# 15 American Option Pricing

## Answers to Questions and Problems

1. Explain why American and European calls on a nondividend stock always have the same value.

   An American option is just like a European option, except the American option carries the right of early exercise. Exercising a call before expiration discards the time value inherent in the option. The only offsetting benefit from early exercise arises from an attempt to capture a dividend. If there is no dividend, there is no incentive to early exercise, so the early exercise feature of an American call on a nondividend stock has no value.

2. Explain why American and European puts on a nondividend stock can have different values.

   The exercise value of a put is $X - S$. On a European put, this value cannot be captured until the expiration date. Therefore, before expiration, the value of the European put will be a function of the present value of these exercise proceeds: $e^{-r(T-t)}(X - S)$. The American put gives immediate access at any time to the full proceeds, $X - S$, through exercise. In certain circumstances, notably on puts that are deep-in-the-money with time remaining until expiration, this differential in exercise conditions can give the American put extra value over the corresponding European put, even in the absence of dividends.

3. Explain the circumstances that might make the early exercise of an American put on a nondividend stock desirable.

   Early exercise of an American put provides the holder with an immediate cash inflow of $X - S$. These proceeds can earn a return from the date of exercise to the expiration date that is not available on a European put. However, early exercise discards the time value of the put. Therefore, the early exercise decision requires trading off the sacrificed time value against the interest that can be earned by investing the exercise value from the date of exercise to the expiration date of the put. For deep-in-the-money puts with time remaining until expiration, the potential interest gained can exceed the time value of the put that is sacrificed.

4. What factors might make an owner exercise an American call?

   The key factor is an approaching dividend, and exercise of an American call should occur only at the moment before an ex-dividend date. The dividend must be "large" relative to the share price, and the call will typically also be deep-in-the-money.

5. Do dividends on the underlying stock make the early exercise of an American put more or less likely? Explain.

   Dividends make early exercise of an American put less likely. Dividends decrease the stock price and increase the exercise value of the put. Thus, the holder of the American put has an incentive to delay exercising and wait for the dividend payments.

6. Do dividends on the underlying stock make the early exercise of an American call more or less likely? Explain.

   Dividends increase the likelihood of early exercise on an American call. In fact, if there are no dividends on the underlying stock, early exercise of an American call is irrational.

7. Explain the strategy behind the pseudo-American call pricing strategy.

   In pseudo-American call pricing, the analysis treats the stock price as the current stock price reduced by the present value of all dividends to occur before the option expires. It then considers potential exercise just prior to each ex-dividend date, by reducing the exercise price by the present value of all dividends to be paid, including the imminent dividend. (The dividends are a reduction from the exercise price because they represent a cash inflow if the option is exercised.) For each dividend date, the analysis values a European option using the Black–Scholes model. The pseudo-American price is the maximum of these European option prices. Implicitly, the pricing strategy assumes exercise on the date that gives the highest European option price.

8. Consider a stock with a price of \$140 and a standard deviation of 0.4. The stock will pay a dividend of \$2 in 40 days and a second dividend of \$2 in 130 days. The current risk-free rate of interest is 10 percent. An American call on this stock has an exercise price of \$150 and expires in 100 days. What is the price of the call according to the pseudo-American approach?

   First, notice that the second dividend is scheduled to be paid in 130 days, after the option expires. Therefore, the second dividend cannot affect the option price and it may be disregarded. To apply the pseudo-American model, we begin by subtracting the present value of the dividend from the stock price to form the adjusted stock price:

$$\text{Adjusted Stock Price} = \$140 - \$2e^{-0.10(40/365)} = \$138.02$$

   For the single dividend date, we reduce the exercise price by the \$2 of dividend so the adjusted exercise price is \$148. Applying the Black–Scholes model with $S = \$138.02$, $E = \$148$, and $T - t = 40$ days gives a price of \$4.05. Applying the Black–Scholes model with $S = \$138.02$, $E = \$150$, and $T - t = 100$ gives a price of \$8.29. The higher price, \$8.29, is the pseudo-American option price.

9. Could the exact American call pricing model be used to price the option in question 8? Explain.

   Yes. Once we notice that the second dividend falls beyond the expiration date of the option, the exact American model fits exactly and gives a price of \$8.28, almost the same as the pseudo-American price of \$8.29.

10. Explain why the exact American call pricing model treats the call as an "option on an option."

    The exact American model applies to call options on stocks with a single dividend occurring before the option expires. Early exercise of an American call is optimal only at the ex-dividend date. At the ex-dividend date, the holder of an American call has a choice: exercise and own the stock or do not exercise and hold what is then equivalent to a European option that expires at the original expiration date of the American call.

(The option that results from not exercising is equivalent to a European call because there are no more dividends occurring before expiration.) Thus, the exact American call model recognizes that the call embodies an option to own a European option at the dividend date. It also embodies the right to acquire the stock at the stated exercise price at the ex-dividend date.

11. Explain the idea of a bivariate cumulative standardized normal distribution. What would be the cumulative probability of observing two variables both with a value of zero, assuming that the correlation between them was zero? Explain.

    The bivariate cumulative distribution considers the probability of two standardized normal variates having values equal to or below a certain threshold at the same time given a certain correlation between the two. Consider first a univariate standardized normal variate. The probability of its value being zero or less equals the chance that it is below its mean of zero, which is 50 percent. Considering two such variates, with a zero correlation between them, the probability that both have a value of zero or less equals $0.5 \times 0.5 = 0.25$. If the two variables had a correlation other than zero, this probability would be different.

12. In the exact American call pricing model, explain why the model can compute the call price with only one dividend.

    The exact American model uses the cumulative bivariate standardized normal distribution, which considers the correlation between a pair of variates. The formula, for example, evaluates the probability of not exercising and the option finishing in-the-money, and of not exercising and the option finishing out-of-the-money. If there were more dividends, the bivariate distribution would be inadequate to handle all of the possible combinations, and higher multivariate normal distributions would have to be considered. For these, no solution has yet been found.

13. What is the critical stock price in the exact American call pricing model?

    The critical stock price, $S*$, is the stock price that makes the call owner indifferent regarding exercise at the ex-dividend date. If the option is not exercised at the ex-dividend date, the American call effectively becomes a European call and the value is simply given by the Black–Scholes model. If the owner exercises, she receives the stock price, plus the dividend, less the exercise price. Therefore, where $D_1$ is the dividend, the critical stock price makes the following equation hold:

    $$S^* + D_1 - X = \text{European Call}$$

14. Explain how the analytical approximation for American option values is analogous to the Merton model.

    Both models pertain to underlying goods with a continuous dividend rate.

15. Explain the role of the critical stock price in the analytic approximation for an American call.

    The critical stock price is the stock price that makes the owner of an American call indifferent regarding exercise. If the stock price exceeds the critical stock price, the owner should exercise. Otherwise, the option should not be exercised.

16. Why should an American call owner exercise if the stock price exceeds the critical price?

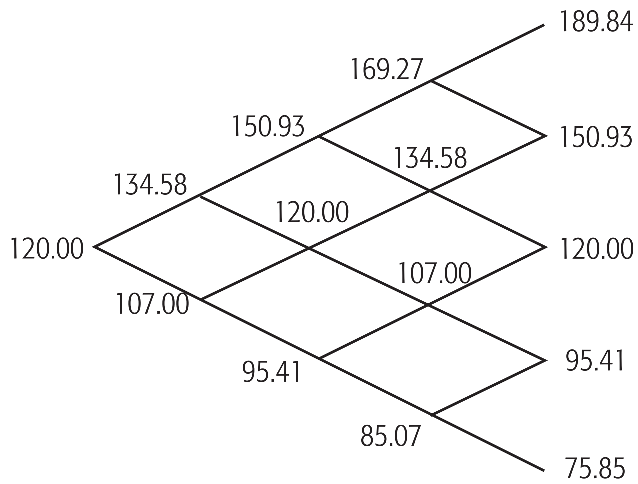
    If the stock price exceeds the critical stock price, the owner should exercise to capture the exercise proceeds. These can be invested to earn a return from the date of exercise to the expiration of the option. The critical stock price is the price at which the benefits of earning that interest just equal the costs of discarding the time value of the option. If the stock price exceeds the critical stock price, the potential interest proceeds are worth more than the time value of the option, and the option should be exercised.

