# 19 Interest Rate Options

## Answers to Questions and Problems

1. Explain the relationship among mortgage-backed securities, mortgage pass-throughs, and collateralized mortgage obligations.

   The mortgage-backed security (MBS) is a security that gives the security owner rights to cash flows from mortgages that underlie the MBS. The MBS comes in two basic types: mortgage pass-through securities and collateralized mortgage obligations. The owner of a pass-through owns a fractional share of the entire pool of mortgages that underlie the pass-through security. The owner of a pass-through participates in all the cash flows from the underlying mortgage pool. A collateralized mortgage obligation (CMO) is another type of MBS. A CMO is created by decomposing the cash flows from a pool of mortgages. For example, some CMOs are backed by interest-only payments from a pool of mortgages, while other CMOs might be backed by principal-only payments from the same mortgage pool.

2. Explain the similarities and differences between the zero-coupon yield curve and the implied forward yield curve.

   Both the zero-coupon yield curve and the forward yield curve show rates of interest that apply to single future payments. The rates from both curves can be used to discount a single payment from a distant future date to an earlier date. The zero-coupon yield curve gives discount rates for discounting a distant payment to the present. The rates from the forward curve are essentially rates for discounting a distant payment from its payment date to a time one period earlier. For example, if the forward curve has annual rates, the forward rate for a period from year 7 to year 8 could be used to discount a payment to be received at year 8 back to year 7. Together, the single-period forward rates that constitute the forward yield curve can be used to discount a distant payment back to any earlier time.

3. Given the zero-coupon yield curve, explain how to find the implied forward yield curve.

   The forward rate between any two periods is a function of the zero-coupon discount rates for a horizon from the present to the initiation point of the forward rate and the zero-coupon rates for a horizon from the present to the termination date of the forward rate. For example, consider a single-period forward rate from year 5 to year 6. This forward rate can be found by using the zero-coupon yield curve to find the zero-coupon factors for years 5 and 6, $Z_{0,5}$ and $Z_{0,6}$. The forward rate factor for this period, $FRF_{5,6}$, is given by:

$$FRF_{5,6} = \frac{Z_{0,6}}{Z_{0,5}}$$

Given the set of one-period *FRF*s, the elements of the forward yield curve can be found quite easily, because the one-period forward rate is simply the one-period *FRF* minus 1:

$$1 + FR_{n,n+1} = FRF_{n,n+1}$$

In general, for any forward rate factor for a period beginning at time $x$ and ending at time $y$, we have:

$$FRF_{x,y} = \frac{Z_{0,y}}{Z_{0,x}}$$

4. Given the implied forward yield curve, explain how to find the zero-coupon yield curve.

   The forward yield curve consists of all one-period forward rates. For a horizon of $n$ periods, the zero-coupon factor is:

$$Z_{0,n} = FRF_{0,1} \times FRF_{1,2} \times \ldots FRF_{n,-1,n}$$

   The $n$-period zero-coupon yield is:

$$n\text{-Period Zero-Coupon Yield} = \sqrt[n]{Z_{0,n}}$$

5. Given the par yield curve, explain how to find the zero-coupon yield curve.

   Finding the zero-coupon yield curve requires the technique of bootstrapping detailed in the text. Bootstrapping uses the sequence of par yields to find the zero-coupon factors for each maturity covered by the yield curve. Given the sequence of zero-coupon factors, the zero-coupon yields that constitute the zero-coupon yield curve are found as in the answer to the previous question.

   In bootstrapping, we begin by noting that the short-term (one-period) par yield is the same as the zero-coupon yield for one period. This gives the information necessary to find the zero-coupon factor for two periods. Let $R_n$ indicate the $n$-period par yield. Because each $R_n$ is from the par yield curve, the coupon rate on an $n$-period instrument must be $R_n$. Therefore, assuming a par value on a bond of 100, we have for the two-period case:

$$100 = \frac{R_2 \times 100}{Z_{0,1}} + \frac{100 + R_2 \times 100}{Z_{0,2}}$$

   Recalling that the one-period par yield is the same as the one-period zero-coupon yield, we can solve this equation for $Z_{0,2}$. We then repeat the process. For a three-period horizon, we solve:

$$100 = \frac{R_3 \times 100}{Z_{0,1}} + \frac{R_3 \times 100}{Z_{0,2}} + \frac{100 + R_3 \times 100}{Z_{0,3}}$$

   We repeat this process until all zero-coupon factors are found.

6. How does Equation 19.2 differ from the usual bond pricing formula?

   The usual bond pricing formula discounts all cash flows from a bond at a single rate, the yield-to-maturity of the bond. In Equation 19.2, each cash flow from the bond is discounted at the zero-coupon rate appropriate for the time until the particular cash flow is to be paid.

7. Explain how to create a forward rate agreement for Treasuries using Treasury strips. Specifically, how would you create a forward rate agreement to cover a period from five to eight years in the future? Assume the five-year zero-coupon rate is 7 percent and the eight-year zero-coupon rate is 7.3 percent. Assume also that you wish to take a long position in the FRA, with a transaction amount of $1,000,000. (That is, the market value of the strips traded will be $1,000,000.)

When a T-bond is stripped, each cash flow from the bond is treated as a security. Therefore, a stripped T-bond is a portfolio of zero-coupon bonds. A forward rate agreement to cover any particular period can be constituted by two zero-coupon instruments, one maturing at the beginning of the forward rate period, one maturing at the end of the forward rate period. In a forward rate agreement, no cash flow occurs at the time of contracting. Instead, a cash flow occurs at the expiration of the forward agreement.

