\left(w_1^{\sigma} + w_2^{\sigma}\right)^{\frac{1}{\rho}}}.$$

(ii) The cost function is the minimum value function of this problem, that is,

$$c(\boldsymbol{w}, y) = w_1\overline{x}_1 + w_2\overline{x}_2 = y\frac{\left(w_1^{\sigma} + w_2^{\sigma}\right)}{\left(w_1^{\sigma} + w_2^{\sigma}\right)^{\frac{1}{\rho}}} = y\left(w_1^{\sigma} + w_2^{\sigma}\right)^{\frac{1}{\sigma}}.$$

(iii) Consider

$$\frac{\partial c}{\partial w_1} = y\left(w_1^{\sigma} + w_2^{\sigma}\right)^{\frac{1-\sigma}{\sigma}} w_1^{\sigma-1} = \overline{x}_1.$$

Similarly $\frac{\partial c}{\partial w_2} = \overline{x}_2$, so Shephard's Lemma holds.

4. (i) The conditional demand functions are the optimal point of the following problem

$$\begin{array}{ll} \min\text{imize} & w_1x_1 + w_2x \\ \text{subject to} & 100x_1^{\frac{1}{2}}x_2^{\frac{1}{4}} = y. \end{array}$$

The Lagrangian function associated with this problem is

$$Z = w_1x_1 + w_2x_2 + \lambda\left(y - 100x_1^{\frac{1}{2}}x_2^{\frac{1}{4}}\right),$$

and the first order conditions are

$$Z_1 = w_1 - 50\lambda x_1^{-\frac{1}{2}}x_2^{\frac{1}{4}} = 0$$

$$Z_2 = w_2 - 25\lambda x_1^{\frac{1}{2}}x_2^{-\frac{3}{4}} = 0$$

$$Z_\lambda = y - 100\lambda x_1^{\frac{1}{2}}x_2^{\frac{1}{4}} = 0.$$

From the first two equations

$$\frac{w_1}{w_2} = \frac{2x_2}{x_1}$$

so

$$x_1 = \frac{2w_2x_2}{w_1}.$$

Substituting into the third equation gives

$$y = 100\left(\frac{2w_2}{w_1}\right)^{\frac{1}{2}} x_2^{\frac{3}{4}}$$

so

$$\overline{x}_2 = \left(\frac{y}{100}\right)^{\frac{4}{3}}\left(\frac{w_1}{2w_2}\right)^{\frac{2}{3}}$$

and

$$\overline{x}_2 = \frac{2w_2}{w_1}\left(\frac{y}{100}\right)^{\frac{4}{3}}\left(\frac{w_1}{2\mathrm{w}_2}\right)^{\frac{2}{3}} = \left(\frac{y}{100}\right)^{\frac{4}{3}}\left(\frac{2w_2}{w_1}\right)^{\frac{1}{3}}.$$

(ii) The cost function is the minimum value function for this problem that is

$$c(\boldsymbol{w}, y) = w_1\overline{x}_1 + w_2\overline{x}_2$$

$$= w_1\left(\frac{y}{100}\right)^{\frac{4}{3}}\left(\frac{w_1}{2w_2}\right)^{-\frac{1}{3}} + w_2\left(\frac{y}{100}\right)^{\frac{4}{3}}\left(\frac{w_1}{2w_2}\right)^{\frac{2}{3}}$$

$$= \frac{3}{2}\left(\frac{y}{100}\right)^{\frac{4}{3}} w_1^{\frac{2}{3}}\left(2w_2\right)^{\frac{1}{3}}.$$

(iii) Marginal cost is

$$\frac{\partial c}{\partial y} = \left(\frac{y}{100}\right)^{\frac{1}{3}} w_1^{\frac{2}{3}}\frac{\left(2w_2\right)^{\frac{1}{3}}}{50}.$$

Now from the first order conditions

$$\overline{\lambda} = \frac{w_1 x_1^{\frac{1}{2}} x_2^{-\frac{1}{4}}}{50}$$

$$= w_1\left(\frac{y}{100}\right)^{\frac{2}{3}}\left(\frac{w_1}{2w_2}\right)^{-\frac{1}{6}}\left(\frac{y}{100}\right)^{-\frac{1}{3}}\left(\frac{w_1}{2w_2}\right)^{-\frac{1}{6}}$$

$$= \frac{\partial c}{\partial y}.$$

(iv) The factor demand functions are the optimal point of the following problem

$$\max\text{imize} \qquad \Pi = py - w_1x_1 - w_2x_2 = 100px_1^{\frac{1}{2}}x_2^{\frac{1}{4}} - w_1x_1 - w_2x_2.$$

The first order conditions are

$$\Pi_1 = 50px_1^{-\frac{1}{2}}x_2^{\frac{1}{4}} - w_1 = 0$$

$$\Pi_2 = 25px_1^{\frac{1}{2}}x_2^{-\frac{3}{4}} - w_2 = 0,$$

$$\Rightarrow \frac{2x_2}{x_1} = \frac{w_1}{w_2} \Rightarrow x_1 = \frac{2w_2x_x}{w_1}$$

so from the first condition

$$50p\left(\frac{2w_2x_2}{w_1}\right)^{-\frac{1}{2}} x_2^{\frac{1}{4}} = w_1$$

thus

$$x_2^{-\frac{1}{4}} = \frac{w_1}{50p}\left(\frac{2w_2}{w_1}\right)^{\frac{1}{2}} = \frac{(2w_1w_2)^{\frac{1}{2}}}{50p}$$

and

$$x_2^* = \frac{(50p)^4}{4w_1^3w_2^2}$$

$$x_1^* = \frac{(50p)^4}{2w_1^3w_2}.$$

The profit function is the maximum value function of this problem that is

$$\Pi^*(w,p) = 100px_1^{*\frac{1}{2}}x_2^{*\frac{1}{4}} - w_1x_1^* - w_2x_2^* = \frac{(50p)^4}{w_1^2w_2} - \frac{(50p)^4}{2w_1^2w_2} - \frac{(50p)^4}{4w_1^2w_2}$$

$$= \frac{(50p)^4}{4w_1^2w_2}.$$

(v) By Hotelling's lemma the firm's supply function is given by

$$y^*(\boldsymbol{w},p) = \frac{\partial \Pi^*}{\partial p} = \frac{(50p)^3\,50}{w_1^2w_2} = \frac{50^4p^3}{w_1^2w_2}.$$

Exercises for Chapter 8
8.2
1. (i) The ordinates for the subintervals are

$$\begin{aligned} y_1 &= 0 \\ y_2 &= \frac{1}{n} \\ y_3 &= \frac{2}{n} \\ &\vdots \\ y_{n+1} &= 1 \end{aligned}$$

so

$$S_R = \frac{1}{n}\left(y_2 + \ldots + y_{n+1}\right) = \frac{1}{n^2}\left(1 + 2 + \ldots + n\right) = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2} \text{ as } n \to \infty$$

(ii) The ordinates for the subintervals are

$$\begin{aligned} y_1 &= 0 \\ y_2 &= \left(\frac{1}{n}\right)^2 \\ y_3 &= \left(\frac{2}{n}\right)^2 \\ &\vdots \\ y_{n+1} &= 1 \end{aligned}$$

so

$$S_R = \frac{1}{n}\left(y_2 + \ldots + y_{n+1}\right) = \frac{1}{n^3}\left(1 + 2^2 + \ldots + n^2\right) = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$
$$\to \frac{1}{3} \text{ as } n \to \infty.$$

(iii) The ordinates for the subintervals are

$$\begin{aligned} y_1 &= 0 \\ y_2 &= \left(\frac{1}{n}\right)^3 \\ y_3 &= \left(\frac{2}{n}\right)^3 \\ &\vdots \\ y_{n+1} &= 1 \end{aligned}$$

so

$$S_R = \frac{1}{n}\left(y_2 + \ldots + y_{n+1}\right) = \frac{1}{n^4}\left(1 + 2 + \ldots + n\right)^2 = \frac{1}{4n^4} n^2 \left(n+1\right)^2$$
$$= \frac{1}{4}\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \to \frac{1}{4} \text{ as } n \to \infty.$$

8.3

1. (i) This function has a maximum at $x = 0$ and cuts the x axis at $x = -4$ and $x = 4$. So we want

$$\mathrm{I} = \int_{-4}^{4} 16 - x^2 dx = 16x - x^3/3 \Big|_{-4}^{4} = 256/3.$$

2. Required areas are given by the following definite integrals:

(i) $\int_1^9 x^{\frac{1}{2}} dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_1^9 = \frac{52}{3}$

(ii) $\int_0^1 7 + x^4 dx = 7x + \frac{x^5}{5} \Big|_0^1 = 7\frac{1}{5}.$

(iii) $\int_{-1}^0 (x+1)^{\frac{2}{3}} dx = \frac{2}{5}(x+1)^{\frac{5}{2}} \Big|_{-1}^0 = \frac{2}{5}.$

(iv) $\int_1^2 x(5x^2+2)^3 = \frac{(5x^2+2)^4}{40} \Bigg|_1^2 = 5856.4 - 60.025 = 5796.375$

(v) $\int_0^1 (x+1)^{-2} dx = -(x+1)^{-1}\Big|_0^2 = -\frac{1}{x+1}\Big|_0^2 = \frac{2}{3}.$

8.4

1. (i) $I = \int 9x^{-5} - 7x^2 dx = -\frac{9}{4}x^{-4} - \frac{7}{3}x^3 + C.$

(ii) $I = \int (1+x)^{-\frac{1}{2}} dx = 2(1+x)^{\frac{1}{2}} + C.$

(iii) $I = \int 3xe^{x^2} dx = \frac{3}{2}e^{x^2} + C.$

(iv) $I = \int x^{-3} + 2x^{-6} dx = -\frac{x^{-2}}{2} - \frac{2}{5}x^{-5} + C$

(v) $I = \int x^{\frac{1}{2}} - x^{-\frac{1}{2}} dx = \frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + C$

(vi) $I = \int (x^2+2x+7)^{\frac{3}{2}}(x+1)dx = \frac{1}{5}(x^2+2x+7)^{\frac{5}{2}} + C.$

(vii) $I = \int \frac{\left(-10x^{\frac{5}{2}} + 6x^{\frac{1}{2}}\right)}{\left(x^{\frac{5}{2}} - x^{\frac{3}{2}}\right)^2 dx = 4\left(x^{\frac{5}{2}} - x^{\frac{3}{2}}\right)^{-1}} + C$

(viii) $I = \int \frac{x^4}{(1-x^5)^{\frac{1}{2}}} dx = -\frac{2}{5}(1-x^5)^{\frac{1}{2}} + C$

2. (i) Let $u=\sqrt{x+1}$ so $x=u^2-1$, $dx/du=2u$ and $dx=2udu$. Then

$$\int x^2\sqrt{x+1}dx=\int\left(u^2-1\right)^2 2u^2du=2\int u^6-2u^4+u^2du$$

$$=2\left(\frac{u^7}{7}-\frac{2}{5}u^5+\frac{1}{3}u^3\right)+C=2\left[\frac{(x+1)^{\frac{7}{2}}}{7}-2(x+1)^{\frac{5}{2}}/5+\frac{(x+1)^{\frac{3}{2}}}{3}\right]+C.$$

(ii) Let $u=\sqrt{3x+2}$ so $x=\frac{(u^2-2)}{3}$ and $dx=\frac{2udu}{3}$ then

$$\int x^2/\sqrt{3x+2}\,dx=\frac{2}{27}\int\left(u^2-2\right)^2u^{-1}udu=\frac{2}{27}\int u^4-4u^2+4du$$

$$=\frac{2}{27}\left(\frac{1}{5}u^5-\frac{4}{3}u^3+4u\right)+C$$

Substituting $u=(3x+2)^{\frac{1}{2}}$ into this expression gives the result.

(iii) Let $u=\sqrt{4x+14}$ so $x=\frac{(u^2-14)}{4}$ and $dx=\frac{udu}{2}$. As *x* ranges from 0 to 2 *u* ranges from $\sqrt{14}$ to $\sqrt{22}$. Moreover $\frac{dx}{du}$ does not change sign. Thus

$$\int_0^2\frac{x}{\sqrt{4x+14}}dx=\frac{1}{8}\int_{\sqrt{14}}^{\sqrt{22}}\frac{(u^2-14)udu}{u}=\frac{1}{8}\left(\frac{u^3}{3}-14u\right)_{\sqrt{14}}^{\sqrt{22}}$$

$$=\frac{1}{8}\left(\frac{1}{3}(22)^{\frac{3}{2}}-14\sqrt{22}-\frac{1}{3}(14)^{\frac{3}{2}}+14\sqrt{14}\right)$$

$$=\frac{1}{24}\left(23\sqrt{14}-20\sqrt{22}\right)=0.4566$$

3. Thus substitution clearly will not work as when x ranges from –1 to 1 *u* remains at the value 2 and we no longer have a definite integral.