17. Consider the binomial model for an American call and put on a stock that pays no dividends. The current stock price is \$120, and the exercise price for both the put and the call is \$110. The standard deviation of the

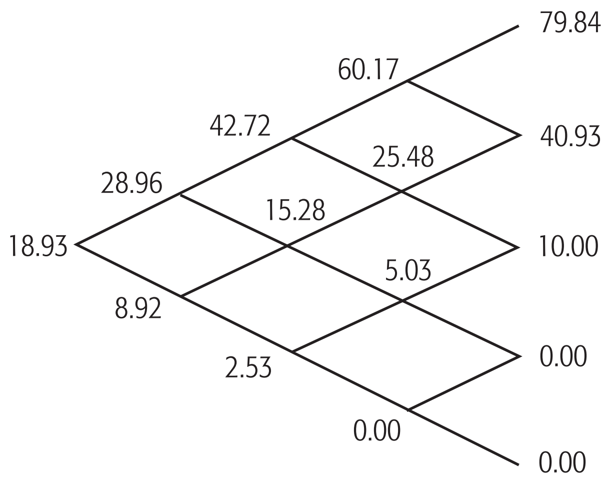
stock returns is 0.4, and the risk-free rate is 10 percent. The options expire in 120 days. Model the price of these options using a four-period tree. Draw the stock tree and the corresponding trees for the call and the put. Explain when, if ever, each option should be exercised. What is the value of a European call in this situation? Can you find the value of the European call without making a separate computation? Explain.

$U = 1.1215$; $D = 0.8917$; $\pi_U = 0.5073$. The American call is worth \$18.93, while the American put is worth \$5.48. With no dividend, the American call should not be exercised at any time. The put should be exercised if the stock price drops three times from \$120.00 to \$85.07. Then the exercisable proceeds would be \$24.93, but the corresponding European put would be worth only \$24.03. The asterisk in the option tree indicates a node at which exercise should occur.

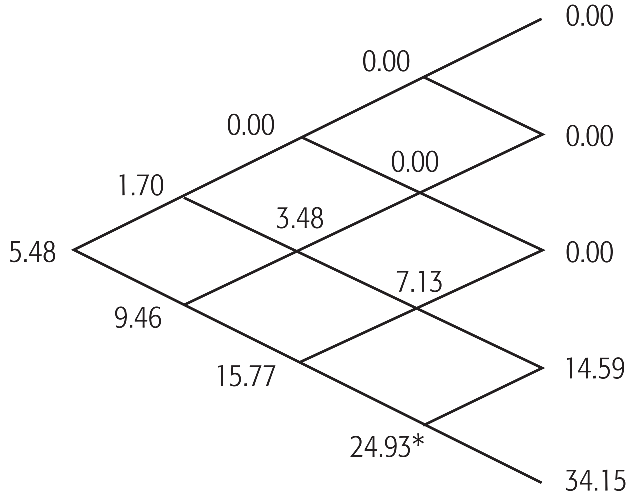
**Stock Price Lattice**




**Call Price Lattice**



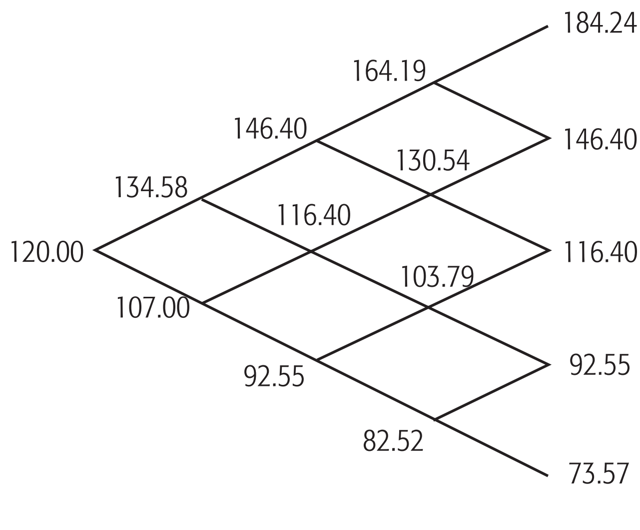
**Put Price Lattice**



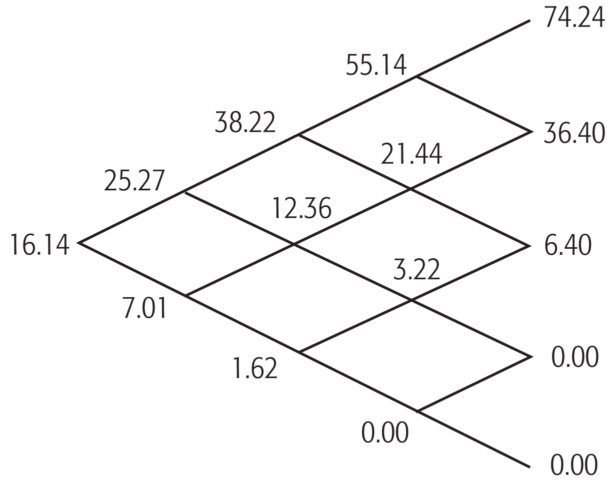

18. Consider the binomial model for an American call and put on a stock whose price is $120. The exercise price for both the put and the call is $110. The standard deviation of the stock returns is 0.4, and the risk-free rate is 10 percent. The options expire in 120 days. The stock will pay a dividend equal to 3 percent of its value in 50 days. Model and compute the price of these options using a four-period tree. Draw the stock tree and the corresponding trees for the call and the put. Explain when, if ever, each option should be exercised.

    $U = 1.1215$; $D = 0.8917$; $\pi_U = 0.5073$. The call is worth $16.14, and the put is worth $6.28. The call should never be exercised. The put should be exercised if the stock price drops three straight times to $82.52. This gives exercisable proceeds of $27.48, compared to a computed value of $26.58. The asterisk in the option tree indicates a node at which exercise should occur.