For a period from five to eight years in the future, one can create a forward rate agreement by buying and selling a five-year and an eight-year zero, with both having the same market value. Given the information in the question, we have:

$$Z_{0,5} = (1 + 0.07)^5 = 1.402552$$

$$Z_{0,8} = (1 + 0.073)^8 = 1.757105$$

These zero-coupon factors imply principal amounts as follows to have market values of \$1,000,000:

$$\$1{,}000{,}000 \times Z_{0,5} = \$1{,}000{,}000 \times 1.402552 = \$1{,}402{,}552$$

$$\$1{,}000{,}000 \times Z_{0,8} = \$1{,}000{,}000 \times 1.757105 = \$1{,}757{,}105$$

To create a long FRA position, one should transact as follows:

> Buy 8-year zero-coupon with principal amount of $1,757,105 at $1,000,000.
> Sell 5-year zero-coupon with principal amount of $1,402,552 at $1,000,000.

These two transactions require zero initial cash flow. At year 5, the 5-year zero will mature, and the trader will owe $1,402,552. This is the forward price of the FRA to cover years 5 to 8. Notice that this will be an FRA that is determined and settled in advance.

8. Assume that you wish to conduct an OAS analysis of a callable corporate bond with an 18-year maturity that is callable in seven years. You cannot find any Treasury bond that matches the cash flows of the callable bond. Explain how you might proceed with your analysis.

   You can create a surrogate T-bond using strips. First, determine the exact amounts and timing of the cash flows on the corporate bond. Second, find the zero-coupon Treasury instrument that best matches each cash flow. Third, create a portfolio of those zeros that match the desired, but unavailable, T-bond. Fourth, price the portfolio of strips. Fifth, find the yield-to-maturity of the surrogate bond. This effectively provides the desired T-bond to allow the OAS analysis to proceed.

9. In various incarnations of the Black model in this chapter, we saw that the role of the stock price in the Black–Scholes–Merton model could be played alternately by the futures price, the forward bond price, and the forward LIBOR interest rate. What underlying assumptions unite these various proxies for the stock price and allow the application of the Black model?

   The implicit assumption is that these different measures (the futures price, the forward bond price, and the forward LIBOR interest rate) are all assumed to have a log-normal distribution at the expiration date of the option.

10. What does it mean for an interest rate agreement to be "determined in advance and settled in arrears"?

   By convention, most interest rate agreements are determined in advance and paid in arrears. The rate on an FRA is determined in advance, that is, at the time the agreement matures and the underlying loan begins. The FRA is settled in arrears, meaning that the cash flow on the FRA occurs at the maturity of the instrument presumed to underlie the FRA. For example, consider a three-month FRA to cover a period extending from five to eight months in the future. The contract is initiated at time zero, the present. The gain or loss on the contract is determined in advance—at the beginning of the period covered by the FRA—which is at month 5. The agreement is settled in arrears—at the maturity date of the underlying instrument—which is at month 8.

11. You are examining a horizon extending from six through nine months in the future. For this period, an FRA is available, as well as calls and puts on LIBOR. Assume that you enter an FRA to receive-fixed and pay LIBOR. You also buy a call on LIBOR and sell a put on LIBOR for the same period, with strike rates equal to the rate on the FRA. All instruments have the same notional amount. Explain the economic import of your transactions. Can the totality of your transactions be analyzed as some simpler instrument? If so, what?

    An FRA is equivalent to a portfolio of a call and put on LIBOR, assuming matching time periods and notional amounts. A receive-fixed/pay LIBOR FRA can be replicated by a long put/short call portfolio. In the terms of the question, the FRA is a receive-fixed/pay LIBOR position. The long call/short put portfolio is equivalent to a pay-fixed/receive LIBOR FRA. With all of these transactions, the net economic effect of the transactions is zero. There should be no net cash flows at the time of contracting or at the expiration date of the FRA and options. All this transacting amounts to nothing, except perhaps transaction costs.

12. For a future period, the FRA rate is 8 percent. Consider the following varieties of a "collarlet" (caplet plus floorlet) for this loan. Assume that the periods and notional amounts of the options match the terms of the FRA.

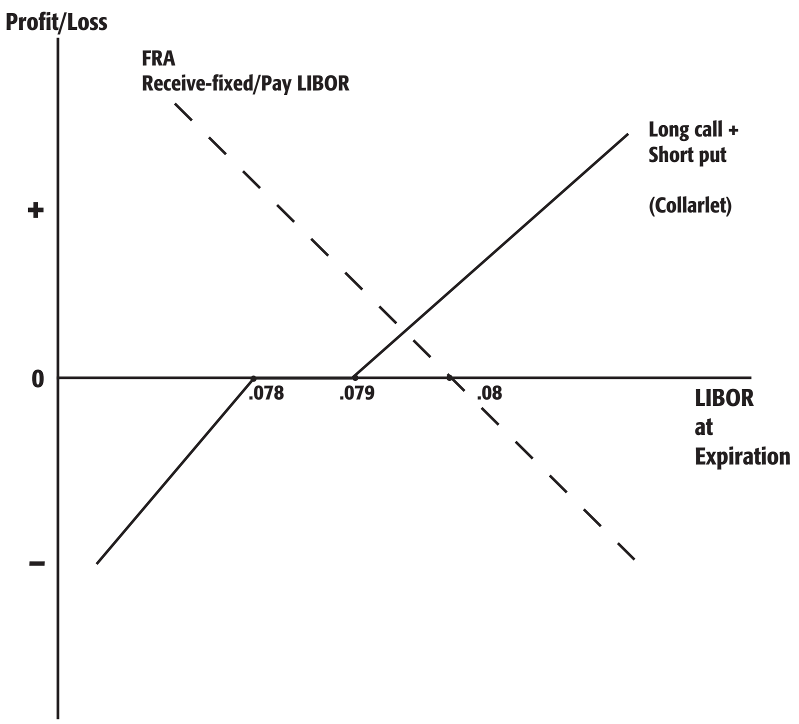
A. A collarlet gives an 8–8 percent collar on the loan. How much should the collarlet cost?

