12 Bounds on Option Prices

## Answers to Questions and Problems

1. What is the maximum theoretical value for a call? Under what conditions does a call reach this maximum value? Explain.

The highest price theoretically possible for a call option is to equal the value of the underlying stock. This happens only for a call option that has a zero exercise price and an infinite time until expiration. With such a call, the option can be instantaneously and costlessly exchanged for the stock at any time. Therefore, the call must have at least the value of the stock itself. Yet it cannot be worth more than the stock, because the option merely gives access to the stock itself. As a consequence, the call must have the same price as the stock.

2. What is the maximum theoretical value for an American put? When does it reach this maximum? Explain.

The maximum value of a put equals the potential inflow of the exercise price minus the associated outflow of the stock price. The maximum value for this quantity occurs when the stock price is zero. At that time, the value of the put will equal the exercise price. In this situation, the put gives immediate potential access to the exercise price because it is an American option.

3. Answer question 2 for a European put.

As with the American put, the European put attains its maximum value when the stock price is zero. However, before expiration, the put cannot be exercised. Therefore, the maximum price for a European put is the present value of the exercise price, when the exercise price is discounted at the risk-free rate from expiration to the present. This discounting reflects the fact that the owner of a European put cannot exercise now and collect the exercise price. Instead, he or she must wait until the option expires.

4. Explain the difference in the theoretical maximum values for an American and a European put.

The exercise value of a put option equals the exercise price (an inflow) minus the value of the stock at the time of exercise (an outflow). In our notation, this exercise value is X - S. For any put, the maximum value occurs when the stock is worthless, S = 0. The American and European puts have different maximum theoretical values because of the different rules governing early exercise. Because an American put can be exercised at any time, its maximum theoretical value equals the exercise price, X. If the stock price is zero at any time, an American put gives its owner immediate access to amount X through exercise. This is not true of a European put, which can be exercised only at expiration. If the option has time remaining until expiration and the stock is worthless, the European put holder must wait until expiration to exercise. With the stock worthless, the exercise will yield X to the European put holder. Because the exercise must wait until expiration, however, the put can be worth only the present value of

the exercise price. Thus, the theoretical maximum value of a European put is  $Xe^{-r(T-t)}$ . In the special case of an option at expiration, t = 0, the maximum value for a European and an American put is X.

5. How does the exercise price affect the price of a call? Explain.

The call price varies inversely with the exercise price. The exercise price is a potential liability that the call owner faces, because the call owner must pay the exercise price in order to exercise. The smaller this potential liability, other factors held constant, the greater will be the value of a call option.

6. Consider two calls with the same time to expiration that are written on the same underlying stock. Call 1 trades for \$7 and has an exercise price of \$100. Call 2 has an exercise price of \$95. What is the maximum price that Call 2 can have? Explain.

A no-arbitrage condition places an upper bound on the value of Call 2. The price of Call 2 cannot exceed the price of the option with the higher exercise price plus the \$5 difference in the two exercise prices. Thus, the upper bound for the value of Call 2 is \$12. If Call 2 is priced above \$12, say, at \$13, the following arbitrage becomes available.

Sell Call 2 for cash flow +\$13 and buy Call 1 for cash flow -\$7. This is a net cash inflow of +\$6. If Call 2 is exercised against you, you can immediately exercise Call 1. This provides the stock to meet the exercise of Call 1 against you. On the double exercise, you receive \$95 and pay \$100, for a net cash flow of -\$5. However, you received \$6 at the time of trading for a net profit of \$1. This is the worst case outcome.

If Call 1 cannot be exercised, the profit is the full \$6 original cash flow from the two trades. Also, if the stock price lies between \$95 and \$100 when Call 1 is exercised against you, it may be optimal to purchase the stock in the market rather than exercise Call 2 to secure the stock. For example, assume the stock trades for \$98 when Call 1 is exercised against you. In this case, you buy the stock for \$98 instead of exercising Call 2 and paying \$100. Then your total cash flows are +\$6 from the two trades, +\$95 when Call 1 is exercised against you, and -\$98 from purchasing the stock to meet the exercise. Now your net arbitrage profit is \$3. In summary, stock prices of \$95 or below give a net profit of \$6, because Call 1 cannot be exercised. Stock prices of \$100 or above give a net profit of \$1, because you will need to exercise Call 2 to meet the exercise of Call 1. Prices between \$95 and \$100 give a profit equal to +\$6 +\$95 - stock price at the time of exercise.