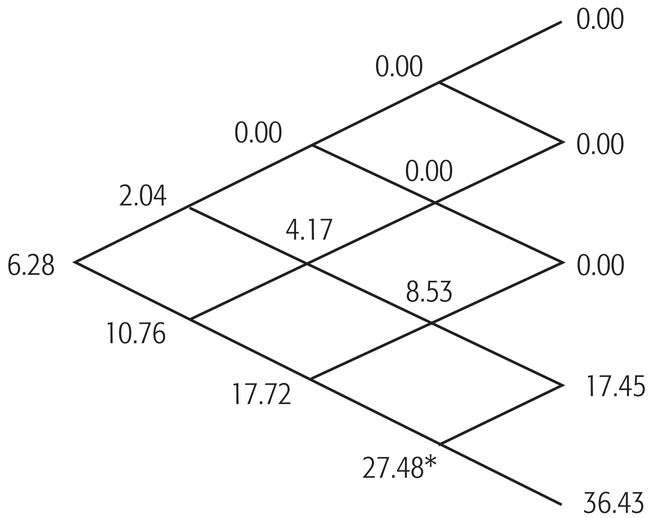
**Stock Price Lattice**



**Call Price Lattice**



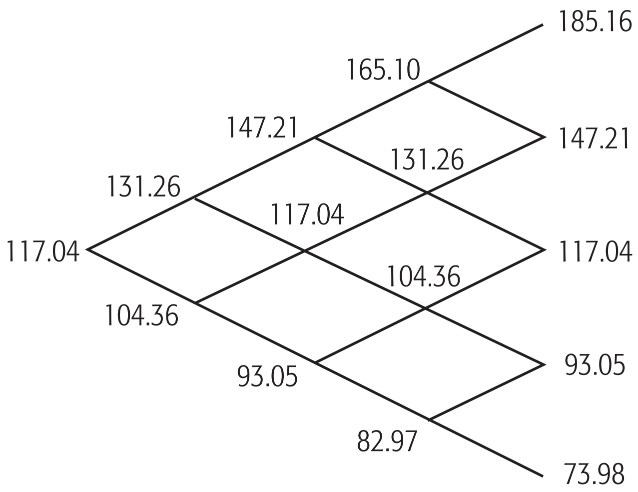

**Put Price Lattice**



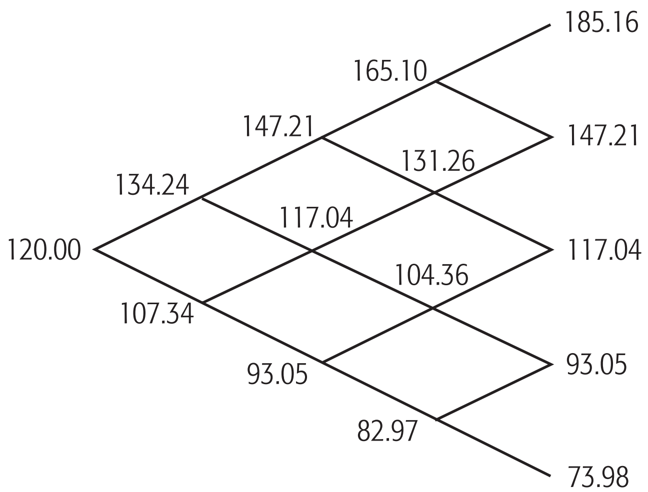

19. Consider the binomial model for an American call and put on a stock whose price is \$120. The exercise price for both the put and the call is \$110. The standard deviation of the stock returns is 0.4, and the risk-free rate is 10 percent. The options expire in 120 days. The stock will pay a \$3 dividend in 50 days. Model and compute the price of these options using a four-period tree. Draw the stock tree and the corresponding trees for the call and the put. Explain when, if ever, each option should be exercised.

$U = 1.1215$; $D = 0.8917$; $\pi_U = 0.5073$. The call is worth \$16.63, while the put is worth \$6.14. The call should never be exercised. The put should be exercised if the stock price drops three straight times to \$82.97. This gives exercisable proceeds of \$27.03, which exceeds the computed value of \$26.13. The asterisk in the option tree indicates a node at which exercise should occur.

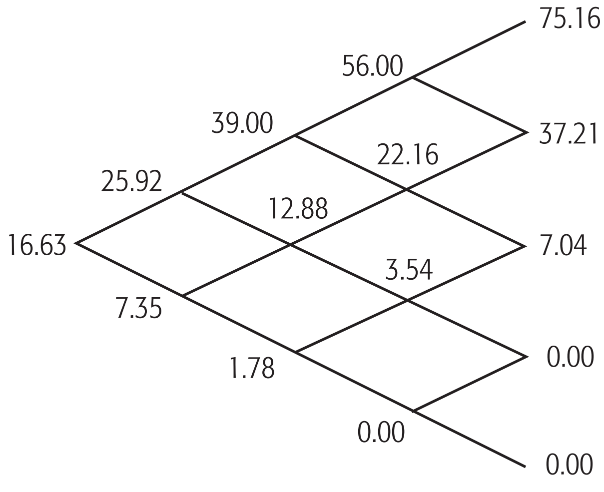
**Stock Price Lattice**



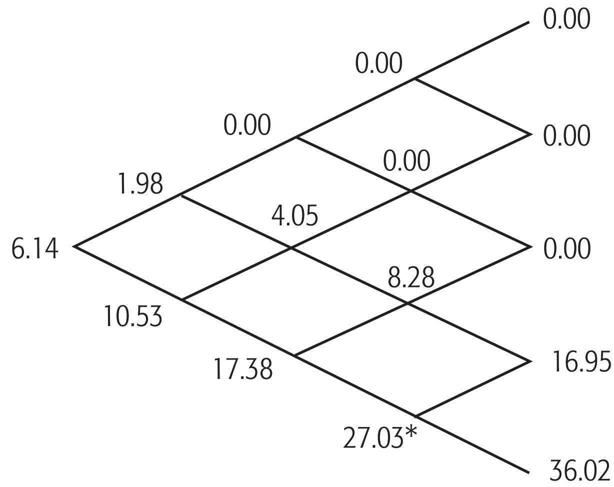

**Adjusted Stock Price Lattice**




**Call Price Lattice**



**Put Price Lattice**




20. Consider the analytic approximation for American options. A stock sells for \$130, has a standard deviation of 0.3, and pays a continuous dividend of 3 percent. An American call and put on this stock both have an exercise price of \$130, and they both expire in 180 days. The risk-free rate is 12 percent. Find the value of the call and put according to this model. Demonstrate that you have found the correct critical stock price for both options.

For the call, the critical price is $S^* = \$604.08$. For the put, the critical price is $S^{**} = \$103.88$. To verify that these critical prices are correct, we need to show that they satisfy the following two equations.

$$\text{Call:} \quad S^* - X = c_t(S^*, X, T - t) + \left\{1 - e^{-\delta(T-t)} N(d_1)\right\}(S^*/q_2)$$

$$\text{Put:} \quad X - S^{**} = p_t(S^{**}, X, T - t) - \left\{1 - e^{-\delta(T-t)} N(-d_1)\right\}(S^{**}/q_1)$$

$$q_1 = \frac{1 - n - \sqrt{(n-1)^2 + 4k}}{2}$$

$$q_2 = \frac{1 - n + \sqrt{(n-1)^2 + 4k}}{2}$$

$$n = \frac{2(r - \delta)}{\sigma^2}, \quad k = \frac{2r}{\sigma^2(1 - e^{-r(T-t)})}$$

With these values:

$$n = 2(0.12 - 0.03)/(0.3 \times 0.3) = 2.00$$

$$k = (2 \times 0.12)/[0.3 \times 0.3(1 - 0.9425)] = 0.24/0.0052 = 46.4082$$

$$q_1 = \frac{1 - 2 - \sqrt{(2-1)^2 + 4(46.4082)}}{2} = -7.3307$$

$$q_2 = \frac{1 - 2 + \sqrt{(2-1)^2 + 4(46.4082)}}{2} = 6.3307$$

For the call, evaluating $d_1$ at the critical price for the call, \$604.08, gives $d_1 = 35.4167$:

$$d_1 = \frac{\ln\left(\frac{604.08}{130}\right) + [0.12 - 0.03 + 0.5(0.3)(0.3)]\left(\frac{180}{365}\right)}{0.3\sqrt{\frac{180}{365}}} = 35.4167$$

For the put, evaluating $d_1$ at the critical price for the put, \$103.88, gives $d_1 = -0.7487$:

$$d_1 = \frac{\ln\left(\frac{103.88}{130}\right) + [0.12 - 0.03 + 0.5(0.3)(0.3)]\left(\frac{180}{365}\right)}{0.3\sqrt{\frac{180}{365}}} = -0.7487$$

For the call, $N(d_1) = N(35.4167) = 1.0000$, while for the put, $N(-d_1) = N(-27.0603) = 0.772981$. The prices of the corresponding European call and put, each evaluated at its critical price, are \$472.68 and \$22.74, respectively.

With these values, we now verify that the specified critical prices are correct. For the call:

$$604.08 - 130.00 = 474.08 = 472.68 + (0.0147)(604.08/6.3307)$$

For the put:

$$130.00 - 103.88 = 26.12 = 22.74 - (0.2384)(103.88/-7.3307)$$

21. An American call and put both have an exercise price of \$100. An acquaintance asserts that the critical stock price for both options is \$90 under the analytic approximation technique. Comment on this claim and explain your reasoning.

Something is amiss. The critical price for a call must lie above the exercise price, while the critical price for a put must lie below the exercise price. Therefore, \$90 might be the critical price for the put, but it cannot be the critical price for the call.

22. Consider a stock with a price of \$80 and a standard deviation of 0.3. The stock will pay a \$5 dividend in 70 days. The current risk-free rate of interest is 10 percent. Options written on this stock have an exercise price of \$80 and expire in 120 days. Model and compute the price of these options using a four-period tree.