4. In the formula for integrations by parts,

(i) Let $u'=\cos x$ and $v=x$. Then

$$\int x\cos xdx=\int u'vdx=x\sin x-\int\sin xdx=x\sin x+\cos x+C$$

(ii) Let $u'=x$, $v=\log 3x$. Then

$$\int x\log 3xdx=\frac{x^2}{2}\log 3x-\int\frac{x^2}{3x}dx+C=\frac{x^2}{2}\log 3x-\frac{x^2}{6}+C$$

(iii) Let $u'=e^x$, $v=x$. Then

$$\int xe^xdx=\int u'vdx=xe^x-\int e^xdx=xe^x-e^x+C.$$

(iv) Let $u'=(1+x)^{\frac{1}{2}}$, $v=x$. Then

$$\int x(1+x)^{\frac{1}{2}}dx=\frac{2}{3}(1+x)^{\frac{3}{2}}x-\frac{2}{3}\int(1+x)^{\frac{3}{2}}dx=\frac{2}{3}(1+x)^{\frac{3}{2}}x-\frac{4}{15}(1+x)^{\frac{5}{2}}+C$$

(v) Let $v = \log x$ and $u' = 1$. Then

$$\int \log x dx = x\log x - \int 1 dx = x\log x - x + C$$

(vi) Let $v = x^3$, $u' = e^{3x}$. Then

$$\int x^3 e^{3x} dx = \frac{1}{3}e^{3x}x^3 - \int x^2 e^{3x} dx.$$

For the integral on the right hand side let $u' = e^{3x}$ and $v = x^2$. Then

$$\int xe^{3x} dx = \frac{1}{3}e^{3x}x^2 - \frac{2}{3}\int xe^{3x} dx. \tag{2}$$

Again for the integral on the right hand side let $u' = e^{3x}$, $v = x$. Then

$$\int xe^{x} dx = \frac{1}{3}e^{3x}x - \frac{1}{3}\int e^{3x} dx = \frac{1}{3}e^{3x}x - \frac{1}{9}e^{3x} + C \tag{3}$$

Substituting (3) into (2) the result thus obtained with (1) gives

$$\int x^3 e^3 dx = \frac{1}{3}e^{3x}\left(x^3 - x^2 + \frac{2}{3}x - \frac{2}{9}\right) + C.$$

(vii) Let $u' = e^{ax}$ and $v = \sin bx$. Then

$$I = \int e^{ax}\sin bx = \frac{e^{ax}}{a}\sin bx - \frac{b}{a}\int e^{ax}\cos bx dx \qquad (4)$$

For the integral on the right hand side let $u' = e^{ax}$ and $v = \cos bx$. Then

$$\int e^{ax}\cos bx dx = \frac{e^{ax}}{a}\cos bx + \frac{b}{a}\int e^{ax}\sin bx$$
$$= \frac{1}{a}\left(e^{ax}\cos bx + bI\right).$$

Substituting back into (4) gives

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}\left(e^{ax}\cos bx + bI\right).$$

Solving for I and adding a constant gives

$$I = \frac{e^{ax}}{a^2 + b^2}\left(a\sin bx - b\cos bx\right) + C$$

8.5

1. (i) $\int_0^\infty e^x dx = \lim_{b\to\infty}\int_0^b e^x dx = \lim_{b\to\infty} e^b - 1 = \infty,$ so our integral is divergent.

(ii) $\int_0^\infty e^{-x} dx = \lim_{b\to\infty}\int_0^b e^{-x} dx = \lim_{b\to\infty}\left[-e^{-x}\right]_0^b = \lim_{b\to\infty}\left[-e^{-b} + 1\right] = 1$

so the integral is convergent.

(iii) $\int_0^1 x^{-\frac{1}{2}} dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-\frac{1}{2}} dx = \lim_{\varepsilon\to 0}\left[2x^{\frac{1}{2}}\right]_\varepsilon^1 = \lim_{\varepsilon\to 0}\left[2 - 2\varepsilon^{\frac{1}{2}}\right] = 2$

so the integral is convergent.

(iv) $\int_0^1 x^{-2}dx = \lim_{\varepsilon\to 0}\int_\varepsilon^1 x^{-2}dx = -\lim_{\varepsilon\to 0} x^{-1}\Big|_\varepsilon^1 = \lim_{\varepsilon\to 0}\left[1-\frac{1}{\varepsilon}\right],$

so the integral is divergent.

(v) $\int_{-1}^2 x^{-2}dx = \int_{-1}^0 x^{-2}dx + \int_0^2 x^{-2}dx.$

Now $\int_{-1}^0 x^{-2}dx = \lim_{\varepsilon\to 0}\int_{-1}^{\varepsilon} x^{-2}dx = \lim_{\varepsilon\to 0} -x^{-1}\Big]_{-1}^{\varepsilon} = \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}-1\right] = \infty,$

so our integral is divergent.

2. (i) $\int_1^2\int_0^1 x^2 + xydxdy = \int_1^2 \frac{x^3}{3}+\frac{x^2y}{2}\Big|_0^1 dy = \int_1^2 \frac{1}{3}+\frac{y}{2}dy = \frac{1}{3}y+\frac{1}{4}y^2\Big]_1^2 = 1\frac{1}{12}$

(ii) $\int_0^1\int_1^2 x^2 + y^2\, dydx = \int_0^1 x^2y+\frac{y^3}{3}\Big|_1^2 dx = \int_0^1 x^2+\frac{7}{3}dx = \frac{x^3}{3}+\frac{7x}{3}\Big|_0^1 = \frac{8}{3}.$

(iii) $\int_0^1\int_0^x x^2 + y^2\, dydx = \int_0^1 x^2y+\frac{y^3}{3}\Big|_0^x dx = \int_0^1 \frac{4}{3}x^3dx = \frac{x^4}{3}\Big|_0^1 = \frac{1}{3}.$

(iv) $\int_0^1\int_0^{y^2} (3x+2y)^2 dx\,dy = \int_0^1\int_0^{y^2} 9x^2+6xy+4y^2dx\,dy = \int_0^1 3x^3+3x^2y+4xy^2\Big|_0^{y^2} dy$

$= \int_0^1 3y^6+3y^5+4y^4dy = \frac{3}{7}y^7+\frac{1}{2}y^6+\frac{4}{5}y^5\Big|_0^1 = \frac{121}{70}.$

8.6

1. (i) The Lagrangian function for this problem is

$Z = x_1x_2 + \lambda(Y - p_1x_1 - p_2x_2).$

The first order conditions are

$Z_1 = x_2 - \lambda p_1 = 0$

$Z_2 = x_1 - \lambda p_2 = 0$

$Z_\lambda = Y - p_1x_1 - p_2x_2 = 0.$

The first two conditions imply that

$$x_2 = \frac{p_1x_1}{p_2}$$

so substituting into the third equation gives the Marshallian demand functions

$x_1^* = Y/2p_1,\ \ x_2^* = Y/2p_2.$

(ii) The indirect utility function is the maximum value function of this problem so it is

$V(\mathbf{p},Y) = x_1^*x_2^* = Y^2/p_1p_2.$

(iii) $CS = \int_1^{1.2} \frac{Y}{2p_1 dp_1} = \frac{Y}{2}\log p_1 \Big|_1^{1.2} = \frac{Y}{2}\log 1.2 = 0.09116Y.$

C is defined by

$$V\left(\mathbf{p}^1, Y + C\right) = V\left(p^0, Y\right)$$

$$\Rightarrow \frac{\left(Y + C\right)^2}{4\left(1.2\right)p_2} = \frac{Y^2}{4p_2} \Rightarrow Y + C = Y\sqrt{1.2}$$

$$\Rightarrow C = Y\left(\sqrt{1.2} - 1\right) = 0.09544Y.$$

E is defined by

$$\mathrm{V}\left(\boldsymbol{p}^1, Y\right) = V\left(\mathbf{p}^0, Y - E\right)$$

$$\Rightarrow \frac{Y^2}{4\left(1.2\right)p_2} = \frac{\left(Y - E\right)^2}{4p_2}$$

$$\Rightarrow E = Y\left(1 - 1/\sqrt{1.2}\right) = 0.08713Y.$$

(iv) As income $Y > 0$ we have
$E < CS < C.$

2. (i) By Roy's identity the Marshallian demand functions are given by

$$x_j^* = \frac{\frac{\partial V}{\partial pj}}{\frac{\partial V}{\partial Y}}.$$

Now

$$\frac{\partial V}{\partial Y} = \frac{1}{\prod_{i=1}^{n} p_i^{\beta i}},$$

and

$$\frac{\partial V}{\partial p_j} = \frac{-\gamma_j}{\prod_{i=1}^{n} p_i^{\beta i}} - \left(y - \sum_{i=1}^{n} \gamma_i p_i\right)\frac{\beta_j}{p_j \prod_{i=1}^{n} p_i^{\beta i}}$$

so

$$\frac{x_j^* = \gamma_j + \beta_j\left(y - \sum_{i=1}^{n} \gamma_i p_i\right)}{p_{\mathrm{j}}} = \frac{\gamma_j\left(1 - \beta_j\right) + \beta_j\left(y - \sum_{i \neq j} \gamma_i p_i\right)}{p_j}.$$

(ii) The loss in consumers' welfare using consumers' surplus is given by

$$CS = \int_{p_1^0}^{1.1p_1^0} x_1^* dp_1 = \int_{p_1^0}^{1.1p_1^0} \gamma_1 (1-\beta_1) + \ell p_1^{-1} dp_1$$

where $\ell = \beta_1 \left(y - \sum_{i \neq 1} \gamma_i p_i \right)$. Thus

$$\begin{aligned} CS &= \gamma_1 (1-\beta_1) p_1 + \ell \log p_1 \Big|_1^{1.1} \\ &= \gamma_1 (1-\beta_1)(1.1) p_1^0 + \ell \log(1.1) p_1^0 - \gamma_1 (1-\beta_1) p_1^0 + \ell \log p_1^0 \\ &= (0.1)\gamma_1 p_1^0 (1-\beta_1) + \ell \log 1.1. \end{aligned}$$

(iii) The loss in consumer welfare measured by compensation variation *C* is defined by

$$\mathrm{V}(\mathbf{p}^1, y + \mathrm{C}) = \mathrm{V}(\mathbf{p}^0, y),$$

where the new prices $\mathbf{p}^1$ are the same as the old prices $\mathbf{p}$ except $p_1^1 = 1.1 p_1^0$. So

$$V(\mathbf{p}^1, y + C) = \frac{y + C - \sum_i \gamma_i p_i^0 - 0.1\gamma_1 p_1^0}{(1.1)^{\beta_1} \prod_i p_i^{0\beta i}}$$

and *C* is given by

$$\frac{y + C - \sum_i \gamma_i p_i^0 - 0.1\gamma_1 p_1^0}{(1.1)^{\beta_1} \prod_i p_i^{0\beta_i}} = \frac{y - \sum_i \gamma_i p_i^0}{\prod_i p_i^{0\beta_i}},$$

that is,

$$y + C - \sum \gamma_i p_i^0 - 0.1\gamma_1 p_1^0 = (1.1)^{\beta_1} \left(y - \sum_i \gamma_i p_i^0 \right),$$

that is,

$$C = 0.1\gamma_1 p_1^0 + \left((1.1)^{\beta_1} - 1 \right) \left(y - \sum_i \gamma_i p_i^0 \right).$$

The loss in consumer welfare measured by equivalence variation is defined by

$$V(\mathbf{p}^1, y) = V(\mathbf{p}^0, y - E)$$

where

$$V(\mathbf{p}^1, y) = \frac{y - \sum_i \gamma_i p_i^0 - 0.1\gamma_1 p_1^0}{(1.1)^{\beta_1} \prod_i p_i^{0\beta i}}$$

and

$$V(\mathbf{p}^0, y - E) = \frac{y - E - \sum_i \gamma_i p_i^0}{\prod_i p_i^{0\beta i}}.$$

Thus *E* is given by

$$y - E - \sum_i \gamma_i p_i^0 = \frac{y - \sum_i \gamma_i p_i^0 - 0.1\gamma_1 p_1^0}{(1.1)^{\beta_1}},$$

that is,

$$E = \frac{0.1\gamma_1 p_1^0 + \left((1.1)^{\beta_1} - 1\right)\left(y - \sum_i \gamma_i p_i^0\right)}{(1.1)^{\beta_1}} = \frac{C}{(1.1)^{\beta_1}}.$$

Exercises for Chapter 9

9.4

1. All the equations are first order linear differential equations with constant coefficients.

(i) The general solution to the homogeneous equation is $y = Ce^{-4x}$ with *C* an arbitrary constant.