This collarlet is equivalent to an FRA with a contract rate of 8 percent. Since the FRA rate stated in the question is also 8 percent, this collarlet should be costless.

B. A 7.8−7.9 percent collarlet is costless. How should you respond as an arbitrageur?

A collarlet with an 8 percent strike rate should be costless, and represents a firm commitment at 8 percent, like an FRA at 8 percent. This collarlet guarantees a rate not less than 7.8 percent and not higher than 7.9 percent. With an FRA rate of 8 percent prevailing, this collarlet has positive economic value. Therefore, since it is costless, the arbitrageur should buy the call and sell the put to acquire the collarlet, and should enter an FRA as the receive-fixed party.

The following figure shows the payoffs from the collarlet as the solid line and the receive-fixed FRA as the dashed line. No matter what LIBOR prevails at the expiration of the options and the FRA, the arbitrageur makes a profit. The profit is equal to the sum of the FRA line and the collarlet line (times the unspecified



fraction of the year times the unspecified notional amount). Graphically, the figure shows that this sum is positive no matter what the Observed LIBOR happens to be.

C. An 8.2–8.5 percent collarlet is fairly priced. What can you infer about its cost?

The collarlet represents a commitment to finance at a rate not higher than 8.5 percent and not lower than 8.2 percent. Since the prevailing FRA rate is 8 percent, taking the collarlet is worse than the FRA. Therefore, rational prices must be such that the collarlet has a negative price—that is, the cost of buying the call and selling the put should give a cash inflow. Said another way, the put must be worth more than the call.

D. A 7.5–7.8 percent collarlet is fairly priced. What can you infer about its cost?

This collarlet gives a financing opportunity of not less than 7.5 percent, but not more than 7.8 percent. These are excellent terms in an environment with an FRA rate of 8 percent. Therefore, this collarlet must have a positive cost. The cost of the call exceeds the cost of the put.

| Maturity (years) | Yield Curve #1 (par annual yields) | Yield Curve #2 (par annual yields) |
|---|---|---|
| 1 | 0.0500 | 0.1200 |
| 2 | 0.0540 | 0.1196 |
| 3 | 0.0545 | 0.1191 |
| 4 | 0.0552 | 0.1171 |
| 5 | 0.0564 | 0.1130 |
| 6 | 0.0594 | 0.1105 |
| 7 | 0.0652 | 0.1070 |
| 8 | 0.0666 | 0.1035 |
| 9 | 0.0714 | 0.0992 |
| 10 | 0.0772 | 0.0953 |

13. Using the yield data in the table above, interpret Yield Curve #1 as the par yield curve. Complete the following table by finding all zero-coupon factors and by finding all one-year forward rate factors. Also, find all four-year forward rates. Assume annual compounding throughout.

| Maturity (years) | Par Yield | Zero-Coupon Factor | One-Year Forward Rate Factors |
|---|---|---|---|
| 1 | 0.0500 | 1.050000 | 1.050000 |
| 2 | 0.0540 | 1.111145 | 1.058233 |
| 3 | 0.0545 | 1.172909 | 1.055586 |
| 4 | 0.0552 | 1.240408 | 1.057549 |
| 5 | 0.0564 | 1.317252 | 1.061950 |
| 6 | 0.0594 | 1.419450 | 1.077584 |
| 7 | 0.0652 | 1.576568 | 1.110690 |
| 8 | 0.0666 | 1.702647 | 1.079970 |
| 9 | 0.0714 | 1.921525 | 1.128551 |
| 10 | 0.0772 | 2.237349 | 1.164361 |

To find the four-year forward rates, we first find all four-year forward rate factors:

$$FRF_{0,4} = \frac{Z_{0,4}}{Z_{0,0}} = \frac{1.240408}{1.000000} = 1.240408$$

$$FRF_{1,5} = \frac{Z_{0,5}}{Z_{0,1}} = \frac{1.317252}{1.050000} = 1.254526$$

$$FRF_{2,6} = \frac{Z_{0,6}}{Z_{0,2}} = \frac{1.419450}{1.111145} = 1.277466$$

$$FRF_{3,7} = \frac{Z_{0,7}}{Z_{0,3}} = \frac{1.576568}{1.172909} = 1.344152$$

$$FRF_{4,8} = \frac{Z_{0,8}}{Z_{0,4}} = \frac{1.702647}{1.240408} = 1.372651$$

$$FRF_{5,9} = \frac{Z_{0,9}}{Z_{0,5}} = \frac{1.921525}{1.317252} = 1.458738$$

$$FRF_{6,10} = \frac{Z_{0,10}}{Z_{0,6}} = \frac{2.237349}{1.419450} = 1.576208$$

Given the various four-year *FRF*s above, each four-year forward rate is the fourth root of the four-year *FRF* minus 1, as given in the following table:

| Starting Date | Ending Date | Four-Year *FRF* | Four-Year Forward Rate |
|---|---|---|---|
| 0 | 4 | 1.240408 | 0.055337 |
| 1 | 5 | 1.254526 | 0.058327 |
| 2 | 6 | 1.277466 | 0.063132 |
| 3 | 7 | 1.344152 | 0.076743 |
| 4 | 8 | 1.372651 | 0.082406 |
| 5 | 9 | 1.458738 | 0.098991 |
| 6 | 10 | 1.576208 | 0.120478 |

14. Using the yield data in the table above, interpret Yield Curve #2 as the zero-coupon yield curve. Find all zero-coupon factors. Find the par yield curve. Assume annual compounding throughout.