A. Draw the stock price trees.

$$U = e^{\sigma\sqrt{\Delta t}} = e^{0.3\sqrt{30/365}} = 1.0898$$

$$D = 1/U = 1/1.0898 = 0.9176$$

$$\pi_U = \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.1 \times 30/365} - 0.9176}{1.0898 - 0.9176} = 0.5264 \qquad \pi_D = 1 - 0.5264 = 0.4736$$

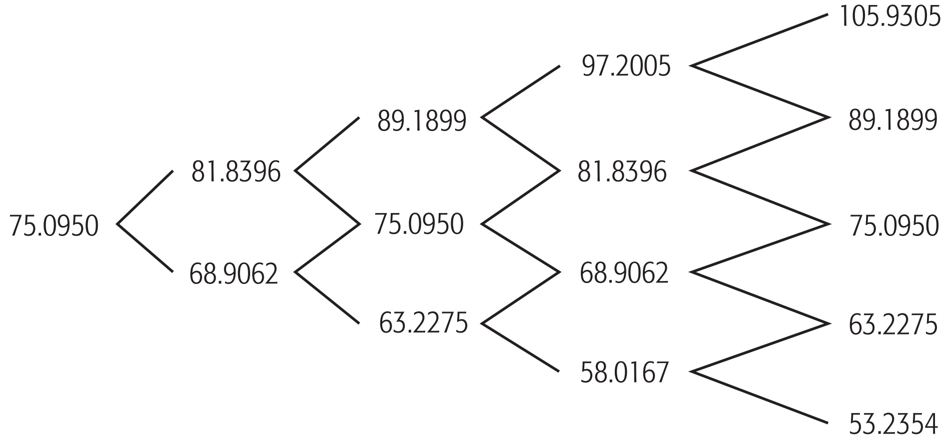
$$e^{-r\Delta t} = 0.9918$$

To construct the stock price tree necessary to calculate the value of this option, we must adjust the current stock price of \$80 downward by the present value of the dividends to be received prior to the option's expiration. In this problem, the dividend of \$5.00 will be paid in 70 days.

$$D_1 X e^{-r(T-t)} = \$5.00 \times e^{-0.1 \times (70/365)} = \$4.9050$$

$$S' = \$80 - \$4.9050 = \$75.0950$$

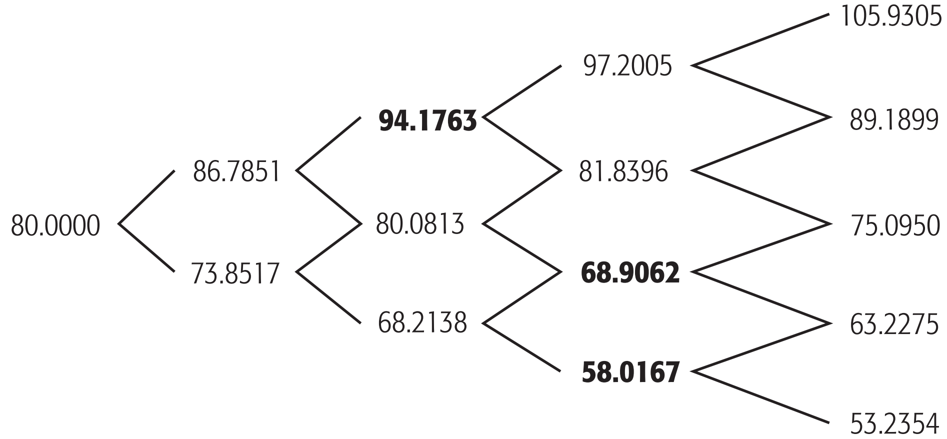
**Stock price tree**




After constructing the stock price tree, the prices in the tree must be adjusted upward by the present value of the dividend yet to be received. That is, we must add the present value of \$5.00 to be received in 40 days to the stock prices at node one, and we must add the present value of \$5.00 to be received in 10 days to the stock prices at node two. Thus, we add \$4.9050 to the node zero stock price, \$4.9455 to the node one stock prices, and \$4.9863 to the node two stock prices.

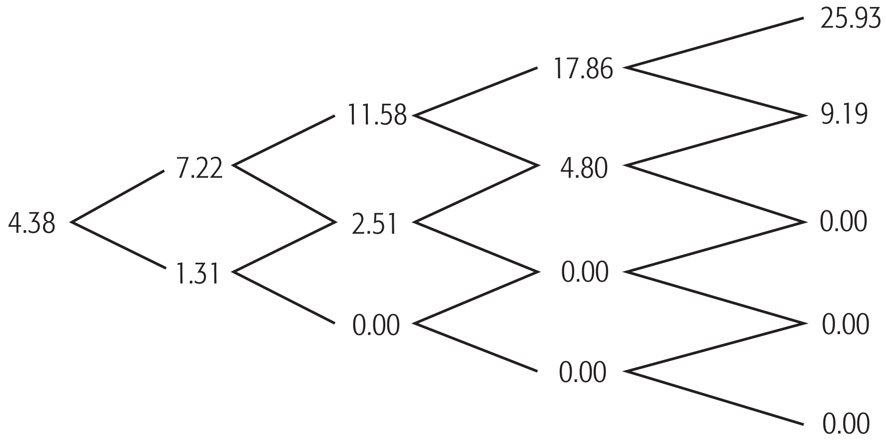
$$D_1 Xe^{-r(T-t)} = \$5.00 \times e^{-0.1 \times (70/365)} = \$4.9050$$
$$D_1 Xe^{-r(T-t)} = \$5.00 \times e^{-0.1 \times (40/365)} = \$4.9455$$
$$D_1 Xe^{-r(T-t)} = \$5.00 \times e^{-0.1 \times (10/365)} = \$4.9863$$

**Dividend-adjusted stock price tree**



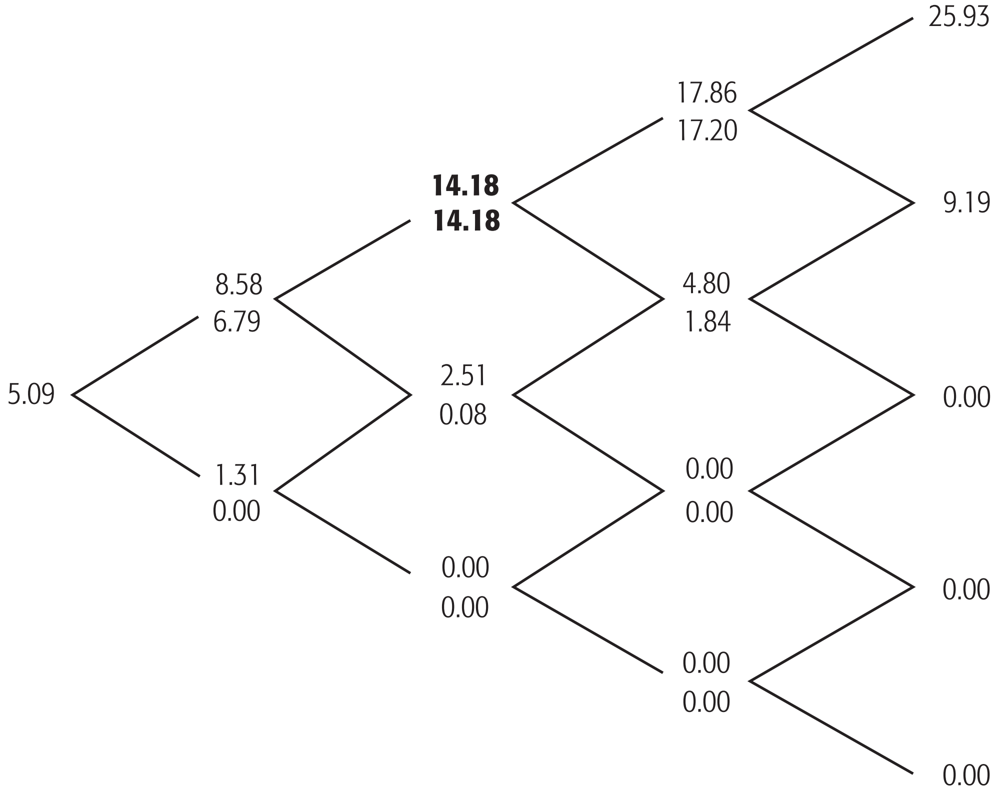

B. Calculate the values of European and American call and put options written on this stock. Value the options using the recursive procedure. Construct the price trees for each option.