A particular solution to the nonhomogeneous equation is

$$y = e^{-4x}\int_0^x e^{4s}e^{2e}\mathrm{d}s = \frac{e^{-4x}}{6}\left(e^{6s}\Big]_0^x\right) = \frac{e^{-4x}}{6}\left(e^{6x} - 1\right) = \frac{1}{6}\left(e^{2x} - e^{-4x}\right).$$

The general solution to our equation then is

$$y(x) = ce^{-4x} + \frac{1}{6}\left(e^{2x} - e^{-4x}\right).$$

Using the initial condition we get $c = 2$ and the required particular solution of

$$y(x) = \frac{2e^{-4x} + \left(e^{2x} - e^{-4x}\right)}{6}.$$

(ii) The general solution to the homogeneous equation is $y = ce^{-6c}$ with *c* an arbitrary constant. A particular solution to the nonhomogeneous equation is

$$y = e^{-7x}\int_0^x e^{7s}e^{-s}ds = \frac{e^{-7x}}{6}\left(e^{6x} - 1\right) = \frac{1}{6}\left(e^{-x} - e^{-7x}\right).$$

The general solution to our equation is then

$$y(x) = ce^{-7x} + \frac{1}{6}\left(e^{-x} - e^{-7x}\right).$$

Using the initial condition we get $c = 1$ so the required particular solution is

$$y(x) = \frac{5e^{-7x}}{6} + \frac{1}{6}e^{-x}$$

(iii) The general solution to the homogenous equation is $y = ce^x$, c an arbitrary constant. A particular solution to the nonhomogeneous equation is

$$y = -e^x \int_0^x s^2 e^{-s} ds.$$

Now using integration by parts we have

$$\int s^2 e^{-s} dx = -s^2 e^{-s} + 2\int se^{-s} ds$$

and

$$\int se^{-s} ds = -se^{-s} + \int e^{-s} ds = -se^{-s} - e^{-s}$$

so

$$\int s^2 e^{-s} ds = -s^2 e^{-s} - 2e^{-s}(s+1) + c.$$

It follows that

$$\int_0^x s^2 e^{-s} ds = -s^2 e^{-s} - 2e^{-s}(s+1)\Big|_0^x = -x^2 e^{-x} - 2\mathrm{e}^{-x}(x+1) + 2$$

and that a particular solution to the nonhomogeneous equation is

$$y = x^2 + 2(x+1) - 2e^x.$$

The general solution to our equation then is

$$y(x) = x^2 + 2(x+1) - 2e^x + ce^x = x^2 + 2(x+1) + c'e^x$$

where c' is an arbitrary constant.

$$y(1) = 1 \Rightarrow 1 = 5 + c'e \Rightarrow c' = -4e^{-1}$$

so required particular solution is

$$y(x) = x^2 + 2(x+1) - 4e^{x-1}.$$

2. (i) We have

$$Y = \gamma Y + v\frac{dY}{dt} + \bar{I}$$

so

$$-v\frac{dY}{dt} + Y(1-\gamma) = \bar{I}$$

or

$$\frac{dY}{dt} + aY = -\frac{\bar{I}}{v} \text{ where } a = \frac{\gamma - 1}{v}.$$

(ii) Potential equilibrium where $Y = \bar{Y}$ and $\frac{dY}{dt} = 0.$

That is at

$$\bar{Y} = -\frac{aI}{v} = \frac{\mathrm{I}}{1-\gamma}.$$

This is a particular solution to the nonhomogeneous equation. The general solution to the homogeneous equation is $Y = ce^{-at}$ with c an arbitrary constant.

Adding we get the general solution to the nonhomogeneous equation given by

$$Y = \bar{Y} + ce^{-at}.$$

and

$$Y \to \bar{Y} \text{ if and only if } e^{-at} \to 0. \text{ That is, if and only if } a > 0.$$

(iii) Want $a > 0$ where $a = \gamma - \frac{1}{v}.$

As γ is the marginal property to consume, $0 < \gamma < 1$ and $\gamma - 1 < 0.$ The accelerator *v* is $v > 0,$ so this condition will not hold and *Y* will be subject to explosive growth. This may be a good representation in a growth model.

3. (i) Consumers when faced with rising prices may buy now to avoid higher prices in the future. If this rationale is correct we would expect $c > 0.$ Alternatively they may reason that rising prices today must be matched by prices falling in the future so they will hold off buying now, in which case we would expect $c < 0.$

(ii) Equating Q_s to Q_d we have

$$a + bP + c\frac{dP}{dt} = \alpha + \beta P$$

or

$$\frac{dP}{dt} + dP = \frac{\alpha - a}{c} \text{ with } d = \frac{b-\beta}{c},$$

which is a first order linear differential equation with constant coefficients. A particular solution to the nonhomogeneous equation is found by letting $P = \bar{P}$ and $\frac{dP}{dt} = 0$ in the equation to give

$$\bar{P} = \frac{\alpha - \mathrm{a}}{b - \beta}.$$

As *P* is at rest at this value $\bar{P}$ is a potential equilibrium.

The general solution to the homogeneous equation is

$$P(t) = \ell e^{-dt} = P(0)e^{-dt}, \ \ell \text{ an arbitrary constant.}$$

Adding gives the general solution to the nonhomogenous equation

$$P(t) = \bar{P} + P(0)e^{-dt}.$$

(iii) Now

$$P \to \overline{P} \text{ iff } d > 0,$$

that is,

$$\text{iff } \frac{b-\beta}{c} > 0.$$

We are given $b < 0$ and $\beta > 0$ so the condition we require is $c < 0$.

If this is the case the resultant equilibrium for *P* is $\overline{P} = \dfrac{(\alpha - a)}{(b-\beta)}$.

4. (i) Consider

$$Y(\lambda K, \lambda L) = (\lambda K)^{\alpha} (\lambda L)^{1-\alpha} = \lambda K^{\alpha} L^{1-\alpha}$$

so $Y(K,L)$ is homogeneous of degree one.

$$\frac{\partial Y}{\partial K} = \alpha K^{\alpha-1} L^{1-\alpha} > 0 \quad \frac{\partial^2 Y}{\partial K^2} = \alpha(\alpha-1) K^{\alpha-1} L^{1-\alpha} < 0.$$

Similarly $\dfrac{\partial Y}{\partial K} > 0$ and $\dfrac{\partial^2 Y}{\partial K^2} < 0.$

Now

$$Y(0,L) = 0L^{1-\alpha} = 0$$

$$Y(K,0) = K^{\alpha} 0 = 0,$$

so the function satisfies all the requirements of a neoclassical production function.

(ii) $\dfrac{Y}{K} = Y\left(\dfrac{K}{L}, 1\right) = \left(\dfrac{K}{L}\right)^{\alpha} = k^{\alpha}$

so $\phi(k) = k^{\alpha}$

(iii) The differential equation is

$$k' = \mathrm{s}k^{\alpha} - (n+\delta)k$$

or

$$k' + (n+\delta)k = \mathrm{s}k^{\alpha}$$

which is clearly a Bernoulli Equation. Dividing through by k^{α} gives

$$k^{-\alpha}k' + (n+\delta)k^{1-\alpha} = \mathrm{s}$$

and letting $z = k^{1-\alpha}$ so

$$z' = (1-\alpha)k^{-\alpha}k'$$

we write our equation in terms of *z* as

$$z' + (n+\delta)(1-\alpha)z = s(1-\alpha).$$

The general solution to this equation is

$$z = k^{1-\alpha} = c_1 e^{-(1-\delta)(n+\delta)t} + \frac{s}{n+\delta},$$

where c_1 is an arbitrary constant.

(iv) As $0 < \alpha < 1$, $(1-\alpha)(n+\delta)$ is positive so

$$e^{-(1-\alpha)(n+\delta)t} \to 0 \text{ as } t \to \infty.$$

Thus

$$k^{1-\alpha} \to \frac{s}{n+\delta}$$

and

$$k^* = \left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}$$

is the steady state level of k.

(v) The steady state level o consumption is

$$c^* = \phi(k^*) - (n-\delta)k^*$$

$$= \left(\frac{s}{n+\delta}\right)^{\frac{\alpha}{1-\alpha}} - (n-\delta)\left(\frac{s}{n+\delta}\right)^{\frac{1}{1-\alpha}}.$$

The golden rule level of k occurs when

$$\phi'(k^*) = n+\delta.$$

That is when

$$\alpha k^{*\alpha-1} = n+\delta$$

so

$$k_g^* = \left(\frac{n+\delta}{\alpha}\right)^{\frac{1}{\alpha-1}} = \left(\frac{\alpha}{n+\delta}\right)^{1-\alpha}.$$

comparing k^* with k_g^* we see that the golden rule level of s is

$$s_g^* = \alpha.$$

9.6

All these equations are second order linear differential equations with constant coefficients.

1. (i) Homogeneous equation with auxiliary equation

$$m^2 - 4m + 4 = 0,$$

which has repeated roots of $m_1 = m_2 = 2$, so the general solution is $y = c_1 e^{2x} + c_2 x e^{2x}$, c_1 and c_2 arbitrary constants.

Now

$$y(0) = c_1 = 2$$

and

$$y' = 2ce^{2x} + 2c_2xe^{2x} + c_2e^{2x} = 4e^{2x} + 2c_2xe^{2x} + c_2e^{2x}$$

so

$$y'(0) = 4 + c_2 = 5 \Rightarrow c_2 = 1.$$

The required particular solution is then

$$y = 2e^{2x} + xe^{2x}.$$

(ii) This homogeneous equation has an auxiliary equation given by

$$m^2 - m - 2 = 0$$

which has roots $m_1 = 2,\ m_2 = -1$ so the general solution to the differential equation is $y = \mathrm{c}_1e^{2x} + c_2e^{-x}$, c_1 and c_2 arbitrary constants.

Now

$$y(0) = c_1 + c_2 = 1,\ y' = 2c_1e^{2x} - c_2e^{-x},\ y'(0) = 2c_1 - c_2 = -5.$$

Solving the two equations in c_1 and c_2 gives $c_1 = -\frac{4}{3}$ and $c_2 = \frac{7}{3}$ so the required particular solution is

$$y = -\frac{4}{3}e^{2x} + \frac{7}{3}e^{-x}.$$

(iii) Nonhomogeneous equation with auxiliary equation

$$m^2 - 2m - 3 = 0$$

which has roots $m_1 = 3,\ \ m_1 = -1$ so the general solution to the homogeneous equation is $y = c_1e^{3x} + c_2e^{-x}$, with c_1 and c_2 arbitrary constants. For a particular solution of the nonhomogeneous equation try

$$y = a_0 + a_1x + a_2x^2$$

so

$$y' = a_1 + 2a_2x \text{ and } y'' = 2a_2.$$

Substituting into our differential equation gives

$$2a_2 - 2(a_1 + 2a_2x) - 3(a_0 + a_1x + a_2x^2) = 9x^2.$$

Equating coefficients gives

$$2a_2 - 2a_1 - 3a_0 = 0,\ -4a_2 - 3a_1 = 0,\ -3a_2 = 9$$

so

$$a_2 = -3,\ a_1 = 4 \text{ and } a_0 = -\frac{14}{3}.$$

Therefore a particular solution to the nonhomogeneous equation is

$$y = -\frac{14}{3} + 4x - 3x^2$$

and the general solution is

$$y = -\frac{14}{3} + 4x - 3x^2 + c_1e^{3x} + c_2e^{-x}.$$

(iv) The auxiliary equation is

$$m^2 - 2m - 3 = 0$$

which has roots $m_1 = 3$ and $m_2 = -1$ so the general solution to the homogeneous equation is

$$y = c_1e^{3x} + c_2e^{-x},\ c_1 \text{ and } c_2 \text{ arbitrary constants.}$$

For a particular solution to the nonhomogeneous equation we try

$$y = cxe^{-x}$$

so

$$y' = ce^{-x} - cxe^{-x},\ y'' = -2ce^{-x} + cxe^{-x}.$$

Substituting into our differential equation gives

$$-2ce^{-x} + cxe^{-x} - 2\left(ce^{-x} - cxe^{-x}\right) - 3cxe^{-x} = 8e^{-x},$$

that is,

$$-4ce^{-x} = 8e^{-x} \Rightarrow c = -2.$$

Thus a particular solution to the nonhomogeneous equation is

$$y = -2xe^{-x}$$

and the general solution is

$$y = -2xe^{-x} + c_1e^{3x} + c_2e^{-x}.$$

(v) This equation has auxiliary equation given by

$$m^2 - m - 2 = 0$$

which has roots $m_1 = 2,\ m_2 = -1,$ so the general solution to the homogeneous equation is $y = c_1e^{-x},\ c_1 \text{ and } c_2$ arbitrary constants.