If the elements of Yield Curve #2 are zero-coupon yields, the zero-coupon factors are found by adding 1 to the yield and raising the quantity to a power reflecting the number of periods that equals the maturity of the rate, as follows:

$$Z_{0,1} = 1.12$$

$$Z_{0,2} = (1.1196)^2 = 1.253504$$

$$Z_{0,3} = (1.1191)^3 = 1.401544$$

$$Z_{0,4} = (1.1171)^4 = 1.557285$$

$$Z_{0,5} = (1.1130)^5 = 1.707953$$

$$Z_{0,6} = (1.1105)^6 = 1.875475$$

$$Z_{0,7} = (1.1070)^7 = 2.037198$$

$$Z_{0,8} = (1.1035)^8 = 2.198764$$

$$Z_{0,9} = (1.0992)^9 = 2.342559$$

$$Z_{0,10} = (1.0953)^{10} = 2.485026$$

We now use the zero-coupon factors to find the par yields for each maturity, assuming a par value of $1,000. To find the par yield, we need to find the coupon, *COUP*, and the corresponding coupon rate on the bond that gives a market value of $1,000, when each cash flow from the bond is discounted by the appropriate zero-coupon factor. In essence, we apply the bond pricing equation:

$$1{,}000 = \sum_{n=1}^{N} \frac{COUP_n}{Z_{0,n}} + \frac{1{,}000}{Z_{0,N}}$$

$$COUP_n = \frac{1{,}000 - \dfrac{1{,}000}{Z_{0,N}}}{\displaystyle\sum_{n=1}^{N} \frac{1}{Z_{0,n}}}$$

As an example, consider the five-year maturity:

$$Z_{0,5} = 1.707953$$

$$\sum_{n=1}^{5} \frac{1}{Z_{0,n}} = 3.631759$$

$$1{,}000 = COUP_t \times 3.631759 + \frac{1{,}000}{1.707953}$$

$$COUP_t = 114.13$$

The coupon is 114.13, for a coupon rate of 11.413 percent for the five-year maturity. This coupon rate is the par yield. The following table gives the result for the entire yield curve:

| Maturity | $Z_{0,n}$ | $\frac{1}{Z_{0,n}}$ | $\sum_{n=1}^{N} \frac{1}{Z_{0,n}}$ | $1{,}000 - \frac{1{,}000}{Z_{0,n}}$ | Coupon | Coupon Rate (percent) |
|---|---|---|---|---|---|---|
| 1 | 1.120000 | 0.892857 | 0.892857 | 107.142857 | 120.00 | 12.000 |
| 2 | 1.253504 | 0.797764 | 1.690621 | 202.236291 | 119.62 | 11.962 |
| 3 | 1.401544 | 0.713499 | 2.404120 | 286.501173 | 119.17 | 11.917 |
| 4 | 1.557285 | 0.642143 | 3.046263 | 357.856783 | 117.47 | 11.747 |
| 5 | 1.707953 | 0.585496 | 3.631759 | 414.503795 | 114.13 | 11.413 |
| 6 | 1.875475 | 0.533198 | 4.164957 | 466.801744 | 112.08 | 11.208 |
| 7 | 2.037198 | 0.490870 | 4.655828 | 509.129697 | 109.35 | 10.935 |
| 8 | 2.198764 | 0.454801 | 5.110629 | 545.199030 | 106.68 | 10.668 |
| 9 | 2.342559 | 0.426884 | 5.537512 | 573.116408 | 103.50 | 10.350 |
| 10 | 2.485026 | 0.402410 | 5.939922 | 597.589723 | 100.61 | 10.061 |

15. Using the yield data in the table above, interpret Yield Curve #1 as the zero-coupon yield curve. Find the price of an 8 percent annual coupon bond that matures in eight years. A callable bond with a price of 98.75 percent of par matures in eight years and has a coupon of 8 percent. Find the OAS between these bonds.

    The following table gives the intermediate calculations for pricing the bond:

| Maturity | Zero-Coupon Rate | Zero-Coupon Factor | Bond Cash Flow | Present Value of Cash Flow |
|---|---|---|---|---|
| 1 | 0.0500 | 1.050000 | 80 | 76.1905 |
| 2 | 0.0540 | 1.110916 | 80 | 72.0126 |
| 3 | 0.0545 | 1.172573 | 80 | 68.2261 |
| 4 | 0.0552 | 1.239764 | 80 | 64.5284 |
| 5 | 0.0564 | 1.315655 | 80 | 60.8062 |
| 6 | 0.0594 | 1.413708 | 80 | 56.5888 |
| 7 | 0.0652 | 1.556030 | 80 | 51.4129 |
| 8 | 0.0666 | 1.674992 | 1080 | 644.7794 |
| | | | | Bond Price = 1094.5449 |

For the OAS analysis, we need to find the yield spread to add to the zero-coupon rates that will give this bond a price of 98.75 percent of par. In terms of the preceding table, it will be the amount to add to the zero-coupon rate in the second column. The following table shows various spreads and the resulting price. The OAS analysis can proceed by trial and error until the computed yield-adjusted T-bond price equals the callable bond price of 98.75.

| Spread | Resulting Bond Price |
|---|---|
| 0.050000 | 826.7478 |
| 0.030000 | 922.0697 |
| 0.010000 | 1032.6465 |
| 0.020000 | 975.2822 |
| 0.019000 | 980.8258 |
| 0.017000 | 992.0389 |
| 0.018000 | 986.4112 |
| 0.017800 | 987.5334 |
| 0.017810 | 987.4772 |
| 0.017805 | 987.5053 |
| 0.017806 | 987.4997 |

A spread of 0.017806 gives a bond price of 987.4997 ≈ 987.500. The OAS is 178.06 basis points.

16. A T-bond futures contract matures in five months, and the corresponding T-bond futures option expires in four months. The current futures price is 103.50. The yield curve is flat at 7 percent. Assume continuous compounding. The standard deviation of the T-bond futures price is 0.2835. The strike price for both a call and a put is 100.00. Using the Black model, find the price of the call and the put.