**European call price tree**

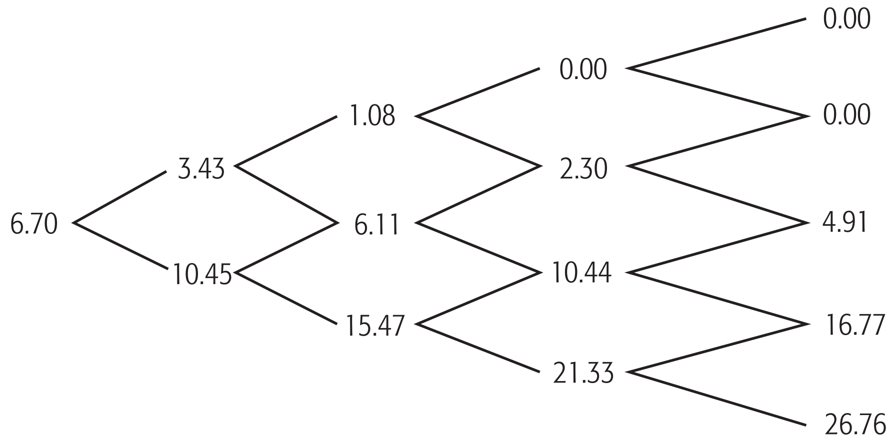



**American call price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.

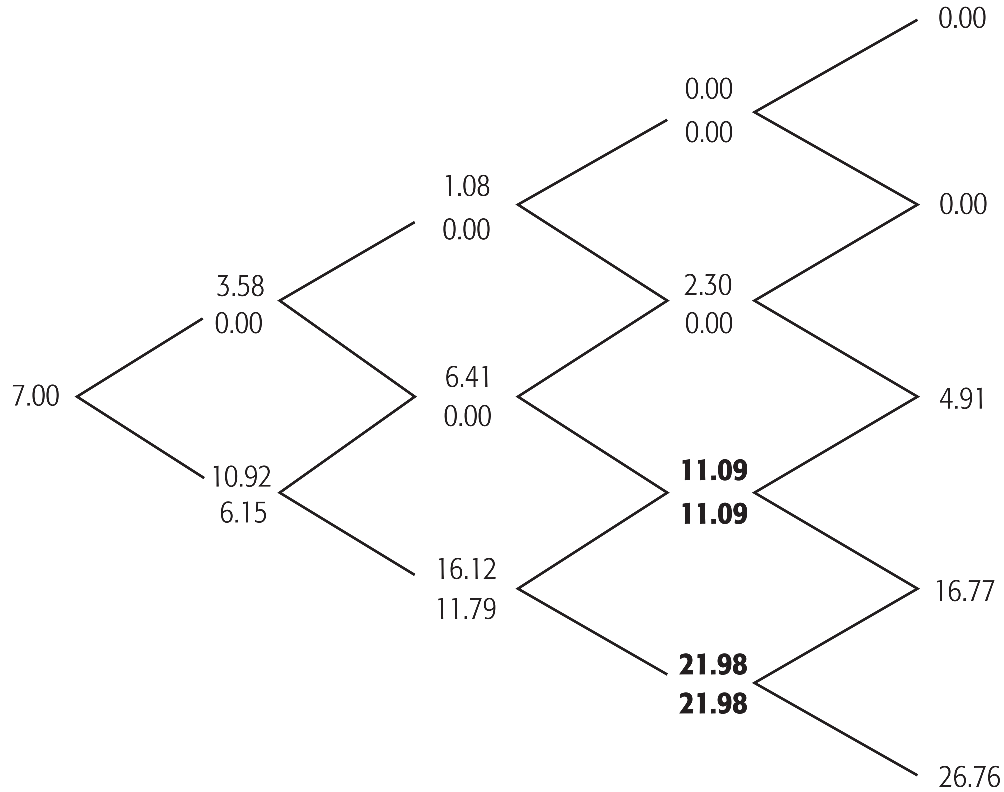


**European put price tree**




**American put price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.




The prices for the American call and put options are \$5.09 and \$7.00, respectively. The prices for the corresponding European call and put options are \$4.38 and \$6.70.

C. Compare the prices of the European and American options. How much value does the right to exercise the option before expiration add to the value of the American options?

In both cases the American options are more valuable than the equivalent European options. The American call is worth \$.71 more than the European call, and the American put is worth \$.29 more than the European put.

D. Explain when, if ever, each option should be exercised.

Theory tells us that it will only be rational for the investor to exercise an American call option immediately before a dividend is paid, and that the rational exercise of an American put will occur immediately after a dividend is paid. The dividend will be paid in 70 days, which is between the second and third branches in the stock price tree used to value the options. Examination of both the stock price and option pricing trees reveals the following: The call option should be exercised early if the stock price rises to \$94.18 after two periods. At this stock price, the intrinsic value of the American option, \$14.18, is greater than the value of the European option, \$11.58. The put option should be exercised if the stock price falls to \$68.91 or lower. It should also be exercised at a stock price of \$58.02 or less after three periods. At this stock price, \$68.91, the intrinsic value of the American put option, \$11.09, is greater than the value of the European option, \$10.44.

23. Consider a stock with a price of \$70 and a standard deviation of 0.4. The stock will pay a dividend of \$2 in 40 days and a second dividend of \$2 in 130 days. The current risk-free rate of interest is 10 percent. An American call on this stock has an exercise price of \$75 and expires in 180 days. What is the price of the call according to the pseudo-American approach?

Theory suggests that the early exercise of a call will occur immediately before a dividend. The pseudo-American pricing "model" views each dividend date as a potential date for early exercise and estimates the value of the American call by evaluating a portfolio of European call options. Because there are two dividends paid during the life of the option, we must determine the value of three European call options to price this call option using the pseudo-American pricing methodology. The valuation technique requires an adjustment to the current stock price equal to the present value of all dividends to be received over the life of the option. In addition, at each potential early exercise date, that is, the dividend date, we decrease the strike price of the option by the present value of the dividend yet to be received. In other words, for the option that expires in forty days, we reduce the strike price of the option by \$2 for the dividend to be paid that day, and the present value of the \$2 dividend that will be paid 90 days in the future. The estimated value of the American call option is equal to the value of the European call option with the largest value.

To calculate the value of the European options using the Black–Scholes model, we must adjust the current stock price of \$70 downward by the present value of the dividends to be received before the option's expiration. In this problem, both dividends are paid before the option's expiration. The first dividend will be received in 40 days, and the second will be received in 130 days.

$$D_1\,Xe^{-r(T-t)} = \$2 \times e^{-0.1 \times (40/365)} = \$1.98$$

$$D_2\,Xe^{-r(T-t)} = \$2 \times e^{-0.1 \times (130/365)} = \$1.93$$

$$S' = \$70 - \$1.98 - \$1.93 = \$66.09$$

**Option #1 that expires in 180 days**

$$d_1 = \frac{\ln(66.09/75) + ([0.10 + 0.5(0.4^2)](0.4932))}{0.40\sqrt{(0.4932)}} = -0.1341$$

$$d_2 = -0.1341 - 0.40\sqrt{(0.4932)} = -0.4150$$

$$N(d_1) = 0.446650 \quad N(d_2) = 0.339061$$

$$c = 66.09 \times 0.446650 - 75e^{-0.1 \times 0.4932} \times 0.339061 = \$5.31$$

**Option #2 that expires in 40 days**

To calculate the value of the European option that expires in 40 days using the Black–Scholes model, we must decrease the strike price of the option, \$75, by the present value of the dividends to be received after

40 days, but before the option's expiration. In this case, both dividends are paid before the option's expiration. The first dividend will be received immediately and is equal to \$2, and the second \$2 dividend will be received in 90 days.

$$D_1 Xe^{-r(T-t)} = \$2 \times e^{-0.1 \times (90/365)} = \$1.95$$

$$X' = \$75 - \$2 - \$1.95 = \$71.05$$

$$d_1 = \frac{\ln(66.09/71.05) + ([0.10 + 0.5(0.4^2)](0.1096))}{0.40\sqrt{(0.1096)}} = -0.3972$$

$$d_2 = -0.3972 - 0.40\sqrt{(0.1096)} = -0.5296$$

$$N(d_1) = 0.345612 \quad N(d_2) = 0.298190$$

$$c = 66.09 \times 0.345612 - 71.05e^{-0.1 \times 0.1096} \times 0.298190 = \$1.89$$

**Option #3 that expires in 130 days**

To calculate the value of the European option that expires in 130 days using the Black–Scholes model, we must decrease the strike price of the option, \$75, by the amount of the dividends to be received prior to the option's expiration. In this case, the second dividend of \$2 is paid on day 130. Thus, the strike price of \$75 will be reduced by \$2 to \$73.

$$X' = \$75 - \$2 = \$73$$

$$d_1 = \frac{\ln(66.09/73) + ([0.10 + 0.5(0.4^2)](0.3562))}{0.40\sqrt{(0.3562)}} = -0.1479$$

$$d_2 = -0.1479 - 0.40\sqrt{(0.3562)} = -0.3866$$

$$N(d_1) = 0.441212 \quad N(d_2) = 0.349521$$

$$c = 66.09 \times 0.441212 - 73e^{-0.1 \times 0.3562} \times 0.349521 = \$4.54$$

The value of the American call using the pseudo-American pricing methodology is the largest of the three option values, \$5.31.

$$C = \text{MAX}(\$5.31, \$1.89, \$4.54) = \$5.31$$

24. Consider a stock with a price of \$140 and a standard deviation of 0.4. The stock will pay a dividend of \$5 in 40 days and a second dividend of \$5 in 130 days. The current risk-free rate of interest is 10 percent. An American call on this stock has an exercise price of \$150 and expires in 100 days.