For a particular solution of the nonhomogeneous equation by

$$y = cxe^{2x}$$

with

$$y' = ce^{2x} + 2cxe^{2x}, y'' = 4ce^{2x} + 4cxe^{2x}.$$

Substituting in our equation gives

$$e^{2x}\left(4c + 4cx - -2cx - 2cx\right) = 4e^{2x} \Rightarrow c = \frac{4}{3},$$

so the general solution to our equation is

$$y = \frac{4}{3}xe^{2x} + c_1e^{2x} + c_2e^{-x}.$$

Now

$$y(0) = c_1 + c_2 = 5$$

$$y' = \frac{4}{3}e^{2x} + \frac{8}{3}xe^{2xc} + 2c_1e^{2x} - c_2e^{-x}$$

$$y'(0) = \frac{4}{3} + 2c_1 - c_2 = 1,$$

so

$$c_1 = \frac{14}{9}, c_2 = \frac{31}{9}$$

and our required particular solution is

$$y = \frac{(12xe^{2x} + 14e^{2x} + 31e^{-x})}{9}.$$

(vi) The auxiliary equation is

$$2m^2 - 2m + 1 = 0$$

which has conjugate complex roots

$$m_1 = \frac{1}{2} + \frac{i}{2},\ m_2 = \frac{1}{2} - \frac{i}{2}$$

so the general solution to our equation is

$$y = 2 + e^{x/2}\left(\frac{c_1\cos x}{2} + \frac{c_2\sin x}{2}\right)$$

with

$$y' = \frac{e^{x/2}}{2}\left(\frac{c_1\cos x}{2} + \frac{c_2\sin x}{2}\right) + e^{x/2}\left(-\frac{c_1}{2}\frac{\sin x}{2} + \frac{c_2}{2}\frac{\cos x}{2}\right).$$

Hence

$$y(0) = 2c_1 = 4 \Rightarrow c_1 = 2$$

$$y'(0) = \frac{(c_1 + c_2)}{2} = 2 \Rightarrow c_2 = 2$$

so the required particular solution is

$$y = +e^{x/2}\left(\frac{2\cos x}{2} + \frac{2\sin x}{2}\right).$$

(vii) The auxiliary equation is

$$m^2 + m = 0$$

which has roots $m_1 = 0,\ m_2 = -1$ so the general solution to the homogeneous equation is $y = c_1 + c_2 e^{-x}$, where c_1 and c_2 are arbitrary constants. For a particular solution to the nonhomogeneous equation try

$$y = a_0 + a_1 x.$$

But a constant appears in the general solution of the homogeneous equation so instead try

$$y = a_0 x + a_1 x^2.$$

Now

$$y' = a_0 + 2a_1 x,\ y'' = 2a_1$$

so substituting into our differential equation gives

$$2a_1 + a_0 + 2a_1 x = x \Rightarrow a_1 = \frac{1}{2},\ a_0 = -1$$

and the general solution to the nonhomogeneous equation is

$$y = \frac{-x + x^2}{2} + c_1 + c_2 e^{-x}.$$

Differentiating we have

$$y' = -1 + 2x - c_2 e^{-x}.$$

Using the initial conditions we have

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = -1 - c_2 = 0$$

$$\Rightarrow c_2 = -1 \text{ and } c_1 = 2,$$

so our required particular solution is

$$y = \frac{-x + x^2}{2} + 2 - e^{-x}.$$

(viii) The auxiliary equation is

$$m^2 + 2m + 1 = 0$$

which has repeated roots $m_1 = m_2 = -1,$ so the general solution to the homogeneous equation is $y = c_1 e^{-x} + c_2 x e^{-x}$, c_1 and c_2 arbitrary constants.

For a particular solution tempted to try $y = ce^{-x}$ but e^{-x} appears in the general solution of the homogeneous equation as does xe^{-x} so instead we try

$$y = cx^2 e^{-x}.$$

Now

$$y' = 2cxe^{-x} - cx^2e^{-x} = ce^{x}\left(2x - x^2\right)$$
$$y'' = -ce^{-x}\left(2x - x^2\right) + ce^{-x}\left(2 - 2x\right) = ce^{-x}\left(2 - 4x + x^2\right).$$

so substituting into our differential equation gives

$$ce^{-x}\left(2 - 4x + x^2 + 4x - 2x^2 + x^2\right) = e^{-x} \Rightarrow c = \frac{1}{2}.$$

The general solution to our equation is then

$$y = \frac{x^2e^{-x}}{2} + c_1e^{-x} + c_2xe^{-x}.$$

2. Consider $y = y_1 + y_2$ and

$$\begin{aligned} ay'' + by' + cy &= a\left(y_1'' + y_2''\right) + b\left(y_1' + y_2'\right) + c\left(\mathrm{y}_1 + \mathrm{y}_2\right) \\ &= \left(ay_1'' + by_1' + cy_1\right) + \left(ay_2'' + by_2' + cy_2\right) \\ &= f_1\left(x\right) + f_2\left(x\right). \end{aligned}$$

(i) The auxiliary equation is

$$m^2 - 3m = 0$$

which has roots $m_1 = 0,\ m_2 = 3$ so the solution to the homogeneous equation is $y = c_1 + c_2e^{3x}$, where c_1, c_2 are arbitrary constants.

For $f_1\left(x\right) = 6$ try $y = a_0x.$ Then $y' = a_0$, $y'' = 0$ and we have $-3a_0 = 6 \Rightarrow a_0 = -2$.

For $f_2\left(x\right) = 3e^{3x}$ try $y = cxe^{3x}$. Then

$$y' = ce^{3x} + 3cxe^{3x} = ce^{3x}\left(1 + 3x\right)$$
$$y'' = 3ce^{3x}\left(1 + 3x\right) + 3ce^{3x} = 3ce^{3x}\left(2 + 3x\right)$$

so

$$3ce^{3x}\left(2 + 3x\right) - 3ce^{3x}\left(1 + 3x\right) = 3e^{3x} \Rightarrow c = 1.$$

Combining we have that a particular solution to our differential equation is

$$-2x + xe^{3x}$$

and the general solution is

$$y = -2x + xe^{3x} + c_1 + c_2e^{3x}.$$

(ii) From (vii) of question 1 the general solution to the homogeneous equation is $y = c_1e^{-x} + c_2xe^{-x}$, c_1 and c_2 arbitrary constants, and a particular solution is $y = \frac{x^2e^{-x}}{2}$. For $f_2(x) = 3x$ we try $y = a_0 + a_1x$ as a particular solution so $y' = a_1,\ y'' = 0$ and $2a_1 + a_0 - a_1x = 3x \Rightarrow a_1 = 3$ and $a_0 = -6.$

Combining we have a particular solution to the nonhomogeneous equation as

$$y = \frac{-6 + 3x + x^2e^{-x}}{2} + c_1e^{-x} + c_2xe^{-x}.$$

(iii) From (v) of question 1 the general solution to the homogeneous equation is $y = c_1e^{2x} + c_2e^{-x}$, with c_1 and c_2 as arbitrary constants.

For $f_1(x) = 6x,$ we try $y = a_0 + a_1x$ so $y' = a_1,\ y'' = 0$ and then $-a_2 - 2a_0 - 2a_1x = 6x \Rightarrow a\ = -3,$ $a_0 = \frac{3}{2}.$

For $f_2(x) = 4e^{-x}$ try $y = cxe^{-x}$
so

$$y' = ce^{-x}(1 - x),\ y'' = ce^{-x}(x - 2)$$

and we have

$$ce^{-x}(x - 2 - 1 + x - 2x) = e^{-x} \Rightarrow c = -\frac{4}{3}.$$

Combining we have the particular solution to the nonhomogeneous equation as

$$y = \frac{3}{2} - 3x - \frac{4}{3}xe^{-x}$$

and the general solution is

$$y = \frac{3}{2} - 3x - \frac{4}{3}xe^{-x} + c_1e^{2x} + c_2e^{-x}.$$

9.7
Routh's theorem states real roots and the real parts of imaginary roots of the nth degree polynomial equation

$$a_0m^n + a_1m^{n-1} + \ldots + a_{n-1}m + a_n = 0$$

are negative if and only if the first n of the determinants

$$a_1, \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix} \ldots$$

are all positive, where it is understood that in these determinants we set $a_r = 0$ for all $r > n$ needed to obtain these determinants.

We know $y(x)$ will be convergent if and only if the real roots or the real parts of complex roots are all negative. Applying Routh's theorem to our equation, this will be the case if and only if

$$a, \begin{vmatrix} a & 0 \\ 1 & b \end{vmatrix} \text{are} > 0 \Rightarrow a > 0,\ ab > 0 \Rightarrow a > 0 \text{ and } b > 0.$$

2. (i) In this equation $a_1 = -21$ so by Routh's theorem the time path is not convergent.

(ii) From this equation

$$a_1 = 21 > 0$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 21 & 20 \\ 1 & 36 \end{vmatrix} > 0$$

$$\begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & 0 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 21 & 20 & 0 \\ 1 & 36 & 0 \\ 0 & 21 & 20 \end{vmatrix} = 20(-1)^{3+3} \begin{vmatrix} 21 & 20 \\ 1 & 36 \end{vmatrix} > 0.$$

The time path for *y* is convergent.

(iii) From this equation

$$a_1 = 8$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 8 & 4 \\ 1 & -7 \end{vmatrix} = -60,$$

so the time path for *y* is divergent.

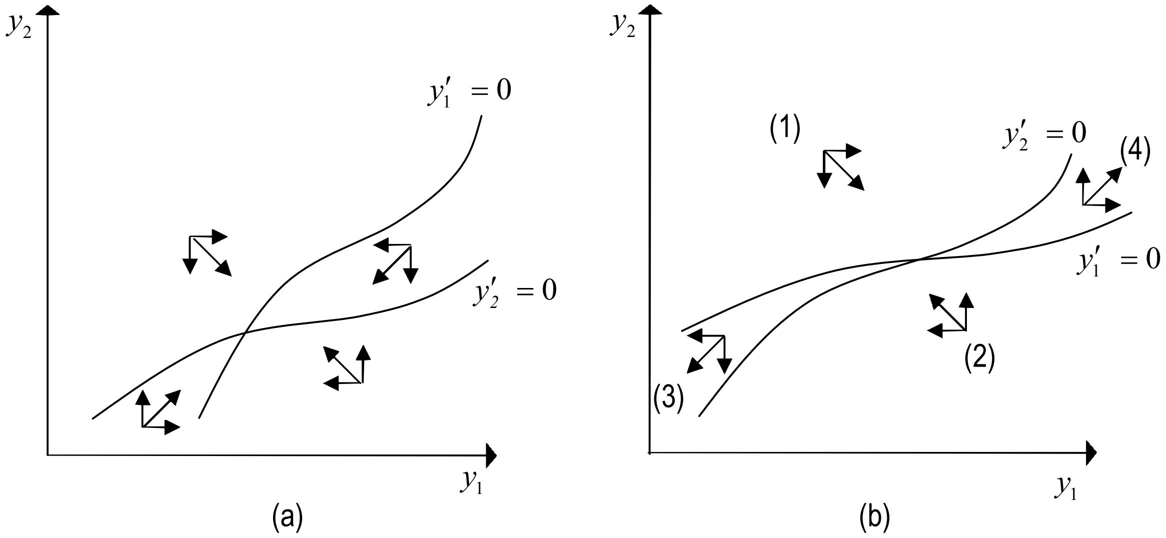
9.8

1. (i) In this example

$$\left.\frac{dy_2}{dy_1}\right|_{y_1'=0} = -\frac{f_1}{f_2} = -\frac{(-\text{ve})}{+\text{ve}} = +\text{ve}$$

so $y_1' = 0$ has a positive slope.

Similarly $y_2' = 0$ has a positive slope, so we have two possible phase diagrams:



Also,

$$\frac{\partial y_1'}{\partial y_2'} = f_1 < 0$$

so y_1 and y_1' move in opposite directions so as y_1 increases y_1' must decrease from positive to zero, to negative which give the directional arrows for y_1 in the two diagrams.

$$\frac{\partial y_2'}{\partial y_2} = g_2 < 0$$

so as y_2 increases, y_2' must decrease from positive, to zero to negative, which gives the directional arrows for y_2 as shown in the diagrams.

Phase diagram (a) gives convergent time paths for our variables without cycles.

Phase diagram (b) gives the same in sections (1) or (2) but divergent time paths in sections (3) or (4).

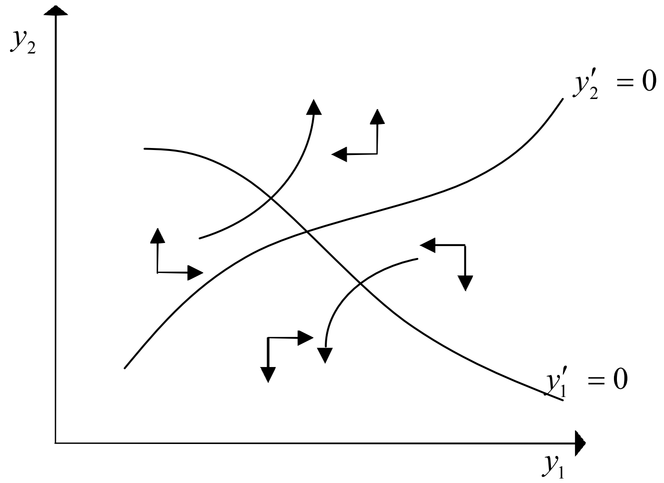
(ii) In this example

$$\left.\frac{\partial y_2}{\partial y_2}\right|_{y_1'=0} = -\frac{f_1}{f_2} = -\frac{(-\text{ve})}{(+\text{ve})} = -\text{ve}$$

and

$$\left.\frac{\partial y_2}{\partial y_1}\right|_{y_2'=0} = -\frac{g_1}{g_2} = -\frac{(-\text{ve})}{(+\text{ve})} = +\text{ve}$$

so the slope of $y_1' = 0$ is negative and the slope of $y_2' = 0$ is positive as shown in the diagram:



Also

$$\frac{\partial y_1'}{\partial y_1} = f_1 < 0$$

so as y_1 increases y_1' must decrease from positive to zero to negative which gives the directional arrows for y_1 as shown in the diagram.