7. Six months remain until a call option expires. The stock price is \$70 and the exercise price is \$65. The option price is \$5. What does this imply about the interest rate?

We know from the no-arbitrage arguments that:  $C \ge S - Xe^{-r(T-t)}$ . In this case, we have C = S - X exactly. Therefore, the interest rate must be zero.

8. Assume the interest rate is 12 percent and four months remain until an option expires. The exercise price of the option is \$70 and the stock that underlies the option is worth \$80. What is the minimum value the option can have based on the no-arbitrage conditions studied in this chapter? Explain.

We know from the no-arbitrage arguments that:  $C \ge S - Xe^{-r(T-t)}$ . Substituting the specified values gives  $C \ge \$80 - \$70e^{-0.12(0.33)} = \$80 - \$67.28 = \$12.72$ . Therefore, the call price must equal or exceed \$12.72 to avoid arbitrage.

9. Two call options are written on the same stock that trades for \$70, and both calls have an exercise price of \$85. Call 1 expires in six months, and Call 2 expires in three months. Assume that Call 1 trades for \$6 and that Call 2 trades for \$7. Do these prices allow arbitrage? Explain. If they do permit arbitrage, explain the arbitrage transactions.

Here we have two calls that are identical except for their time to expiration. In this situation, the call with the longer time until expiration must have a price equal to or exceeding the price of the shorter-lived option. These values violate this condition, so arbitrage is possible as follows:

Sell Call 2 and buy Call 1 for a net cash inflow of \$1. If Call 2 is exercised at any time, the trader can exercise Call 1 and meet the exercise obligation for a net zero cash flow. This retains the \$1 profit no matter what

## 92 CHAPTER 12 BOUNDS ON OPTION PRICES

happens. It may also occur that the profit exceeds \$1. For example, assume that Call 2 cannot be exercised in the first three months and expires worthless. This leaves the trader with the \$1 initial cash inflow plus a call option with a three-month life, so the trader has an arbitrage profit of at least \$1, and perhaps much more.

10. Explain the circumstances that make early exercise of a call rational. Under what circumstances is early exercise of a call irrational?

Exercising a call before expiration discards the time value of the option. If the underlying stock pays a dividend, it can be rational to discard the time value to capture the dividend. If there is no dividend, it will always be irrational to exercise a call, because the trader can always sell the call in the market instead. Exercising a call on a no-dividend stock discards the time value, while selling the option in the market retains it. Thus, only the presence of a dividend can justify early exercise. Even in this case, the dividend must be large enough to warrant the sacrifice of the time value.

11. Consider a European and an American call with the same expiration and the same exercise price that are written on the same stock. What relationship must hold between their prices? Explain.

Because the American option gives every benefit that the European option does, the price of the American option must be at least as great as that of the European option. The right of early exercise inherent in the American option can give extra value if a dividend payment is possible before the common expiration date. Thus, if there is no dividend to consider, the two prices will be the same. If a dividend is possible before expiration, the price of the American call may exceed that of the European call.

12. Before exercise, what is the minimum value of an American put?

The minimum value of an American put must equal its value for immediate exercise, which is X - S. A lower price results in arbitrage. For example, assume X = \$100, S = \$90, and P = \$8. To exploit the arbitrage inherent in these prices, buy the put and exercise for a net cash outflow of -\\$98. Sell the stock for +\\$100 for an arbitrage profit of \$2.

13. Before exercise, what is the minimum value of a European put?

For a put, the exercise value is X - S. However, a European put can be exercised only at expiration. Therefore, the present value of the exercise value is  $Xe^{-r(T-t)} - S$ , and this is the minimum price of a European put. For example, consider X = \$100, S = \$90, T - t = 0.5 years, and r = 0.10. The no-arbitrage condition implies the put should be worth at least \$5.13.