A. What is the price of the call according to the pseudo-American approach?

Theory suggests that the early exercise of a call will occur immediately before a dividend. The pseudo-American pricing "model" views each dividend date as a potential date for early exercise and estimates the value of the American call by evaluating a portfolio of European call options. Since there is only one dividend paid during the life of the option, we must determine the value of two European call options to price this call option using the pseudo-American pricing methodology. The valuation technique requires an adjustment to the current stock price equal to the present value of all dividends to be received over the life of the option. In addition, at each potential early exercise date, that is, the first dividend date, we decrease the strike price of the option by the present value of the dividend yet to be received before the option expires. The estimated value of the American call option is equal to the value of the European call option with the largest value.

To calculate the value of the European options using the Black–Scholes model, we must adjust the current stock price of \$140 downward by the present value of the dividends to be received before the option's expiration. In this problem, the second dividend is paid after the option expires and is irrelevant in the pricing of this option. The dividend of \$5 will be received in 40 days.

$$D_1\, Xe^{-r(T-t)} = \$5 \times e^{-0.1 \times (40/365)} = \$4.95$$
$$S' = \$140 - \$4.95 = \$135.05$$

**Option #1 that expires in 100 days**

$$d_1 = \frac{\ln(135.05/150) + ([0.10 + 0.5(0.4^2)](0.2740))}{0.40\sqrt{(0.2740)}} = -0.2658$$

$$d_2 = -0.2658 - 0.40\sqrt{(0.2740)} = -0.4751$$

$$N(d_1) = 0.395212 \quad N(d_2) = 0.317348$$

$$c = 135.05 \times 0.395212 - 150e^{-0.1 \times 0.2740} \times 0.317348 = \$7.06$$

**Option #2 that expires in 40 days**

To calculate the value of the European option that expires in 40 days using the Black–Scholes model, we must decrease the strike price of the option, \$150, by the amount of the dividends to be received after day 40 but before the option expires. In this case, the second dividend of \$5 is paid on day 40. Thus, the strike price of \$150 will be reduced by \$5 to \$145.

$$X' = \$150 - \$5 = \$145$$

$$d_1 = \frac{\ln(135.05/145) + ([0.10 + 0.5(0.4^2)](0.1096))}{0.40\sqrt{(0.1096)}} - 0.3876$$

$$d_2 = -0.3876 - 0.40\sqrt{(0.1096)} = -0.5201$$

$$N(d_1) = 0.349143 \quad N(d_2) = 0.301514$$

$$c = 135.05 \times 0.349143 - 145e^{-0.1 \times 0.1096} \times 0.301514 = \$3.91$$

The value of the American call using the pseudo-American pricing methodology is the largest of the two option values, \$7.06.

$$C = \text{MAX}(\$7.06, \$3.91) = \$7.06$$

B. What is the price of the call according to the compound option pricing model?

The first step necessary to value this American call option is to determine the critical stock price, $S^*$. The critical stock price is the stock price that makes the investor indifferent between holding an option until expiration, and exercising the option—thereby receiving the stock and the dividend. The critical stock price, $S^*$, is determined by solving the following equation, $S^* + D - X = c$, where the European call option, $c$, has a life of 60 days beginning 40 days in the future. That is, if the American call option is not exercised on the dividend date, then the investor holds an American call option written on a stock that does not pay a dividend. We can then value the American call option as a European call option. This option has 60 days until expiration, and we assume that the interest rate and the volatility of the stock remain constant. The critical stock price, $S^*$, is \$170.90.

$$d_1 = \frac{\ln(170.90/150) + ([0.10 + 0.5(0.4^2)](60/365))}{0.40\sqrt{(60/365)}} = 0.9868$$

$$d_2 = 0.9868 - 0.40\sqrt{60/365} = 0.8246$$

$$N(d_1) = 0.838127 \quad N(d_2) = 0.795204$$

$$c = 170.90 \times 0.838127 - 150e^{-0.1 \times 60/365} \times 0.795204 = \$25.90$$

$$S^* + D - X - c = 0 \quad \$170.90 + \$5 - \$150 - \$25.90 = 0$$

$$a_1 = \frac{\ln\left(\dfrac{140 - 5e^{-0.1(40/365)}}{150}\right) + [0.1 + 0.5(0.4)(0.4)](100/365)}{0.4\sqrt{100/365}} = -0.2658$$

$$a_2 = -0.2658 - 0.4\sqrt{100/365} = -0.4751$$

$$b_1 = \frac{\ln\left(\dfrac{140 - e^{-0.1(40/365)}}{170.90}\right) + [0.1 + 0.5(0.4)(0.4)](40/365)}{0.4\sqrt{40/365}} = -1.6288$$

$$b_2 = -1.6288 - 0.4\sqrt{40/365} = -1.7612 - \sqrt{\frac{t_1 - t}{T - t}} = -\sqrt{\frac{40}{180}} = -0.6325$$

$$N(b_1) = 0.051681 \quad N(b_2) = 0.039104$$

$$N_2[-0.2658; 1.6288; -0.6325] = 0.348740$$

$$N_2[-0.4751; 1.7612; -0.6325] = 0.283474$$

The price of the American call using the compound option pricing model is \$7.10.

C. What is the critical stock price, $S^*$? Discuss the implications of this finding on the likelihood of exercising the call option early.

As computed earlier, the critical stock price is \$170.90, which is \$30.90 higher than the current stock price of \$140. Thus, it is highly unlikely that the call option will be exercised early. That is, it is very unlikely that the stock price will rise \$30.90 in forty days. Thus, the value of the early exercise premium is very small.

D. Compare the prices calculated using the three option pricing methods.

Since the critical stock price of \$170.90 is \$30.90 higher than the current stock price of \$140, it is highly unlikely that the call option will be exercised early. That is, it is very unlikely that the stock price will rise \$30.90 in forty days. In this problem, the approximations provided by the pseudo-American call option pricing model and the dividend-adjusted Black–Scholes European call pricing model are very close to the value produced by the compound option pricing model. Consequently, the prices for the American call options calculated using the three different option pricing models differ by four cents. Thus, the value of the right to exercise this option early is very small.

25. Consider the binomial model for an American call and put on a stock that pays no dividends. The current stock price is \$120, and the exercise price for both the put and the call is \$110. The standard deviation of the stock returns is 0.4, and the risk-free rate is 10 percent. The options expire in 120 days. Model the price of these options using a four-period tree.