Similarly

$$\frac{\partial y_2'}{\partial y_2} = g_2 > 0$$

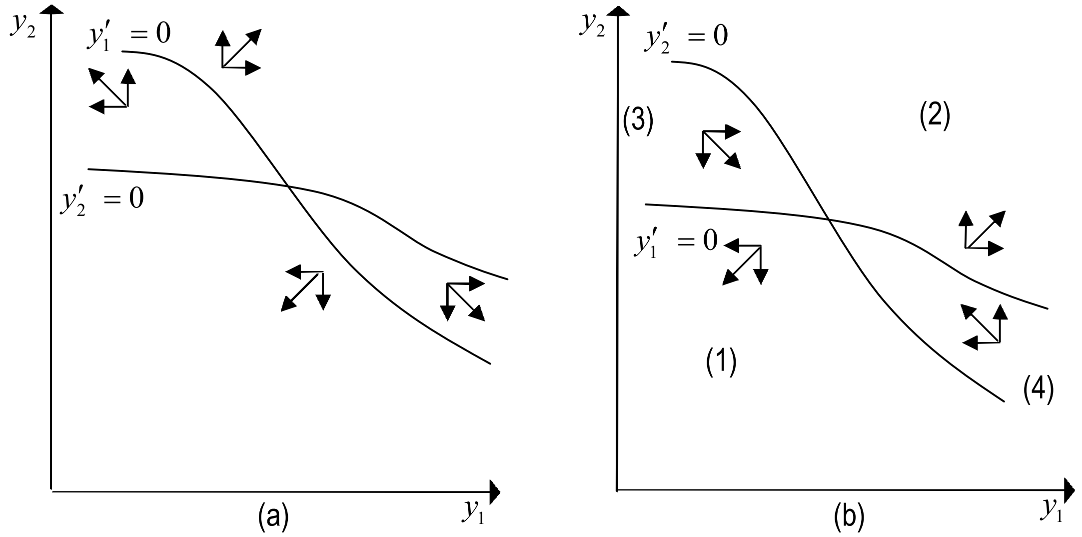
so as y_2 increases, y_2' must also increase from negative to zero to positive which gives the directional arrows for y_2 as shown in the diagram.

Thus the time paths for our variables is divergent without cycles.

(iii) In this example

$$\left.\frac{\partial y_2'}{\partial y_1}\right|_{y_1'=0} = -\frac{f_1}{f_2} = -\frac{(+\text{ve})}{(+\text{ve})} = -\text{ve}$$

so the slope of $y_1' = 0$ is negative, so we have two possible phase diagrams:



The directional arrows are obtained by considering

$$\frac{\partial y_1'}{\partial y_1} = f_1 > 0;$$

so as y_1 increases y_1' must increase going from negative to zero to positive.

Thus the directional arrows for y_1 are as given in the diagrams.

By the same reasoning, as y_2 increases y_2' must increase going from negative to zero to positive thus giving the directional arrows for y_2 as shown in the diagrams.

Phase diagram (a) gives rise to divergent time paths for our variables without cycles.

Phase diagrams (b) gives convergent time paths for our variables, without cycles in sections (3) and (4) but divergent time paths without cycles in sections (1) and (2).

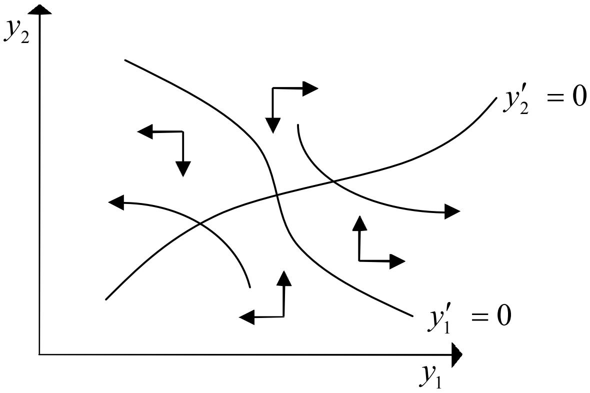
(iv) In this example

$$\left.\frac{\partial y_2'}{\partial y_1}\right|_{y_1'=0} = -\frac{f_1}{f_2} = -\frac{(+\text{ve})}{(+\text{ve})} = -\text{ve},$$

so the slope of $y_1' = 0$ is negative
and

$$\left.\frac{\partial y_2'}{\partial y_1}\right|_{y_2'=0} = -\frac{g_1}{g_2} = -\frac{(+\text{ve})}{(-\text{ve})} = +\text{ve},$$

so the slope of $y_2' = 0$ is positive. Thus we get the following phase diagram:



As

$$\frac{\partial y_1'}{\partial y_1} = f_1 > 0,$$

as we increase y_1, y_1' must also increase going from negative to zero to positive which gives the directional arrows for y_1 as shown in the diagram.

$$\frac{\partial y_2'}{\partial y_2} = f_2 < 0$$

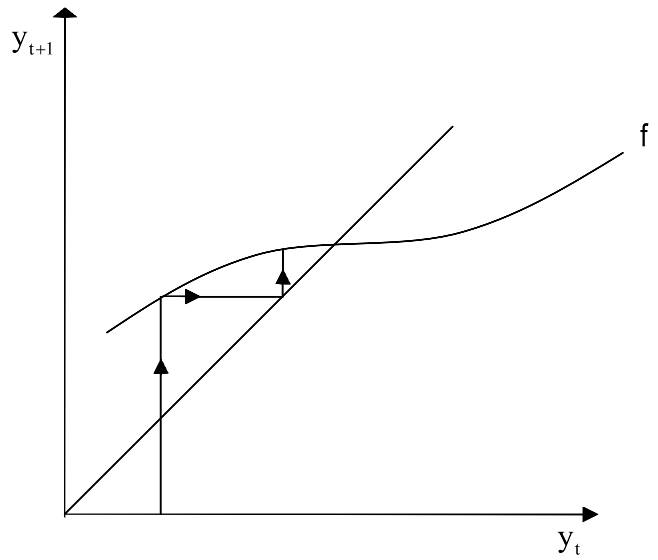
as we increase y_2, y_2' must decrease giving the directional arrow as shown in the diagram.

Thus the time paths of our variables diverge without cycles.

Exercises for Chapter 10

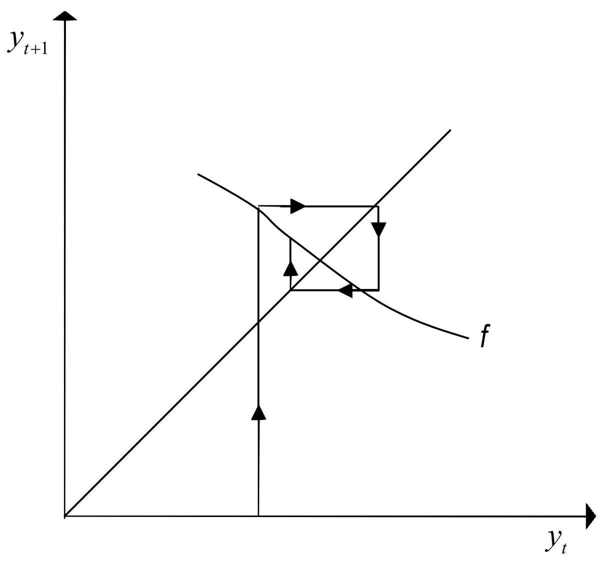
10.2

1. (i)



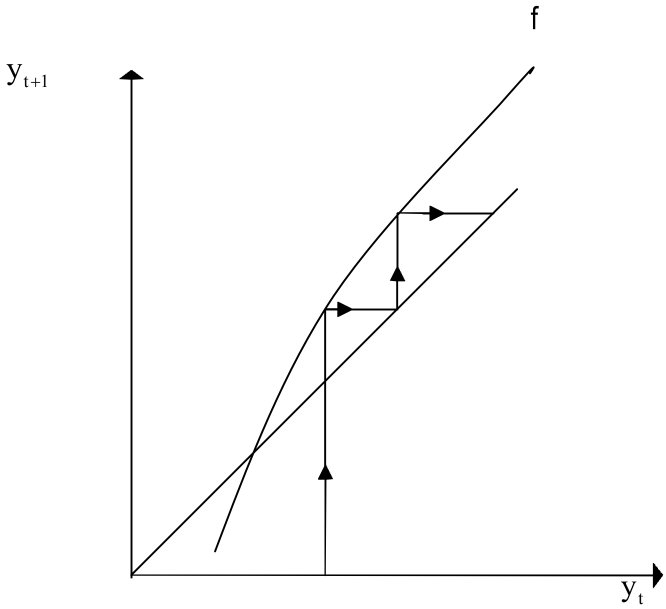
Clearly we require $f' > 0$ and $f' < 1$.

(ii)



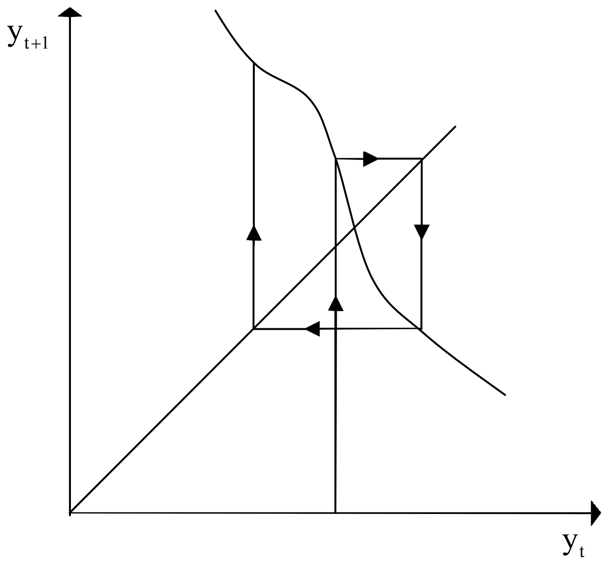
We require $f' < 0$ but $f' > -1$.

(iii)



We require $f' > 1$.

(iv)



We require $f' < -1$.

(a) $f = 10 + 5 \log y_t$ so $f' = \frac{5}{y_t} > 1$ for $0 < y_t < 1.$

The time path of *y* is divergent without oscillations.

(b) $f = 3 - 8 \cos y_t$ so $f' = 8 \sin y_t > 0$ for $0 < y_t < \pi,$ so the time path has no oscillations.

For $0.122\pi < y_t < -.878\pi,\ 8 \sin y_t > 1$ and the time path is divergent, otherwise convergent.

(c) $f = 10 + 8y_t^5$ so $f' = 40y_t^4 > 1$ for $y_t > 1$ so the time path has no oscillations and is divergent.

10.3

1. All the equations are second order linear difference equations with constant coefficients.

(i) We have

$$\Delta y(x) = y(x+1) - y(x),\ \Delta^2 y(x) = y(x+2) - 2y(x+1) + y(x)$$

so writing our difference equation in standard format gives

$$y(x+2) - 4yy(x+1) + 3y(x) = 0$$

which is a homogeneous equation whose auxiliary equation is

$$m^2 - 4m + 3 = 0,$$

with roots $m_1 = 1$ and $m_2 = 3.$ The general solution is

$$y = c_1 + c_2 3^x,\ c_1 \text{ and } c_2 \text{ arbitrary constants.}$$

(ii) We have

$$\nabla y(x-1) = y(x-1) - y(x-2)$$
$$\nabla^2 y(x) = y(x) - 2y(x-1) + y(x-2)$$

so our difference equation is

$$y(x) - 4y(x-1) + 4y(x-2) = x^2$$

with auxiliary equation

$$m^2 - 4m + 4 = 0.$$

This equation has repeated roots

$$m_1 = m_2 = 2$$

so the general solution to the homogeneous equation is

$$y = (c_1 + c_2 x)2^x.$$

For a particular solution try

$$y = a_0 + a_1 x + a_2 x^2.$$

Substituting into our equation gives

$$a_0 + a_1 x + a_2 x^2 - 4\left[a_0 + a_1 (x-1) + a_2 (x-1)^2 \right]$$
$$+4\left[a_0 + a_1 (x-2) + a_2 (x-2)^2 \right] = x^2.$$

Equating coefficients gives

$$a_0 - 4a_1 + 12a_2 = 0$$
$$a_1 - 8a_2 = 0$$
$$a_2 = 1$$
$$\Rightarrow a_1 = 8,\ a_0 = 20,$$

so a particular solution to the nonhomogeneous equation is

$$y = 20 + 8x + x^2$$

and the general solution is

$$y = 20 + 8x + x^2 + (c_1 + c_2 x)2^x.$$

(iii) Writing our equation in standard format gives

$$y(x+2) - 2y(x+1) + 2y(x) = 0$$

which has an auxiliary equation

$$m^2 - 2m + 2 = 0.$$

The roots of this equation are conjugate complex numbers

$$m_1 = 1 + i \quad \text{and} \quad m_2 = 1 - i$$

so the general solution to the homogeneous equation is

$$y = 2^{\frac{x}{2}}\left(c_1 \cos\frac{\pi}{4}x + c_2 \sin\frac{\pi}{4}x \right).$$

For a particular solution of the nonhomogeneous equation try

$$y = a_0 + a_1 x + a_2 x^2.$$

Substituting into our differential equation gives

$$a_0 + a_1 (x+2) + a_2 (x+2)^2 - 2\left[a_0 + a_1 (x+1) + a_2 (x+1)^2 \right]$$
$$+2\left(a_0 + a_1 x + a_2 x^2 \right) = 6x + x^2.$$

Equating coefficients gives

$$a_0 + 2a_2 = 0,\quad a_1 = 6,\quad a_2 = 1$$

so a particular solution to the nonhomogeneous equation is

$$y = -2 + 6x + x^2$$

and the general solution is

$$y = -2 + 6x + x^2 + 2^{\frac{x}{2}}\left(c_1 \cos\frac{\pi}{4}x + c_2 \sin\frac{\pi}{4}x\right).$$

(iv) The auxiliary equation is

$$m^2 - 4m + 4 = 0$$

which has repeated roots $m_1 = m_2 = 2$ so the general solution to the homogeneous equation is

$$y = c_1 2^x + c_2 x 2^x, \quad c_1 \text{ and } c_2 \text{ arbitrary constants.}$$

For $f_1(x) = 2^x$ try $y = cx^2 2^x$ which gives

$$cx^2 2^x - 4c(x-1)^2 2^{x-1} + 4(x-2)^2 2^{x-2} = 2^x.$$

Equating the coefficient of 2^x gives

$$2c = 1 \Rightarrow c = \frac{1}{2}.$$

For $f_2(x) = 8x$ try $y = a_0 + a_1 x$ which gives

$$a_0 + a_1 x - 4\left[a_0 + a_1(x-1)\right] + 4\left[a_0 + a_1(x-2)\right] = 8x.$$

Equating coefficients gives

$$a_0 - 4a_1 =, \quad a_1 = 8, \quad a_0 = 32.$$

Combining we have that a particular solution for the nonhomogeneous equation is

$$y = cx^2 2^{x-1} + 32 + 8x$$

and the general solution is

$$y = cx^2 2^{x-1} + 32 + 8x + c_1 2^x + c_2 x 2^x.$$

(v) The auxiliary equation is

$$m^2 - 3m + 2 = 0$$

which has roots $m_1 = 2, \ m_2 = 1$ so the general solution to the homogeneous equation is $y = c_1 2^x + c_2, \ c_1 \ and \ c_2$ arbitrary constants.