Assume that the put actually trades for \$5. With these prices, an arbitrageur could trade as follows. Borrow \$95 at 10 percent for six months and buy the stock and the put. This gives an initial net zero cash flow. At expiration, the profit depends upon the price of the stock. First, there will be a debt to pay of \$99.87 in all cases. If the stock price is \$100 or above, the put is worthless and the profit equals S -\$99.87. Thus, the profit will be at least \$.13, and possibly much more. For stock prices below \$100, exercise of the put yields \$100, which is enough to pay the debt of \$99.87 and keep \$.13 profit.

14. Explain the differences in the minimum values of American and European puts before expiration.

The difference in minimum values for American and European puts stems from the restrictions on exercising a put. An American put offers the immediate access to the exercise value X if the put owner chooses to exercise. Because the European put cannot be exercised until expiration, the cash inflow associated with exercise must be discounted to  $Xe^{-r(T-t)}$ . The difference in minimum values equals the time value of the exercise price.

15. How does the price of an American put vary with time until expiration? Explain.

The value of an American put increases with the time until expiration. A longer-lived put offers every advantage that the shorter-lived put does. Therefore, a longer-lived put must be worth at least as much as the shorter-lived put. This implies that value increases with time until expiration. Violation of this condition leads to arbitrage.

16. What relationship holds between time until expiration and the price of a European put?

For a European put, the value may or may not increase with time until expiration. Upon exercise, the put holder receives X - S. If the European put holder cannot exercise immediately, the inflow represented by the exercise price is deferred. For this reason, the value of a European put can be lower the longer the time until expiration. However, having a longer term until expiration also adds value to a put, because it allows more time for something beneficial to happen to the stock price. Thus, the net effect of time until expiration depends on these two opposing forces. Under some circumstances, the value of a European put will increase as time until expiration increases, but it will not always do so.

17. Consider two puts with the same term to expiration (six months). One put has an exercise price of \$110, and the other has an exercise price of \$100. Assume the interest rate is 12 percent. What is the maximum price difference between the two puts if they are European? If they are American? Explain the difference, if any.

For two European puts, the price differential cannot exceed the difference in the present value of the exercise prices. With our data, the difference cannot exceed  $(\$110 - \$100)e^{-0.12(0.5)} = \$9.42$ . If the price differential on the European puts exceeds \$9.42, we have an arbitrage opportunity. To capture the arbitrage profit, we sell the relatively overpriced put with the exercise price of \$110 and buy the put with the \$100 exercise price. If the put we sold is exercised against us, we accept the stock and dispose of it by exercising the put we bought. This will always guarantee a profit. For example, assume that the put with X = \$100 trades for \$5 and the put with X = \$110 sells for \$15, giving a \$10 differential. We sell the put with X = \$100 trades for  $\$10e^{-.12(.5)} = \$10.62$ . If the put we sold is exercised against us, we pay \$110 and receive the stock. We can then exercise our put to dispose of the stock and receive \$10. This gives a \$10 loss on the double exercise. However, our maturing bond is worth \$10.62, so we still have a profit \$.62.

For two American puts, the price differential cannot exceed the difference in the exercise prices. If it does, we conduct the same arbitrage. However, we do not have to worry about the discounted value of the differential, because the American puts carry the opportunity to exercise immediately and to gain access to the value of the mispricing at any time, not just at expiration.

18. How does the price of a call vary with interest rates? Explain.

For a call, the price increases with interest rates. The easiest way to see this is to consider the no-arbitrage condition:  $C \ge S - Xe^{-r(T-t)}$ . The higher the interest rate, the smaller will be the present value of the exercise price, a potential liability. With extremely high interest rates, the exercise price will have an insignificant present value and the call price will approach the stock price.

19. Explain how a put price varies with interest rates. Does the relationship vary for European and American puts? Explain.

Put prices vary inversely with interest rates. This holds true for both American and European puts. For the put owner, the exercise price is a potential inflow. The present value of this inflow, and the market value of the put, increases as the interest rate falls. Therefore, put prices rise as interest rates fall.

20. What is the relationship between the risk of the underlying stock and the call price? Explain in intuitive terms.

Call prices rise as the riskiness of the underlying stock increases. A call option embodies insurance against extremely bad outcomes. Insurance is more valuable the greater the risk it insures against. Therefore, if the underlying stock is very risky, the insurance embedded in the call is more valuable. As a consequence, call prices vary directly with the risk of the underlying good.