    Applying the Black model from Equation 19.5, we first compute $d_1^F$ and $d_2^F$:

$$d_1^F = \frac{\ln\left(\frac{F_t}{X}\right) + (.5\,\sigma^2)(T-t)}{\sigma\sqrt{T-t}} = \frac{\ln\left(\frac{103.50}{100}\right) + (0.5 \times 0.2835^2)(0.3333)}{0.2835 \times \sqrt{0.3333}} = 0.292022$$

$$d_2^F = d_1^F - \sigma\sqrt{T-t} = 0.292022 - 0.163671 = 0.128351$$

    Finding the cumulative normal values for the put and call, we have $N(d_1^F) = 0.614865$; $N(d_2^F) = 0.551065$; $N(-d_1^F) = 0.385135$; and $N(-d_2^F) = 0.448935$.

    Assuming continuous discounting at the rate of 7.00 percent, we have:

$$\begin{aligned} c_t^F &= e^{-r(T-t)}[F_t N(d_1^F) - XN(d_2^F)] \\ &= e^{-0.07(0.3333)}[103.50\,(0.614865) - 100\,(0.551065)] \\ &= 8.3353 \\ p_t^F &= e^{-r(T-t)}[XN(-d_2^F) - F_t N(-d_1^F)] \\ &= e^{-0.07(0.3333)}[100\,(0.448935) - 103.50\,(0.385135)] \\ &= 4.9160 \end{aligned}$$

17. Consider European call and put options on a Treasury bond with a coupon of 6 percent paid semiannually that matures in 40 months and has a par value of $1,000. The current term structure environment is given by the downward-sloping yields of Table 19.6. Use these yields and monthly compounding throughout. Both the call and put expire in three months. The call has an exercise price of $970, and the put has an exercise price of $1,010. The standard deviation of Treasury yields is given as follows: one-year maturity, 0.30; two-year maturity, 0.28; three-year maturity, 0.25; and five-year maturity, 0.19. (Use linear interpolation to find the appropriate yield volatility for the maturity of this bond.) Find the prices of the call and put options.

    We begin by finding the actual cash price of the bond, and then the forward price of the bond in three months when the options expire. The following table gives the relevant cash flow and discounting information:

| Month | Cash Flow | Zero-Coupon Factor | PV of Cash Flow |
|---|---|---|---|
| 4 | 30 | 1.0262 | 29.2341 |
| 10 | 30 | 1.0661 | 28.1399 |
| 16 | 30 | 1.1066 | 27.1101 |
| 22 | 30 | 1.1478 | 26.1370 |
| 28 | 30 | 1.1896 | 25.2186 |
| 34 | 30 | 1.2318 | 24.3546 |
| 40 | 1030 | 1.2745 | 808.1601 |
| | | | Total = 968.3543 |

The actual bond price is 968.3543, assuming a par value of 1000. $Z_{0,\,3}$ =1.0196. Therefore, the forward bond price when the options expire in three months is:

$$968.3543 \times 1.0196 = 987.3340$$

We next need to find the volatility of the forward bond price. To do so, we first need to find the Macaulay duration for the forward bond, which requires finding the monthly yield-to-maturity consistent with the forward bond price. We value the forward bond at month 3, finding the yield-to-maturity for the forward bond at the point, and finding the duration of the forward bond at that point as well. Viewed from the perspective of month 3, the remaining timing and cash flows are given in the following table:

| Month | Cash Flow | PV of Cash Flow | Weighted PV of Cash Flow |
|---|---|---|---|
| 1 | 30 | 29.8192 | 29.8192 |
| 7 | 30 | 28.7573 | 201.3011 |
| 13 | 30 | 27.7332 | 360.5313 |
| 19 | 30 | 26.7455 | 508.1649 |
| 25 | 30 | 25.7930 | 644.8261 |
| 31 | 30 | 24.8745 | 771.1091 |
| 37 | 1030 | 823.6101 | 30,473.5720 |
| | | Computed Forward Bond Price = 987.3328 | Sum = 32,989.3237 |

With a forward bond price of 987.3340, the monthly yield to maturity on the forward bond, found by trial and error, is 0.006062, which gives a computed forward bond price of 987.3328 ≈ 987.3340, as shown in the third column of the table. The fourth column gives the present values of the cash flows weighted by the number of months until each cash flow is received, all from the perspective of month 3. The Macaulay's duration, $D$, is the sum of these weighted cash flows, 32989.3237, divided by the forward bond price, 987.3340:

$$D = \frac{32{,}989.3237}{987.3340} = 33.4125 \text{ months}$$

The modified Macaulay's duration, $MD$, of the forward bond is:

$$MD = \frac{D}{1 + r} = \frac{33.4125}{1.006062} = 33.2112 \text{ months} = 2.7676 \text{ years}$$

We are now ready to find the standard deviation of the forward bond price, $\sigma_P$. We know that the three-year yield volatility is 0.25 and the five-year volatility is 0.19. The linearly interpolated volatility for a 40-month maturity is:

$$0.25 \times (20/24) + 0.19 \times (4/24) = 0.24$$

The monthly yield to maturity for the forward bond is 0.006062, implying an annualized yield of 0.075219, assuming monthly compounding. Therefore,

$$\sigma_P \approx MD\, r\, \sigma_r = 2.7676 \times 0.075219 \times 0.24 = 0.049962$$