For $f_1(x) = c_2$ try $y = a_0 x$ which gives

$$2a_0(x+2) - 6a_0(x+1) + 4a_0 x = 5 \Rightarrow a_0 = -\frac{5}{2}.$$

For $f_2(x) = 2^x$ try $y = cx2^x$ which gives

$$2c(x+2)2^{x+2} - 6cx2^{x+1} + 4cx2^x = 2^x.$$

Equating the coefficients of 2^x gives

$$16c = 11 \Rightarrow c = \frac{1}{16}.$$

Combining we have that a particular solution to the nonhomogeneous equation is

$$y = -\frac{5}{2}x + x2^{x-4}$$

and the general solution is

$$y = c_1 2^x + c_2 - \frac{5}{2}x + x2^{x-4}.$$

Now

$$y(0) = c_1 + c_2 = 0$$
$$y(1) = 2c_1 + c_2 - 5/2 + 1/8 = 5/8$$

$$\Rightarrow c_2 = -3, \quad c_1 = 3.$$

Hence the required solution is

$$y = 3.2^x - 3 - \frac{5}{2}x + x2^{x-4}.$$

2. In this model we have the following second order linear difference equation with constant coefficients

$$y(t) - \beta(2+\alpha)y(t-1) + \beta(1+\alpha)y(t-2) = v$$

which has an auxiliary equation

$$m^2 - \beta(2+\alpha)m + \beta(1+\alpha) = 0.$$

(i) The roots of the auxiliary equation are real and distinct if

$$(2+\alpha)^2 \beta^2 > 4(1+\alpha)\beta \Rightarrow \beta > \frac{4(1+\alpha)}{(2+\alpha)^2} \text{ as } \beta > 0.$$

$$\text{Now } \frac{4(1+\alpha)}{(2+\alpha)^2} \geq \frac{2(2+\alpha)}{(2+\alpha)^2} \text{ as } \alpha \geq 0$$
$$= 2/(2+\alpha),$$

so our condition on β implies that

$$\beta > 2/(2+\alpha).$$

Let $m_1 + m_2$ be the roots of the auxiliary equation. Then

$$m_1 + m_2 = (2+\alpha)\beta > 2$$

by our implied condition on β and

$$m_1 m_2 = \beta(1+\alpha) > 0,$$

so both roots are positive and at least one root is ≥ 1. Hence the time path is divergent and nonoscillatory.

(ii) The roots of the auxiliary equation are real and equal if

$$\beta = \frac{4(1+\alpha)}{(2+\alpha)^2},$$

in which case the common root is

$$m = (2+\alpha)\beta/2 > 1$$

from our implied condition on β. Again the time path for output is divergent and nonoscillatory.

(iii) We get cycles in this model only if the roots of the auxiliary equation are conjugate complex and the condition that ensures this is

$$\beta < \frac{4(1+\alpha)}{(2+\alpha)^2}.$$

The absolute value of these roots will be

$$\sqrt{\frac{(2+\alpha)^2\beta^2}{4} + \frac{\beta(1+\alpha) - (2+\alpha)^2\beta^2}{4}} = \sqrt{\beta(1+\alpha)}$$

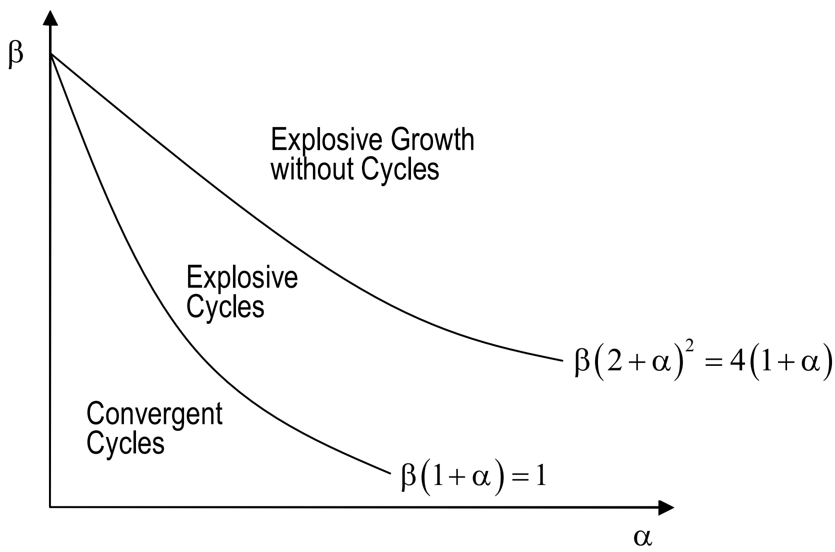
For convergent cycles we require then

$$\sqrt{\beta(1+\alpha)} < 1 \Rightarrow \beta(1+\alpha) < 1 \text{ as } \beta(1+\alpha) > 0.$$

For divergent cycles we require

$$\beta(1+\alpha) > 1.$$

(iv.)



3. When time is treated as a discrete variable we regard our economic variables as changing only after the passage of discrete intervals of time. If $y(x)$ is our economic variables then $y(x)$ is defined for $x = 0, 1, 2, \ldots$

For continuous time the economic variables are regarded as changing continuously.

In a second order linear differential equation with constant coefficients cycles can only occur in the case where the roots of the auxiliary equation are conjugate complex numbers. In a second order linear difference equation with constant coefficients, cycles can occur in all three cases: where the roots of the auxiliary equation are real and distinct, real and equal, and conjugate complex. For example, in the first case if m_1 and m_2 are the distinct roots and $|m_1| > |m_2|$ with m_1 negative then eventually m_1 dominates and we have cycles. This means that for difference equations, unlike differential equations we must look at the conditions on the parameters of our model that ensure

(i) m_1 and m_2 are both positive

(ii) m_1 and m_2 are both negative

(iii) m_1 negative, m_2 positive but $|m_1| > m_2$.

10.5

(i) Substituting into the definitional equation we have

$$Y_t = a + bY_t + v\left(Y_{t-1} - Y_{t-2}\right),$$

that is,

$$Y_t - cY_{t-1} + cY_{t-2} = \frac{a}{(1-b)} \text{ where } c = \frac{v}{(1-b).}$$

The auxiliary equation is

$$m^2 - cm + c = 0.$$

The potential equilibrium is found by setting $Y_t = Y_{t-1} = Y_{t-2} = \bar{Y}$ in our definitional equation to obtain

$$\bar{Y} = \frac{a}{\left(1-b\right)}.$$

(ii) The auxiliary equation has real and distinct roots if $c^2 - 4c > 0$,
that is,

$$c\left(c-4\right) > 0 .$$

As c is positive, this requires $c > 4$.

If m_1 and m_2 are the roots of the auxiliary equation then

$$m_1 + m_2 = c > 0$$
$$m_1 m_2 = c^2 - c^2 + 4c = 4c > 0.$$

so both roots are positive and the time path for *Y* is not subject to oscillations.

The larger root is $\dfrac{c+\sqrt{c(c-4)}}{2} > 1$ as $c > 4,$ so we have explosive growth for *Y*.

(iii) Real and equal roots of the auxiliary equation if

$$c(c-4) = 0 \Rightarrow c = 4.$$

The common root then is

$$m = \frac{c}{2} = 2 > 1,$$

so we have explosive growth with no oscillations.

(iv) We have cycles if the roots of the auxiliary equation are conjugate complex numbers. The condition needed to ensure this case is

$$c(c-4) < 0 \Rightarrow c < 4,$$

and the roots will then be given by

$$m_1, m_2 = \frac{c \pm \sqrt{c(4-c)}}{2}\ \text{i}.$$

The absolute value of these roots is

$$\sqrt{\frac{c^2}{4} + \frac{4c - c^2}{4}} = \sqrt{\text{c}}.$$

If $\sqrt{c} < 1 \Rightarrow c < 1$ then we have convergent cycles. If $c > 1$ divergent cycles.

v) $c = \frac{v}{s}$, where *v* is the accelerator and $s = 1 - b$ is the marginal propensity to save. Then we have the following cases:

(a) $\underline{v \geq 4s}$

Explosive growth for *Y* with no oscillations

(b) $\underline{v < 4s}$

Oscillations for *Y*

$v > s$ explosive oscillations

$v < s$ damped oscillations.

10.6

1. (i) Substituting $\Delta^2 y(x) = y(x+2) - 2y(x+1) + y(x)$ and $\Delta y(x) = y(x+1) + y(x)$ into the equation gives the difference equation in standard format:

$$y(x+2) - 3y(x+1) + 2y(x) = 6.$$

The auxiliary equation is

$$\beta^2 - 3\beta + 2 = 0.$$

Consider

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3,$$

so by Schur's theorem the time path for y will be divergent.

(ii) Substituting $\nabla^2 y(x) = y(x) - 2y(x-1) + y(x-2)$ and $\nabla y(x) = y(x) - y(x-1)$ gives the equation in standard format:

$$-y(x) + y(x-1) + y(x-2) = -7.$$

Consider

$$\begin{vmatrix} 1 & -1 \\ -1 & - \end{vmatrix} = 0,$$

so by Schur's theorem the time path for *y* is divergent.

(iii) The auxiliary equation is

$$3m^2 - 2m + 1 = 0.$$

Consider

$$\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} = 8$$

$$\begin{vmatrix} 3 & 0 & 1 & -2 \\ -2 & 3 & 0 & 1 \\ 1 & 0 & 3 & -2 \\ -2 & 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 1 & -2 \\ -2 & 3 & 0 & 1 \\ -8 & 0 & 0 & 4 \\ -2 & 1 & 0 & 3 \end{vmatrix}$$

$$= 1(-1)^{1+3} \begin{vmatrix} -2 & 3 & 1 \\ -8 & 0 & 4 \\ -2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 0 & -8 \\ -8 & 0 & 4 \\ -2 & 1 & 3 \end{vmatrix}$$

$$= 1(-1)^{3+2} \begin{vmatrix} 4 & -8 \\ -8 & 4 \end{vmatrix}$$

$$= 48,$$

so by Schur's theorem the time path for y is convergent.

3. We must first write our difference equation in its alternative form. Now

$$\Delta^3 y(x) = y(x+3) - 3y(x+2) + 3y(x+1) - y(x)$$
$$\Delta^2 y(x) = y(x+1) - 2y(x+1) + y(x)$$
$$\Delta y(x) = y(x+1) - y(x),$$

so substituting back into our difference equation gives

$$\begin{aligned} & y(x+3) - 3y(x+2) + 3y(x+1) - y(x) \\ & + a_1\left[y(x+2) - 2y(x+1) + y(x)\right] \\ & + a_2\left[y(x+1) - y(x)\right] \\ & + a_3 y(x) = \mathrm{c}. \end{aligned}$$

Collecting terms gives

$$\begin{aligned} & y(x+3) + y(x+2)(a_1 - 3) + y(x+1)(3 - 2a_1 + a_2) \\ & \qquad + y(x)(a_1 + a_3 - a_2 - 1) = \mathrm{c}. \end{aligned}$$

The auxiliary equation is then

$$\begin{aligned} & m^3 + (a_1 - 3)m^2 + (3 - 2a_1 + a_2)m \\ & \qquad\qquad + (a_1 + a_3 - a_2 - 1) = 0. \end{aligned}$$

Schur's theorem states that the roots of this polynomial will all have absolute values less than 1 if and only if

$$\Delta_1 = \begin{vmatrix} 1 & a_1 + a_3 - a_2 - 1 \\ a_1 + a_3 - a_2 - 1 & 1 \end{vmatrix}$$

$$\Delta_2 = \left|\begin{array}{cc|cc} 1 & 0 & a_1 + a_3 - a_2 - 1 & 3 - 2a_1 + a_2 \\ a_1 - 3 & 1 & 0 & a_1 + a_3 - a_2 - 1 \\ \hline a_1 + a_3 - a_2 - 1 & 0 & 1 & a_1 - 3 \\ 3 - 2a_1 + a_2 & a_1 + a_3 - a_2 - 1 & 0 & 1 \end{array}\right|$$

$$\Delta_3 = \left|\begin{array}{ccc|ccc} 1 & 0 & 0 & a_1 + a_3 - a_2 - 1 & 3 - 2a_1 + a_2 & a_1 - 3 \\ a_1 - 3 & 1 & 0 & 0 & a_1 + a_3 - a_2 - 1 & 3 - 2a_1 + a_2 \\ 3 - 2a_1 + a_2 & a_1 - 3 & 1 & 0 & 0 & a_1 + a_3 - a_2 - 1 \\ \hline a_1 + a_3 - a_2 - 1 & 0 & 0 & 1 & a_1 - 3 & 3 - 2a_1 + a_2 \\ 3 - 2a_1 + a_2 & a_1 + a_3 - a_2 - 1 & 0 & 0 & 1 & a_1 - 3 \\ a_1 - 3 & 3 - 2a_1 + a_2 & a_1 + a_3 - a_2 - 1 & 0 & 0 & 1 \end{array}\right|$$

are all positive.