A. Draw the stock price tree and the corresponding trees for the call and the put options.

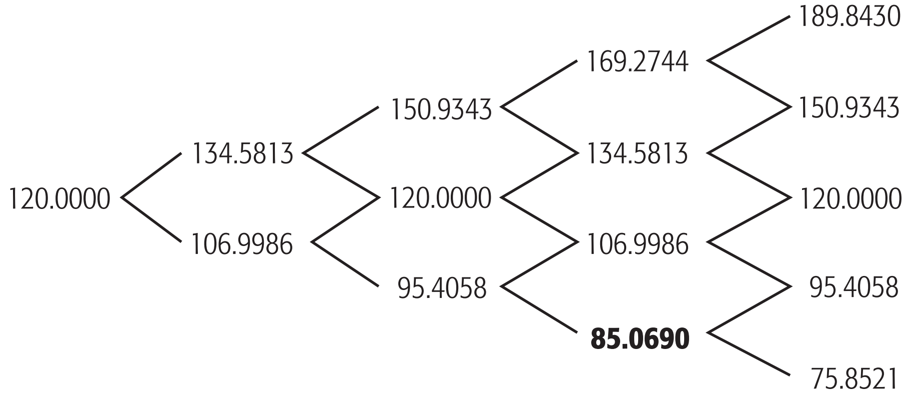
$$U = e^{\sigma\sqrt{\Delta t}} = e^{0.4\sqrt{30/365}} = 1.1215$$

$$D = 1/U = 1/1.1215 = 0.8917$$

$$\pi_U = \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.1 \times 30/365} - 0.8917}{1.1215 - 0.8917} = 0.5073 \quad \pi_D = 1 - 0.5073 = 0.4927$$

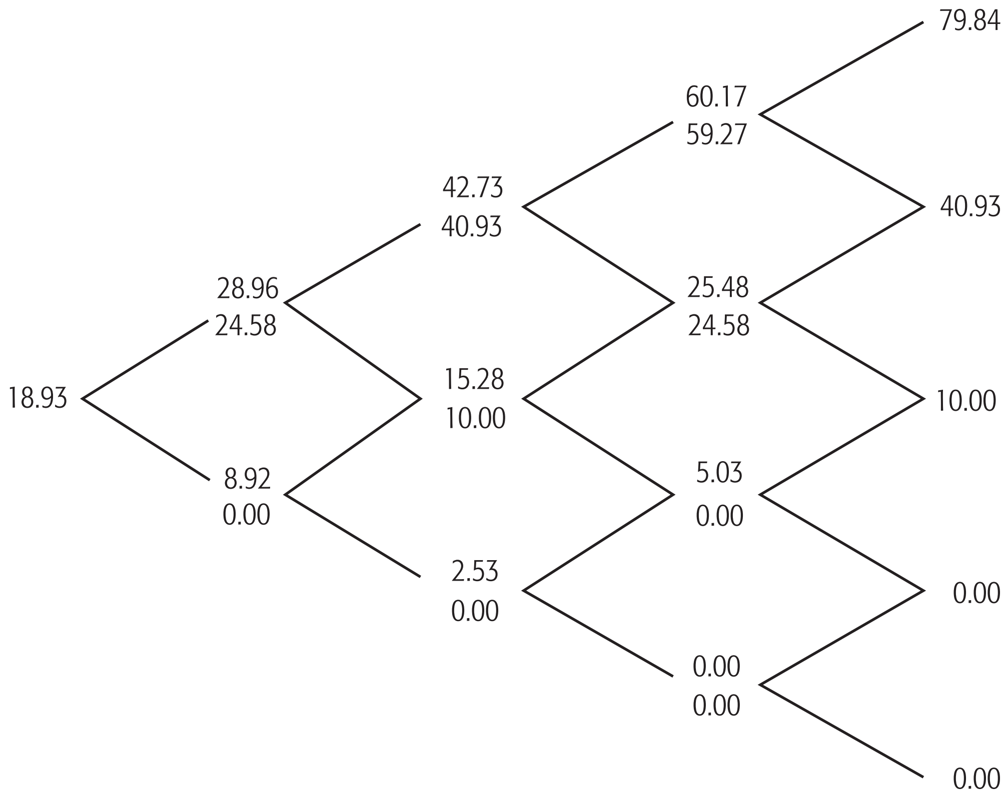
$$e^{-r\Delta t} = 0.9918$$

**Stock price tree**



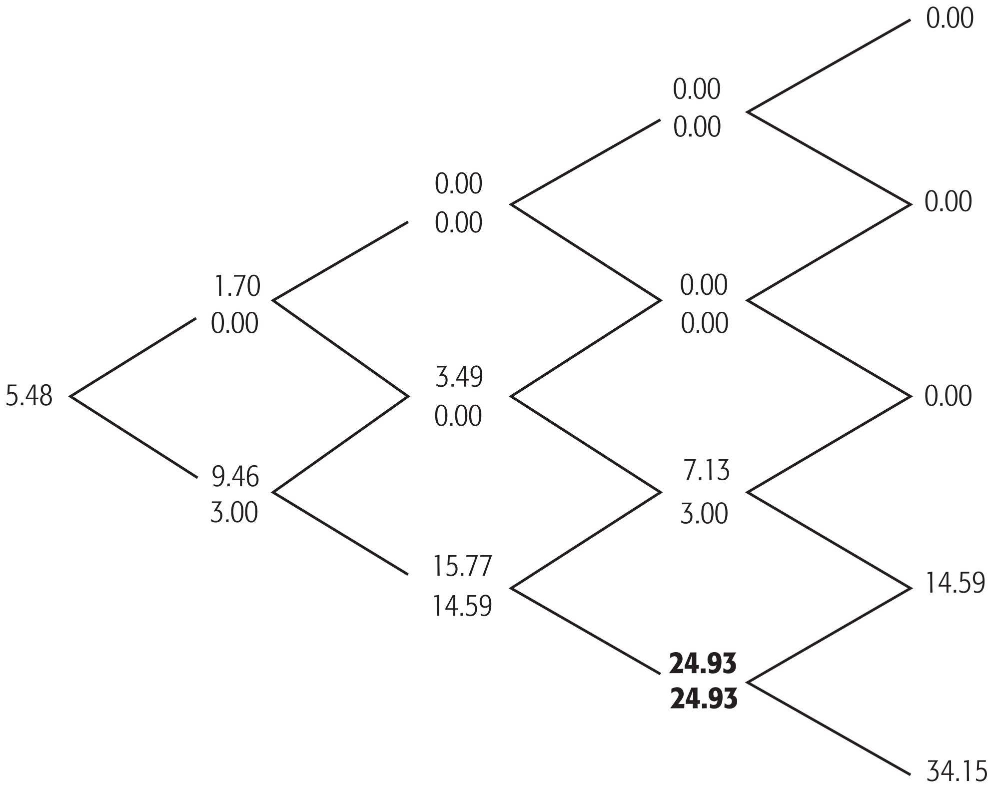

**American call price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.



**American put price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.




B. What is the value of each of the two options? Value the options using the recursive procedure.

The value of the American call option is $18.93, and the American put option is $5.48.

C. Explain when, if ever, each option should be exercised.

Since the stock underlying the options does not pay dividends, it would never be rational to exercise the American call early. However, this is not true for the American put option. If the stock price falls to $85.07, then the investor should exercise the option.

D. What is the value of a European call written on this stock? Can you find the value of the European call without making a separate computation? Explain.

Since exercising the American call early is not rational, the right to exercise the option before expiration is worthless. Thus, the price of the European call is the same as the price of the American call, $18.93.

26. Consider the binomial model for an American call and put on a stock whose price is $50. The exercise price for both the put and the call is $55. The standard deviation of the stock returns is 0.35, and the risk-free rate is 10 percent. The options expire in 160 days. The stock will pay a dividend equal to 4 percent of its value in 65 days. Model and compute the price of these options using a four-period tree.