We now have (finally!) all of the required inputs for the Black model. The forward bond price is 987.3340; the time to expiration is three months, or 0.25 years; the exercise price for the call is 970; the exercise price for the put is 1010; the standard deviation of the forward price is 0.049962; and the three-month factor is 1.0196. We begin by computing the $d_1^F$ and $d_2^F$ terms for the call:

$$d_1^F = \frac{\ln\left(\frac{F_t}{X}\right) + (.5\,\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{987.3340}{970.00}\right) + (0.5 \times 0.049962^2)\,(0.25)}{0.049962 \times \sqrt{0.25}}$$

$$= 0.721522$$

$$d_2^F = d_1^F - \sigma\sqrt{T-t}$$

$$= 0.721522 - 0.024981$$

$$= 0.696541$$

The cumulative normal terms that we require for the call are: $N(d_1^F) = N(0.721522) = 0.764706$; $N(d_2^F) = N(0.696541) = 0.756955$. The value of the call is

$$c_t^F = \frac{1}{1.0196}\,[F_t N(d_1^F) - X N(d_2^F)]$$

$$= 0.980777\,[987.3340\,(0.764706) - 970\,(0.756955)]$$

$$= 20.374543$$

As the put has a different exercise price, we must compute the $d_1$ and $d_2$ terms for it separately:

$$d_1^F = \frac{\ln\left(\frac{F_t}{X}\right) + (.5\,\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{987.3340}{1010.00}\right) + (0.5 \times 0.049962^2)\,(0.25)}{0.049962 \times \sqrt{0.25}}$$

$$= -0.896089$$

$$d_2^F = d_1^F - \sigma\sqrt{T-t}$$

$$= -0.896089 - 0.024981$$

$$= -0.921070$$

The cumulative normal terms that we require for the put are: $N(-d_1^F) = N(0.896089) = 0.814897$; $N(=d_2^F) = N(0.921070) = 0.821493$. The value of the call is

$$p_t^F = \frac{1}{1.0196}\,[X N(-d_2^F) - F_t N(-d_1^F)]$$

$$= 0.980777\,[1010\,(0.821493) - 987.3340\,(0.814897)]$$

$$= 24.649289$$

18. A call and put on three-month LIBOR expire in seven months. The yield curve environment is given by the upward-sloping yield curve of Table 19.6. The notional principal for the option is \$100,000,000. The standard deviation of the three-month forward rate is 0.24. Assuming the strike rate on both the put and the call is 5.6 percent, price the two options. What would the prices be if the options were determined and paid in advance? Assume monthly compounding throughout.

We first need to find the forward rate that underlies the option. This would be the forward rate to cover the time from the expiration of the option at month 7 out to the maturity of the underlying instrument, which is month 10:

$$FRF_{7,10} = \frac{Z_{0,10}}{Z_{0,7}} = \frac{1.0493}{1.0339} = 1.014895$$

This forward rate factor covers three months, so the annualized *FRF* is:

$$1.014895^4 = 1.060925$$

for a forward rate of interest of 0.060925. We begin by computing the $d_1$ and $d_2$ terms with the relevant cumulative probabilities:

$$d_1^{LIBOR} = \frac{\ln\left(\frac{FLIBOR_t}{SR}\right) + (.5\,\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{0.060925}{0.0560}\right) + 0.5\,(0.24)^2\,(0.5833)}{0.24\sqrt{0.5833}}$$

$$= 0.551512$$

$$d_2^{LIBOR} = d_1^{LIBOR} - \sigma\sqrt{T-t}$$

$$= 0.551512 - 0.183298 = 0.368214$$

The relevant cumulative probabilities are: $N(d_1^{LIBOR}) = N(0.551512) = 0.709359$; $N(d_2^{LIBOR}) = N(0.368214) = 0.643643$; $N(-d_1^{LIBOR}) = N(-0.551512) = 0.290641$; and $N(-d_2^{LIBOR}) = N(-0.368214) = 0.356357$.

$$c_t^{FLIBOR} = NP \times FRAC \times \frac{1}{Z_{t,\,T+FRAC}}[FLIBOR_t\,N(d_1^{LIBOR}) - SR\,N(d_2^{LIBOR})]$$

$$= \$100{,}000{,}000 \times 0.25 \times \frac{1}{1.0493}[0.060925\,(0.709359) - 0.0560\,(0.643643)]$$

$$= \$170{,}916.07$$

$$p_t^{LIBOR} = NP \times FRAC \times \frac{1}{Z_{t,T+FRAC}}[SR\,N(-d_2^{LIBOR}) - FLIBOR_t\,N(-d_1^{LIBOR})]$$

$$= \$100{,}000{,}000 \times 0.25 \times \frac{1}{1.0493}[0.0560\,(0.356357) - 0.060925\,(0.290641)]$$

$$= \$53{,}575.93$$

If the option is determined and paid in advance, the option payoffs will occur at month 7, so the zero-coupon factor will be 1.0339. The resulting call and put values are \$173,461.87 and \$54,373.95, respectively.

19. Consider the downward-sloping yield curve of Table 19.6. Find $FRF_{11,\,17}$. Find $FRA_{11,\,17}$ assuming monthly compounding. For a call and put on six-month LIBOR that both expire in eleven months, what is the value of a long call/short put portfolio with a strike rate of 6.00 percent and a notional principal of \$20,000,000?

Assume that the standard deviation of the six-month forward rate is 0.27. How would you construct a riskless bond from the put, call, and FRA? How does this relate to forward put-call parity?

$$FRF_{11,17} = \frac{Z_{0,17}}{Z_{0,11}} = \frac{1.1134}{1.0728} = 1.037845$$

As this is a factor for six months, the corresponding forward rate is:

$$FRA_{11,17} = 1.037845^2 - 1 = 0.077122$$

We next make the intermediate computations to price the option.