Exercises for Chapter 11

11.3

1. (i) The Hamiltonian is

$$H = u - u^2 + \lambda u,$$

which is concave in *u*. Moreover

$$\frac{\partial H}{\partial u} = 1 - 2u + \lambda,$$

$$\frac{\partial^2 H}{\partial u^2} = -2,$$

so the value of *u* that minimizes *H* is

$$u = \frac{(\lambda + 1)}{2}.$$

Now

$$\lambda' = -\frac{\partial H}{\partial x} = 0,$$

so $\lambda = c,$ an arbitrary constant, and

$$x' = u = \frac{(c_1 + 1)}{2}.$$

Hence $x = \frac{c_2 + (c_1 + 1)t}{2},$ where c_2 is another arbitrary constant. But

$$x(0) = 4 = c_2$$

$$x(1) = 2 = \frac{4 + (c_1 + 1)}{2} \Rightarrow c_1 = -5,$$

so the optimal time paths are

$$x^*(t) = 4 - 2t$$

$$u^*(t) = -2$$

$$\lambda^*(t) = -5.$$

(ii) The Hamiltonian is

$$H = -4u - u^2 + \lambda(x - 2u)$$

which is concave in *u* and *x*. Now

$$\frac{\partial H}{\partial u} = -4 - 2u - 2\lambda,$$

$$\frac{\partial^2 H}{\partial u^2} = -2,$$

so the value of *u* that maximizes *H* is

$$u = -2 - \lambda.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -\lambda,$$

so $\lambda = c_1 e^{-t}$,

where c_1 is an arbitrary constant. The state equation is

$$x' = x - 2u = x + 4 + 2c_1 e^{-t}$$

so we have the first order linear differential equation

$$x' - x = 4 + 2c_1 e^{-t}.$$

A particular solution to this equation is

$$x = e^t \int_0^t e^{-s}\left(4 + 2c_1 e^{-s}\right) = e^t \int_0^t 4e^{-s} + 2c_1 e^{-2s} ds = e^t \left[-4e^{-s} - c_1 e^{-2s}\right]_0^t$$

$$= e^t\left(-4e^{-t} - c_1 e^{-2t} + 4 + c_1\right) = -4 - c_1 e^{-t} + \left(4 + c_1\right)e^t,$$

so the general solution is

$$x = c_2 e^t - 4 - c_1 e^{-t} + \left(4 + c_1\right)e^t = -4 - c_1 e^{-t} + \left(4 + c_1 + c_2\right)e^t,$$

where c_2 is an arbitrary constant. Using the end point conditions we have

$$x(0) = 2 \Rightarrow -4 - c_1 + 4 + c_1 + c_2 = 2$$

$$\Rightarrow c_2 = 2,$$

and

$$x(1) = 10 \Rightarrow -4 - c_1 e^{-1} + \left(6 + c_1\right)e = 10 \Rightarrow c_1\left(e - e^{-1}\right) = 14 - 6e \Rightarrow c_1 = -0.9827.$$

So our optimal time paths are

$$x^*(t) = -4 + 0.9827e^{-t} + 5.0173e^t$$

$$\lambda^*(t) = -0.9827e^{-t}$$

$$u^*(t) = -2 + 0.9827e^{-t}.$$

(iii) The Hamiltonian is

$$H = x - u^2 + \lambda\left(u - 3x\right),$$

which is concave in *x* and *u*. As

$$\frac{\partial H}{\partial u} = -2u + \lambda$$

and

$$\frac{\partial^2 H}{\partial u^2} = -2,$$

the value of *u* that maximizes *H* is

$$u = \frac{\lambda}{2}.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -1 + 3\lambda$$

yielding the differential equation

$$\lambda' - 3\lambda = -1$$

whose general solution is

$$\lambda = \frac{1}{3} + c_1 e^{3t},$$

with c_1 being an arbitrary constant. The state equation can then be written as

$$x' + 3x = \frac{1}{6} + \frac{c_1}{2} e^{et}.$$

The general solution to the homogeneous form of this equation is $x = c_2 e^{-3t}$ with c_2 an arbitrary constant. A particular solution to the nonhomogeneous equation can be obtained as in 1 (ii). Alternatively we can use the method of undetermined coefficients and use as our trial particular solution

$$x = k_0 + k_1 e^{3t}.$$

Then $x' = 3k_1 e^{3t}$ and substituting into our equation we get

$$6k_1 e^{3t} + 3k_0 = \frac{1}{6} + \frac{c_1}{2} e^{3t}$$

$$\Rightarrow k_0 = \frac{1}{18}, \; k_1 = \frac{c_1}{12}.$$

Thus the general solution to the state equation is

$$x = c_2 e^{-3t} + \frac{c_1 e^{3t}}{12} + \frac{1}{18}.$$

Using the end point conditions we have

$$x(0) = 1 \Rightarrow c_2 + \frac{c_1}{12} + \frac{1}{18} = 1 \qquad (1)$$

$$x(1) = 4 \Rightarrow c_2 e^{-3} + \frac{c_1 e^3}{12} + \frac{1}{18} = 4. \qquad (2)$$

From equation (1) we have $c_2 = \frac{17}{18} - \frac{c_1}{12}$. Substituting into equation (2) we have

$$\left(\frac{17}{18} - \frac{c_1}{12}\right) e^{-3} + \frac{c_1 e^3}{12} = \frac{71}{18}$$

$$\Rightarrow \frac{c_1}{12}\left(e^3 - e^{-3}\right) = \frac{1}{18}\left(71 - 17e^{-3}\right)$$

$$\Rightarrow c_1 = \frac{2\left(71 - 17e^{-3}\right)}{3\left(e^3 - e^{-3}\right)} = \frac{140.3071}{60.1071} = 2.3343$$

$$c_2 = 0.9444 - 0.1945 = 0.7499.$$

Thus our optimal time paths are

$$x^*(t) = 0.7499e^{-3t} + 0.1945e^{3t} + 0.0556$$
$$\lambda^*(t) = 0.3333 + 2.3343e^{3t}$$
$$u^*(t) = 0.1667 + 1.1672e^{3t}.$$

2. (i) The consumer's problem is

$$\max \int_0^{40} \log C(t) e^{-0.02t} dt$$

subject to

$$W(t) = 0.05W(\mathrm{t}) - C(t)$$
$$W(0) = 1, \; W(40) = 0.5,$$

W and *C* expressed in millions of dollars. The control variable is consumption $C(t)$, the state variable is wealth $W(t)$.

(ii) The Hamiltonian is

$$\mathrm{H} = \log Ce^{-0.02t} + \lambda(0.05W - C).$$

The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{e^{-0.02t}}{C} - \lambda = 0$$

as $\frac{\partial^2 H}{\partial C^2} < 0,$

$$\lambda' = -\frac{\partial H}{\partial w} = -0.05\lambda$$
$$W' = 0.05\, W - C$$
$$W(0) = 1, \; W(40) = 0.5.$$

As the Hameltonian is concave in *C* and *W*, (the sum of concave functions) these conditions are also sufficient.

(iii) The value of *C* that maximizes the Hameltonian is

$$C = \frac{e^{-0.02t}}{\lambda}.$$

From the costate equation we have the first order linear differential equation

$$\lambda' + 0.05\lambda = 0$$

which has a general solution given by

$$\lambda(t) = c_1 e^{-0.05t}$$

where c_1 is an arbitrary constant, so

$$C = \frac{1}{c_1} e^{0.03t} = c_2 e^{0.03t},$$

say.
The state equation gives the differential equation

$$W' - 0.05W = -c_2 e^{0.03t}$$

which has a general solution given by

$$W = c_3 e^{0.05t} + 50c_2 \left(e^{0.03t} - e^{0.05t} \right).$$

$$W(0) = 1 \qquad \Rightarrow c_3 = 1$$

$$W(40) = 0.5 \Rightarrow 0.5 = e^2 + 50c_2 \left(e^{1.2} - e^2 \right) \Rightarrow c_2 = 0.03386.$$

The consumer's optimal time path for consumption is then

$$C^*(t) = 0.03386 e^{0.03t}$$

and the accompanying time path for the consumer's wealth is

$$W^*(t) = 1.693 e^{0.03t} - 0.693 e^{0.05t}.$$

3. (i) Substituting $u(t)$ for $x'(t)$ in the objective function we have

$$\text{Maximize} \int_a^b f\left(u(t), x(t), t\right) dt$$

$$\text{subject to } x'(t) = u(t)$$

$$x(a) = x_a, \; x(b) = x_b.$$

(ii) The Hamiltonian for this problem is

$$H = f\left(u(t), x(t), t\right) + \lambda u(t),$$

and the necessary conditions for an optimal solution are

$$\frac{\partial H}{\partial u} = \frac{\partial f(u, x, t)}{\partial u} + \lambda = 0 \tag{1}$$

$$\lambda' = \frac{\partial H}{\partial x} = -\frac{\partial f(u, x, t)}{\partial x} \tag{2}$$

$$x' = u \tag{3}$$

$$x(a) = x_a, \quad x(b) = x_b.$$

Differentiating both sides of equation (1) with respect to *t* yields

$$\frac{d}{dt}\left(\frac{\partial f(u,x,t)}{\partial u}\right) = -\lambda'.$$

Using equations (2) and (3) renders Euler's equation

11.4

1. (i) The Hamiltonian is

$$H = 2x - u - \frac{u^2}{2} + \lambda(x-u)$$

which is concave in *x* and *u* and as

$$\frac{\partial H}{\partial u} = -1 - u - \lambda = 0$$

$$\frac{\partial^2 H}{\partial u^2} = -1,$$

the value of *u* that maximizes *H* is

$$u = -(1+\lambda).$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -2 - \lambda$$

which has a solution given by

$$\lambda = -2 + c_1 e^{-t}$$

with c_1 an arbitrary constant. As $x(10)$ is free the transversality condition is

$$\lambda(10) = 0 \Rightarrow 0 = -2 + c_1 e^{-10} \Rightarrow c_1 = 2e^{10},$$

so

$$\lambda = -2 + 2e^{10-t}.$$

The state equation is

$$x' = x - u = x - 1 + 2e^{10-t}$$

which is a nonhomogeneous linear differential equation of degree 1 whose general solution is

$$x = c_2 e^{t} - e^{10-t} + 1,$$

with c_2 an arbitrary constant. But

$$x(0) = 5 \Rightarrow c_2 = 4 + e^{10}.$$

Hence our optimal time paths are

$$x^*(t) = 4e^t + e^{t+10} - e^{10-t} + 1$$
$$\lambda^*(t) = -2 + 2e^{10-t}$$
$$u^*(t) = 1 - 2e^{10-t}.$$

(ii) The Hamiltonian is

$$H = 4x - 3u - 2u^2 + \lambda(x+u)$$

which is concave in x and u and has derivatives

$$\frac{\partial H}{\partial u} = -3 - 4u + \lambda, \quad \frac{\partial^2 H}{\partial u^2} = -4,$$

so the value of u that maximizes *H* is

$$u = \frac{(x-3)}{4}.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -(4+\lambda),$$

which has as its general solution

$$\lambda = -4 + c_1 e^{-t}.$$

As *x*(1) is free

$$\lambda(1) = 0 \Rightarrow c_1 = 4e$$

so we have

$$\lambda = -4 + 4e^{1-t}.$$

The state equation is

$$x' = x + u = x - \frac{7}{4} + e^{1-t}$$

which has as its solution

$$x = c_2 e^t + \frac{7}{4} - \frac{1}{2}e^{1-t}.$$

As $x(0) = 2$ we have $c_2 = \frac{1}{4} + \frac{1}{2}e$ so our optimal solutions are

$$x^*(t) = \frac{1}{4}e^t + \frac{1}{2}e^{t+1} - \frac{1}{2}e^{1-t} + \frac{7}{4}$$
$$\lambda^*(t) = -4 + 4e^{1-t}$$
$$u^*(t) = -\frac{7}{4} + e^{1-t}.$$