A. Draw the stock price tree.

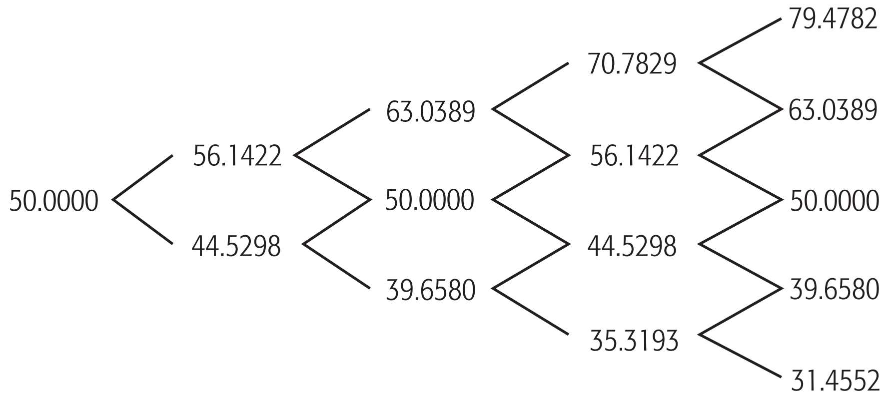
$$U = e^{\sigma\sqrt{\Delta t}} = e^{0.35\sqrt{40/365}} = 1.1228$$

$$D = 1/U = 1/1.1228 = 0.8906$$

$$\pi_U = \frac{e^{r\Delta t} - D}{U - D} = \frac{e^{0.1 \times 40/365} - 0.8906}{1.1228 - 0.8906} = 0.5185 \qquad \pi_D = 1 - 0.5185 = 0.4815$$

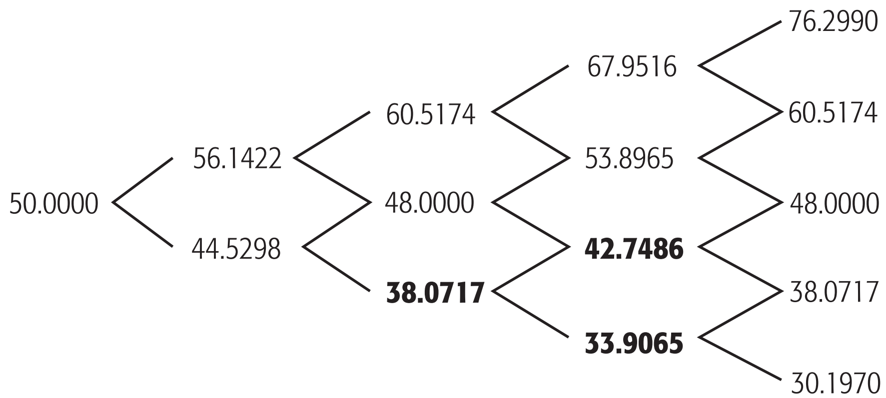
$$e^{-r\Delta t} = 0.9891$$

**Stock price tree**




The stock prices in the tree must be adjusted for the dividend to be paid in 65 days before calculating the value of the options. Therefore, the stock prices in the tree in periods two, three, and four must be adjusted downward by one minus the dividend yield paid by the firm (1 − 4%).

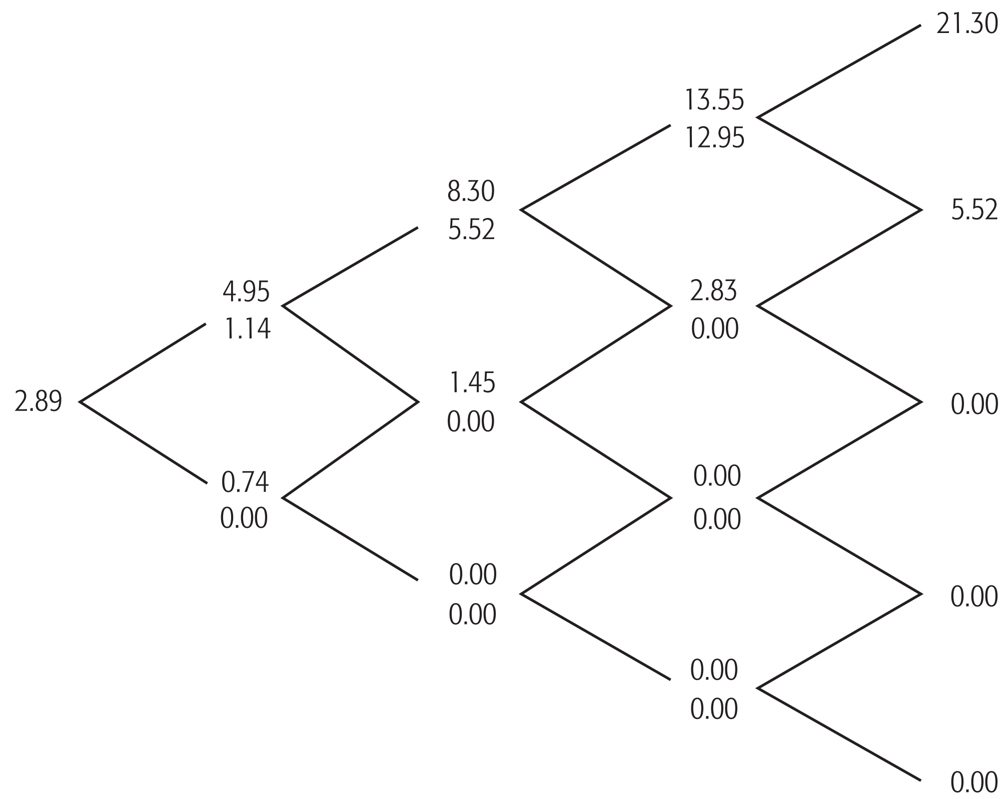
**Dividend-adjusted stock price tree**




B. What is the value of each of the two options? Value the options using the recursive procedure. Draw the tree for each option.

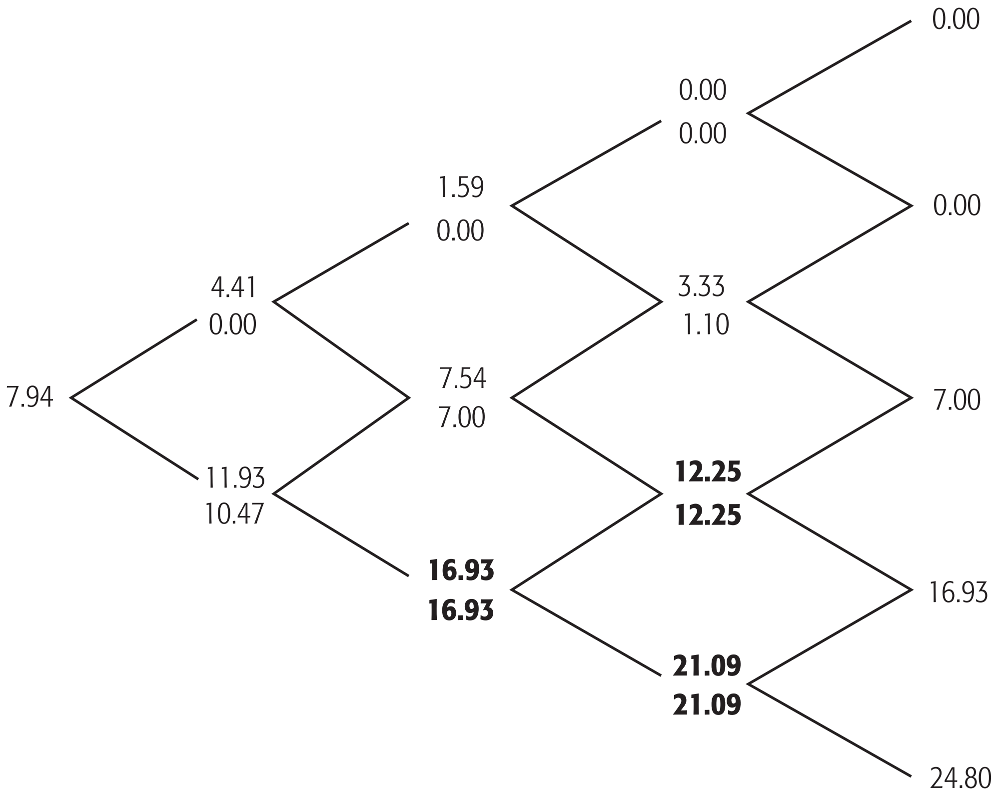
**Call option price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.



**Put option price tree**

Note: The second value at each node in the tree is the intrinsic value of the option.




The price of the American call option is \$2.89, and the price of the American put option is \$7.94.

C. Explain when, if ever, each option should be exercised.

Theory tells us that it will only be rational for the investor to exercise an American call option immediately before a dividend is paid, and that the rational exercise of an American put will occur immediately after a dividend is paid. The dividend will be paid in 65 days, which is between the first and second branches in the stock price tree used to value the options. Examination of both the stock price and option pricing trees reveals the following. The call option should not be exercised early. The put option should be exercised if the stock price falls to \$42.75 or lower after two periods. At the stock prices of \$42.75, \$38.07, and \$33.91, the intrinsic values of the put options are greater than the value of the corresponding options, and the option should be exercised early if the stock price falls to these levels.