$$d_1^{LIBOR} = \frac{\ln\left(\frac{FLIBOR_t}{SR}\right) + (.5\,\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln\left(\frac{0.077122}{0.06}\right) + 0.5\,(0.27)^2\,(0.9167)}{0.27\sqrt{0.9167}}$$

$$= 1.100374$$

$$d_2^{LIBOR} = d_1^{LIBOR} - \sigma\sqrt{T-t}$$

$$= 1.100374 - 0.258510 = 0.841864$$

The relevant cumulative probabilities are: $N(d_1^{LIBOR}) = N(1.100374) = 0.864415$; $N(d_2^{LIBOR}) = N(0.841864) = 0.800068$; $N(-d_1^{LIBOR}) = N(-1.100374) = 0.135585$; and $N(-d_2^{LIBOR}) = N(-0.841864) = 0.199932$. The option prices are:

$$c_t^{FLIBOR} = NP \times FRAC \times \frac{1}{Z_{t,T+FRAC}}[FLIBOR_t\,N(d_1^{LIBOR}) - SR\,N(d_2^{LIBOR})]$$

$$= \$20{,}000{,}000 \times 0.5 \times \frac{1}{1.1134}[0.077122\,(0.864415) - 0.06\,(0.800068)]$$

$$= \$167{,}606.73$$

$$p_t^{LIBOR} = NP \times FRAC \times \frac{1}{Z_{t,T+FRAC}}[SR\,N(-d_2^{LIBOR}) - FLIBOR_t\,N(-d_1^{LIBOR})]$$

$$= \$20{,}000{,}000 \times 0.5 \times \frac{1}{1.1134}[0.06\,(0.199932) - 0.077122\,(0.135585)]$$

$$= \$13{,}825.52$$

The long call/short put portfolio with a common strike rate of 0.06 is equivalent to an FRA with a rate of 6 percent. The actual rate in the market for this FRA is 0.077122. One could create a riskless investment by buying the call, selling the put, and entering an FRA to receive-fixed at 0.077122. These transactions will have a payoff in 17 months of:

$$(0.077122 - 0.06) \times \$20{,}000{,}000 \times 0.5 = \$171{,}220.00$$

The present value of this payoff is \$171,220/1.1134 = \$153,781.21. This is exactly the cost of the long call/short put payoff. Thus, these transactions synthesize a risk-free bond, and this technique is just a straightforward application of forward put-call parity.

20. The upward-sloping term structure of Table 19.6 prevails. Consider a loan based on six-month LIBOR with a maturity of 24 months and a loan amount of \$100,000,000. The loan is to start immediately. Assume that the standard deviation of all relevant interest rates is 0.23.

A. What is the cost of capping this loan at 7 percent?

The loan will have determination dates at months 0, 6, 12, and 18, with corresponding payment dates at 6, 12, 18, and 24 months. Therefore, we need the zero-coupon factors corresponding to each payment date, and we need the forward rates corresponding to each determination date. The zero-coupon factors are: $Z_{0,6} = 1.0289$; $Z_{0,12} = 1.0599$; $Z_{0,18} = 1.0931$; $Z_{0,24} = 1.1287$. The forward rate factors are:

$$FRF_{0,6} = Z_{0,6} = 1.0289$$
$$FRF_{6,12} = \frac{Z_{0,12}}{Z_{0,6}} = \frac{1.0599}{1.0289} = 1.030129$$
$$FRF_{12,18} = \frac{Z_{0,18}}{Z_{0,12}} = \frac{1.0931}{1.0599} = 1.031324$$
$$FRF_{18,24} = \frac{Z_{0,24}}{Z_{0,18}} = \frac{1.1287}{1.0931} = 1.03258$$

These are semiannual *FRF*s, so squaring each and subtracting 1.0 gives the annualized forward rates to cover each period as: $FR_{0,6} = 0.058635$; $FR_{6,12} = 0.061166$; $FR_{12,18} = 0.063629$; $FR_{18,24} = 0.066197$.

With these figures, we now turn to pricing the cap, which consists of four call options on LIBOR. The first matures immediately; the other three mature in 6, 12, and 18 months. The following table presents the salient intermediate results:

| Option Expiration | Forward Rate | Zero-Coupon Factor | $d_1$ | $d_2$ | $N(d_1)$ | $N(d_2)$ |
|---|---|---|---|---|---|---|
| 0 | 0.058635 | 1.0289 | −99.99 | −99.99 | 0.00 | 0.00 |
| 6 | 0.061166 | 1.0599 | −0.748173 | −0.910807 | 0.227178 | 0.181198 |
| 12 | 0.063629 | 1.0931 | −0.299895 | −0.529895 | 0.382129 | 0.298092 |
| 18 | 0.066197 | 1.1287 | −0.057457 | −0.339148 | 0.477091 | 0.367249 |

**Note:** The forward rate begins at the option expiration and continues for six months. The zero-coupon factor is for the option expiration plus the six months until the payment is received.

The first option has a strike rate of 7 percent and is at expiration in an environment where the market rate is 0.058635, so it is worthless. The pricing problem is to price the other three calls. We price the option expiring in one year as an example. First, we calculate the $d_1$ and $d_2$ terms and their cumulative probabilities:

$$d_1^{LIBOR} = \frac{\ln\left(\frac{FLIBOR_t}{SR}\right) + (0.5\,\sigma^2)(T-t)}{\sigma\sqrt{T-t)}}$$
$$= \frac{\ln\left(\frac{0.063629}{0.07}\right) + 0.5\,(0.23)^2\,(1.0)}{0.23\sqrt{1.0}}$$
$$= -0.299895$$
$$d_2^{LIBOR} = d_1^{LIBOR} - \sigma\sqrt{T-t}$$
$$= -0.299895 - 0.23 = -0.529895$$

$N(d_1) = N(-0.299895) = 0.382129$, and $N(d_2) = N(-0.529895) = 0.298092$. The value of the option expiring in 12 months is:

$$c_t^{FLIBOR} = NP \times FRAC \times \frac{1}{Z_{t,T+FRAC}}[FLIBOR_t\,N(d_1^{LIBOR}) - SR\,N(d_2^{LIBOR})]$$

$$= \$100{,}000{,}000 \times 0.5 \times \frac{1}{1.0931}[0.063629\,(0.382129) - 0.07\,(0.298092)]$$
$$= \$157{,}718.70$$

Applying the same calculations, we find that the option that expires in six months is worth \$57,159.95 and the option that expires in 18 months is worth \$260,234.57. Given that the first option is worthless, the total cost of capping the loan at 7 percent equals the value of these three calls, for a total of \$475,110.95.