(iii) Proceeding as we did for exercise 1.(iii) of Section 11.3 we have

$$\lambda = \frac{1}{3} + c_1 e^{3t}$$

where c_1 is an arbitrary constant. Try $\lambda(1) = 0.$ Then

$$c_1 = -\frac{e^{-3}}{3}$$

and using the end point condition $x(0) = 1$ as we did in the previous exercise we have

$$c_2 = \frac{17}{18} + \frac{e^{-3}}{36}.$$

These values for the constants in our solutions would yield

$$x^*(t) = \left(\frac{17}{18} + \frac{e^{-3}}{36}\right)e^{-3t} - \frac{e^{3(t-1)}}{36} + \frac{1}{18}.$$

But then

$$x^*(1) = \left(\frac{17}{18} + \frac{e^{-3}}{36}\right)e^{-3} + \frac{1}{36}$$

which clearly is not greater or equal to 3.
Rerunning the problem as we did for exercise 1. (iii) of Section 11.3 with $x(0) = 1$ and $x(1) = 3,$ we get after a little arithmetic that

$$c_1 = \frac{2\left(53 - 17e^{-3}\right)}{3\left(e^3 - e^{-3}\right)} = \frac{104.3071}{60.1071} = 1.7354$$

$$c_2 = 0.9444 - 0.1446 = 0.7999.$$

Thus our optimal time paths are

$$x^*(t) = 0.7998e^{-3t} + 0.1446e^{3t} + 0.0556$$
$$\lambda^*(t) = 0.3333 + 1.7354e^{3t}$$
$$u^*(t) = 0.1667 + 0.8677e^{3t}.$$

(iv) The Hamiltonian is

$$H = tu + x - u^2 + \lambda(u + x)$$

which is concave in *u* and *x*. Differentiating with respect to *u* we have

$$\frac{\partial H}{\partial u} = t - 2u + \lambda, \quad \frac{\partial^2 H}{\partial u^2} = -2,$$

so the value of *u* that maximizes *H* is

$$u = \frac{(t + \lambda)}{2}.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -(1 - \lambda)$$

which has as its solution

$$\lambda = -1 + c_1 e^{-t},$$

where c_1 is an arbitrary constant. The state equation is

$$x' = \frac{u + x = \left(t - 1 + c_1 e^{-t}\right)}{2 + x},$$

which has as its general solution

$$x = c_2 e^t - \frac{c_1}{4} e^{-t} - \frac{t}{2}$$

where c_2 is another arbitrary constant.
Try $\lambda(1) = 0 \Rightarrow c_1 = e$ which would yield

$$x = c_2 e^t - \frac{e^{1-t}}{4} - \frac{t}{2}.$$

From the end point condition $x(0) = 1$ we have that

$$1 = c_2 - \frac{e}{4} \Rightarrow c_2 = 1 + \frac{e}{4} = 1.6796$$

rendering

$$x = 1.6796e^t - \frac{e^{1-t}}{4} - \frac{t}{2}.$$

Now

$$x(1) = 1.6796e - \frac{3}{4} = 3.8156 > 2$$

so our option yields the optimal time paths which are

$$x^*(t) = 1.6796e^t - \frac{e^{1-t}}{4} - \frac{t}{2}$$

$$\lambda^*(t) = -1 + e^{1-t}$$

$$u^*(t) = \frac{\left(t - 1 + e^{1-t}\right)}{2}.$$

(v) The Hamiltonian is

$$H = 10x - 20u + \lambda(x + u) = (10 + \lambda)x + u(\lambda - 20)$$

which being linear in *x* and *u* is concave in *x* and *u*. Moreover it is an increasing function in *u* for $\lambda > 20$ but a decreasing function in *u* for $\lambda < 20$. Thus

$$u^* = 3 \quad \lambda > 20$$

$$u^* = 0 \quad \lambda < 20.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -(10 + \lambda)$$

which has as its general solution

$$\lambda = -10 + c_1 e^{-t},$$

where c_1 is an arbitrary constant. But as $x(1)$ is free $\lambda(1)=0$ so

$$c_1 = 10e = 27.1828$$

and

$$\lambda^* = -10 + 27.1828e^{-t}.$$

This is a monotonically decreasing function in *t* and its maximum value on our time horizon is

$$\lambda^*(0) = -10 + 27.1828 = 17.1828.$$

Thus λ^* is always less than 20 and

$$u^* = 0.$$

The state equation is

$$x' = x + u = x$$

so the general solution for *x* is

$$x = c_2 e^t$$

where c_2 is an arbitrary constant. But $x(0)=2$ so $c_2 = 2$. The optimal time paths then are

$$u^* = 0$$

$$x^*(t) = 2e^t$$

$$\lambda^*(t) = -10 + 27.1828e^{-t}.$$

(vi) The Hamiltonian is

$$H = 10x - 50u + \lambda(x+u) = (10-\lambda)x + u(\lambda - 50)$$

which is concave in *x* and *u* and is a monotonically increasing function in u for $\lambda > 50$ but a monotonically decreasing function in *u* for $\lambda < 50$. So

$$u^* = 4 \quad \lambda > 50$$

$$u^* = 2 \quad \lambda < 50.$$

The costate equation is

$$\lambda' = -\frac{\partial H}{\partial x} = -(10+\lambda)$$

which has as its solution

$$\lambda = -10 + c_1 e^{-t}.$$

As $x(2)$ is free $\lambda(2) = 0$ so

$$c_1 = 10e^2 = 73.8904.$$

It follows that

$$\lambda^*(t) = -10 + 73.8904e^{-t},$$

a monotonically decreasing function in *t* starting from the value $\lambda^*(0) = 63.8904,$ so λ^* becomes 50 at $t = \overline{t}$ where

$$\lambda^*(\overline{t}) = 50 = -10 + 73.8904e^{-\overline{t}}$$

rendering $\overline{t} = 0.2082$. Thus

$$u^* = 4 \quad 0 \le t \le 0.2082$$
$$u^* = 2 \quad 0.2082 < t \le 2.$$

The state equation is

$$x' = x + u^* .$$

For the time interval [0, 0.2082]

$$x = 4 + c_1 e^{t}$$

and as $x(0) = 1,\ c_1 = -3.$ So for this interval

$$x^* = 4 - 3e^t.$$

For the interval (0.2082, 2]

$$x = 2 + c_2 e^t.$$

But

$$x^*(0.2082) = 4 - 3e^{0.2082} = 0.3056$$

so

$$0.3056 = 2 + c_2 e^{0.2082} \Rightarrow c_2 = -1.3758.$$

For this interval

$$x^* = 2 - 1.3758e^t .$$

2. (i) The consumer's problem is

$$\max\text{imize} \int_0^T \log C(t) e^{-\delta t} dt$$
$$\text{subject to } W' = rW - C$$
$$W(0) = W_0, \quad W(T) \ge 0.$$

(ii) The Hamiltonian is

$$H = \log C(t) e^{-\delta t} + \lambda(rW - C).$$

The function log *C* is concave in *C*, and the linear function $\lambda rW - \lambda C$ is concave in *W* and *C*. Hence *H* is concave in *W* and *C*.

(iii) The necessary conditions are

$$\frac{\partial H}{\partial C} = \frac{e^{-\delta t}}{C} - \lambda = 0$$

$$\lambda' = -\frac{\partial H}{\partial W} = -\lambda r$$

$$W' = rW - C$$

$$W(0) = W_0,\ W(T) \geq 0,\ \lambda(T) \geq 0\ \ \lambda(T)W(T) = 0.$$

Now $\frac{\partial^2 H}{\partial C^2} = -\frac{e^{-\delta t}}{C^2} < 0$

and *H* is concave in *W* and *C* so these conditions are also sufficient.

(iv) The value of *C* that maximizes *H* is

$$C = \frac{e^{-\delta t}}{\lambda}.$$

From the costate equation

$$\lambda = c_1 e^{-rt}$$

where c_1 is an arbitrary constant of integration and $c_1 = \lambda(0) = \lambda_0$ say. It follows that

$$C^*(t) = \frac{1}{\lambda_0} e^{t(r-\delta)}.$$

(v) As $\lambda(T) = \lambda_0 e^{-rt}$ and the transversality condition requires $\lambda(T) \geq 0$ we must have $\lambda_0 \geq 0$. Moreover the optimal solution for consumption rules out $\lambda_0 = 0$. Hence $\lambda_0 > 0$ and $\lambda(T) > 0$. The transversality condition then requires that $W(T) = 0$.

(vi) Optimal consumption will rise over time if $r > \delta$. That is if the rate of interest is greater than the consumer's personal rate of time preference. It will fall if $r < \delta$.

3. (i) The problem facing the community is

$$\text{maximize} \int_0^{100} \log au^{\alpha}\, dt$$

$$\text{subject to } x' = -u$$

$$x(0) = 1000,\quad x(100) \geq 0.$$

(ii) The Hamiltonian is

$$H = \log a + \alpha \log u - \lambda u$$

which is clearly concave in *u*. A set of necessary and sufficient conditions for an optimal solution is

$$\frac{\partial H}{\partial u} = \frac{\alpha}{u} - \lambda = 0$$

$$\lambda' = -\frac{\partial H}{\partial x} = 0$$

$$x' = -u$$

$$x(0) = 1000,\ x(100) \geq 0,\ \lambda(100) \geq 0,\ x(100)\lambda(100) = 0.$$

(iii) The value of *u* that maximizes *H* is $u = \frac{\alpha}{\lambda}$. From the costate equation

$$\lambda(t) = c_1$$

an arbitrary constant. The transversality condition requires that $c_1 \geq 0$ and $u = \frac{\alpha}{c_1}$ prohibits $c_1 = 0$ so $\lambda(100) = c > 0$. The state equation renders

$$x' = -\frac{\alpha}{c_1},$$

which has as its solution

$$x = -\frac{\alpha t}{c_1} + c_2,$$

where c_2 is another arbitrary constant. The end point condition $x(0) = 1000$ yields $c_2 = 1000$ and as $\lambda(100) > 0$ the transversality condition requires that

$$x(100) = -\frac{100\alpha}{c_1} + 1000 = 0$$

so $c_1 = \frac{\alpha}{100}$.

The optimal rate of extraction is then

$$u^* = 100$$

with the accompanying time path for the resource given by

$$x^*(t) = 1000 - 100t.$$

11.5

(i) The elasticity of marginal utility is

$$\eta(c) = -\frac{U''(c)c}{U'(c)}.$$

Now $U'(c) = \frac{1}{c}$ and $U''(c) = -\frac{1}{c^2}$ so $\eta(c) = 1$.

The instantaneous elasticity of substitution is

$$\sigma(c) = \frac{1}{\eta(c)} = 1.$$

(ii) We have

$$\frac{Y}{L} = \left(\frac{K}{L}\right)^{\frac{3}{4}} \left(\frac{L}{L}\right)^{\frac{1}{4}}$$

so

$$y = k^{\frac{3}{4}}.$$

The problem facing the economy is

$$\begin{aligned} &\text{Maximize} \quad \int_0^{\infty} e^{-0.05t} \log c \, dt \\ &\text{subject to} \; k' = k^{\frac{3}{4}} - c - 0.11k \\ &k(0) = k_0, \; k(\infty) \geq 0, \; 0 \leq c(t) \leq \phi(k). \end{aligned}$$

(iii) The Hamiltonian is

$$H = e^{-0.05t} \log c + \lambda \left(k^{\frac{3}{4}} - c - 0.11k \right)$$

and the current value Hameltonian is

$$\mathrm{H}_c = He^{0.05t} = \log c + \mu \left(k^{\frac{3}{4}} - c - 0.11k \right)$$

(iv) The necessary conditions for an optimal solution are

$$\frac{\partial H_C}{\partial C} = \frac{1}{C} - \mu = 0 \tag{1}$$

$$\mu' = -\frac{\partial H_C}{\partial k} + 0.05\mu = -\mu \left(\frac{3}{4} k^{-\frac{1}{4}} - 0.11 \right) + 0.05\mu = -\mu \left(\frac{3}{4} k^{-\frac{1}{4}} - 0.16 \right) \tag{2}$$

$$k' = k^{\frac{3}{4}} - \mathrm{c} - 0.11k$$

$$k(0) = k_0, \; \lim_{T \to \infty} \lambda(T) \geq 0, \; \lim_{T \to \infty} k(T) \geq 0, \; \lim_{T \to \infty} \lambda(T) k(T) = 0.$$

From equation (1)

$$\mu = \frac{1}{c} \text{ and } \mu' = -\frac{1}{c^2} c'$$

so equation (2) yields

$$c'(t) = \left(\frac{3}{4} k(t)^{-\frac{1}{4}} - 0.16 \right) c(t).$$

This equation with the state equation gives the required system of differential equations.

$$c'(t) = 0 \Rightarrow \frac{3}{4} k(t)^{-\frac{1}{4}} = 0.16.$$

Solving for k gives the steady state level if k

$$k^* = 482.7976.$$

The steady state level of *c* is found by substituting this value for $k(t)$ in the equation $k'(t) = 0,$ and solving for *c*. That is

$$c^{**} = (482.7976)^{\frac{3}{4}} - 0.11(482.7976) = 49.8893.$$

(v) At the steady state

$$\mu = \frac{1}{c^{**}} = 0.02, \quad \lambda = 0.02e^{-0.05t}$$

so

$$\lambda(T) = 0.02e^{-0.05T} \to 0 \qquad \text{as } T \to \infty$$

$$k(T) = k^* = 482.7976$$

$$k(T)\lambda(T) = 9.6774e^{-0.05T} \to 0 \text{ as } T \to \infty.$$

It follows that the transversality condition holds at the steady state.