21. A stock is priced at \$50, and the risk-free rate of interest is 10 percent. A European call and a European put on this stock both have exercise prices of \$40 and expire in six months. What is the difference between the call and put prices? (Assume continuous compounding.) From the information supplied in this question, can you say what the call and put prices must be? If not, explain what information is lacking.

From put–call parity,  $S - Xe^{-r(T-t)} = C - P$ . Therefore, the C - P must equal:

 $50 - 40e^{-0.5(0.1)} = 11.95$ 

We cannot determine the two option prices. Information about how the stock price might move is lacking.

22. A stock is priced at \$50, and the risk-free rate of interest is 10 percent. A European call and a European put on this stock both have exercise prices of \$40 and expire in six months. Assume that the call price exceeds the put price by \$7. Does this represent an arbitrage opportunity? If so, explain why and state the transactions you would make to take advantage of the pricing discrepancy.

From question 21, we saw that the call price must exceed the put price by \$11.95 according to put-call parity. Therefore, if the difference is only \$7, there is an arbitrage opportunity, and the call price is cheap relative to the put. The long call/short put position is supposed to be worth the same as the long stock/short bond position. But the long call/short put portfolio costs only \$7, not the theoretically required \$11.95. To perform the arbitrage, we would buy the relatively underpriced portfolio and sell the relatively overpriced portfolio. Specifically, we would: buy the call, sell the put, sell the stock, and buy the risk-free bond that pays the exercise price in six months. From these transactions, we would have the following cash flows: from buying the call and selling the put -\$7; from selling the stock +\$50, and from buying the risk-free bond -\$38.05, for a net cash flow of \$4.95. This net cash flow exactly equals the pricing discrepancy.

At expiration, we can fulfill all of our obligations with no further cash flows. If the stock price is below the exercise price, the put we sold will be exercised against us and we must pay \$40 and receive the stock. We will have the \$40 from the maturing bond, and we use the stock that we receive to repay our short sale on the stock. If the stock price exceeds \$40, we exercise our call and use the \$40 proceeds to pay the exercise price. We then fulfill our short sale by returning the share.

23. Cursory examination of the table below shows a violation of a basic option pricing rule. Which pricing rule has been violated? Discuss the limitations of using a newspaper as the source of prices used to make inferences about pricing.

Option IBM	Strike	Exp.	Call		Put	
			Vol.	Last	Vol.	Last
101 <u>5</u>	115	Sep	1632	<u>3</u> 16	10	13 <u>1</u>
101 <sup>5</sup> / <sub>8</sub>	115	Oct	861	$1\frac{1}{4}$	10	$14\frac{1}{4}$
101 <u>5</u>	115	Jan	225	4	3	14

The pricing rule violated is the rule that the price of an American put option increases as the maturity of the option increases. That is, the longer the time until expiration, the more valuable a put option. The only characteristic that varies across the put options in the table is the time until expiration. This is a classic problem of stale prices. The information reported in the paper is historical information. Examining the reported trading volume reveals the likely source of this apparent problem. Very few put options were traded that day. In fact, only 3 Jan IBM 115 put options were traded on the day reported in the table. It is highly likely that the prices reported in the table represent prices recorded at different times during the day. To take advantage of mispricings, one needs to know the prices of all the relevant options at the same point in time.

24. Explain why an American call is always worth at least as much as its intrinsic value. Explain why this is not true for European calls.

The owner of an American call can exercise the option anytime over the life of the option and capture the option's intrinsic value (S - X). Therefore, the price of the American call can never be less that the intrinsic value of the option. The owner of a European call cannot exercise the option until the option's expiration. If the option is in-the-money, the investor cannot capture this gain until the option's expiration; therefore, it is possible for

the value of a European call to be less than the intrinsic value of the option. This is likely to happen if the option is deep-in-the-money and has a long time until expiration.

25. Give an intuitive explanation of why the early exercise of an in-the-money American put option becomes more attractive as the volatility of the underlying stock decreases, and the risk-free interest rate increases.