B. What is the cost of a floor for this loan at 6 percent?

A floor at 6 percent consists of four put options. The analysis parallels that just applied to the calls, except the first option will have considerable value. We consider it first. By selling this first call, the initiator of the floor agrees to pay the floor rate of 6 percent even though the prevailing rate for the first six months is 0.058635. For the owner of the put, the payoff will occur in six months and will be:

$$(0.06 - 0.058635) \times \$100{,}000{,}000 \times 0.5 = \$68{,}250$$

The value of the put is the present value of this payoff that occurs in six months, which is \$68,250/1.0289 = \$66,332.98.

The other puts can be priced using the Black model. As with the call options, the intermediate calculations are shown in the following table:

| Option Expiration | Forward Rate | Zero-Coupon Factor | $d_1$ | $d_2$ | $N(-d_1)$ | $N(-d_2)$ |
|---|---|---|---|---|---|---|
| 0 | 0.058635 | 1.0289 | N/A | N/A | N/A | N/A |
| 6 | 0.061166 | 1.0599 | 0.199662 | 0.037027 | 0.420873 | 0.485232 |
| 12 | 0.063629 | 1.0931 | 0.370325 | 0.140325 | 0.355570 | 0.444202 |
| 18 | 0.066197 | 1.1287 | 0.489776 | 0.208084 | 0.312146 | 0.417582 |

**Note:** The forward rate begins at the option expiration and continues for six months. The zero-coupon factor is for the option expiration plus the six months until the payment is received.

Again, as an example, we show the computations for the put expiring in one year:

$$d_1^{LIBOR} = \frac{\ln\left(\frac{FLIBOR_t}{SR}\right) + (.5\,\sigma^2)\,(T-t)}{\sigma\sqrt{T-t}}$$
$$= \frac{\ln\left(\frac{0.063629}{0.06}\right) + 0.5\,(0.23)^2\,(1.0)}{0.23\sqrt{1.0}}$$
$$= 0.370325$$
$$d_2^{LIBOR} = d_1^{LIBOR} - \sigma\sqrt{T-t}$$
$$= 0.370325 - 0.23 = 0.140325$$

$N(-d_1) = N(-0.370325) = 0.355570$; $N(-d_2) = N(-0.140325) = 0.444202$. The value of the put is:

$$p_t^{LIBOR} = NP \times FRAC \times \frac{1}{Z_{t,\,T+FRAC}}[SR\,N(-d_2^{LIBOR}) - FLIBOR_t\,N(-d_1^{LIBOR})]$$
$$= \$100{,}000{,}000 \times 0.5 \times \frac{1}{1.0931}[0.06\,(0.444202) - 0.063629\,(0.355570)]$$
$$= \$184{,}224.76$$

The put expiring in six months is worth \$159,015.40, while the put expiring in 18 months is worth \$194,548.73. Together the four puts are worth \$604,121.87. Therefore, faced with this yield curve environment, the borrower should receive a payment of \$604,127.87 for agreeing to accept a floating rate that cannot go below 6 percent.

C. What is the cost of a 5−7.5 percent collar for this loan?

The collar consists of a portfolio of puts and calls. The calls have a strike rate of 7.5 percent, while the puts have a strike rate of 5 percent. The first expiration date is immediate, and with a prevailing 6-month rate of 0.058635, both the call and the put are out-of-the-money and expire worthless. They can be ignored. For the calls, we have:

| Option Expiration | Forward Rate | Zero-Coupon Factor | $d_1$ | $d_2$ | $N(d_1)$ | $N(d_2)$ |
|---|---|---|---|---|---|---|
| 0 | 0.058635 | 1.0289 | N/A | N/A | N/A | N/A |
| 6 | 0.061166 | 1.0599 | −1.172393 | −1.335028 | 0.120520 | 0.090934 |
| 12 | 0.063629 | 1.0931 | −0.599864 | −0.829864 | 0.274298 | 0.203308 |
| 18 | 0.066197 | 1.1287 | −0.302381 | −0.584072 | 0.381181 | 0.279586 |

**Note:** The forward rate begins at the option expiration and continues for six months. The zero-coupon factor is for the option expiration plus the six months until the payment is received.

For the puts, the intermediate calculations are:

| Option Expiration | Forward Rate | Zero-Coupon Factor | $d_1$ | $d_2$ | $N(-d_1)$ | $N(-d_2)$ |
|---|---|---|---|---|---|---|
| 0 | 0.058635 | 1.0289 | N/A | N/A | N/A | N/A |
| 6 | 0.061166 | 1.0599 | 1.320712 | 1.158078 | 0.093299 | 0.123416 |
| 12 | 0.063629 | 1.0931 | 1.163028 | 0.933028 | 0.122409 | 0.175403 |
| 18 | 0.066197 | 1.1287 | 1.137015 | 0.855323 | 0.127766 | 0.196186 |

**Note:** The forward rate begins at the option expiration and continues for six months. The zero-coupon factor is for the option expiration plus the six months until the payment is received.

With these intermediate values, the corresponding option values are shown in the following table:

| Option Expiration | Call Value | Put Value | Collarlet Value |
|---|---|---|---|
| 6 | \$26,025.12 | \$21,893.68 | \$4,131.44 |
| 12 | 100,871.25 | 44,889.35 | 55,981.90 |
| 18 | 188,893.84 | 59,873.00 | 129,020.84 |
| Totals | \$315,790.21 | \$126,656.03 | \$189,134.18 |

The cost of placing a 5–7.5 percent collar on the loan is \$189,134.18.