If the stock's volatility decreases, then the probability of a very large change in the stock price decreases. It is a large stock price move that will take the put deeper-into-the-money, making the put more valuable. If large price decreases are not very likely, then the probability of a large payoff from holding the put is very small and the value of the put will not increase. If at the same time that volatility is increasing the risk-free interest rate is increasing, then the opportunity cost of holding the put will increase. If the put were exercised, the exercise proceeds could be invested at the new higher risk-free rate.

26. Suppose that  $c_1$ ,  $c_2$ , and  $c_3$  are the prices of three European call options written on the same share of stock that are identical in all respects except their strike prices. The strike prices for the three call options are  $X_1$ ,  $X_2$ , and  $X_3$ , respectively, where  $X_3 > X_2 > X_1$  and  $X_3 - X_2 = X_2 - X_1$ . All options have the same maturity. Assume that all portfolios are held to expiration. Show that:

$$c_2 \leq 0.5 \times (c_1 + c_3)$$

Consider two portfolios. The first portfolio, A, consists of a long position in two call options with the strike price  $X_2$ . The second portfolio, B, contains a long position in one call option with a strike price  $X_1$  and a long position in one call option with a strike price of  $X_3$ . Consider the payoffs that are possible at the option's expiration. Remember that there is a fixed dollar difference between the strike prices of the options, for example,  $X_1 = 50$ ,  $X_2 = 55$ , and  $X_3 = 60$ .

	Current Value of Portfolio	Stock Price at Expiration				
Portfolio		<b>S</b> ≤ <b>X</b> <sub>1</sub>	$X_1 < S \leq X_2$	$X_2 < S \leq X_3$	<b>S</b> > X <sub>3</sub>	
A B	$2 c_2 c_1 + c_3$	0 0	$\begin{array}{c} 0\\ S-X_1 \end{array}$	$2(S - X_2)$ $S - X_1$	$2(S - X_2)$ (S - X <sub>1</sub> ) + (S - X <sub>3</sub> )	
Relative value of the portfolio at expiration		$V_A = V_B$	$V_B \ge V_A$	$V_B \ge V_A$	$V_A = V_B$	

Since the value of portfolio B is never less than the value of portfolio A at the expiration of the options, then the current value of portfolio B must be at least as large as the current value of portfolio A. If this were not the case, arbitrage would be possible. Thus,  $c_1 + c_3 - 2$   $c_2 \ge 0$  and  $c_2 \le 0.5 \times (c_1 + c_3)$ .

27. Suppose that  $p_1$ ,  $p_2$ , and  $p_3$  are the prices of three European put options written on the same share of stock that are identical in all respects except their strike prices. The strike prices for the three put options are  $X_1$ ,  $X_2$ , and  $X_3$ , respectively, where  $X_3 > X_2 > X_1$  and  $X_3 - X_2 = X_2 - X_1$ . All options have the same maturity. Assume that all portfolios are held to expiration. Show that:

$$p_2 \le 0.5 \times (p_1 + p_3)$$

Consider two portfolios, A and B. Portfolio A consists of a long position in two put options with a strike price of  $X_2$ . Portfolio B consists of a long position in one put option with a strike price  $X_1$  plus a long position in one put option with a strike price of  $X_3$ . Consider the payoffs that are possible at the option's expiration. Remember that there is a fixed dollar difference between the strike prices of the options, for example,  $X_1 = 50, X_2 = 55$ , and  $X_3 = 60$ .

## 96 CHAPTER 12 BOUNDS ON OPTION PRICES

		Stock Price at Expiration				
Portfolio	Current Value of Portfolio	$S \leq X_1$	$X_1 < S \leq X_2$	$X_2 < S \leq X_3$	<b>S</b> > <b>X</b> <sub>3</sub>	
A	2 p <sub>2</sub>	$2(X_2 - S)$	$2(X_2 - S)$	0	0	
В	$p_1 + p_3$	$(X_1 - S) + (X_3 - S)$	$X_3 - S$	X <sub>3</sub> – S	0	
Relative value of the portfolio at expiration		$V_A = V_B$	$V_B \ge V_A$	$V_B \ge V_A$	$V_A = V_B$	

Because the value of portfolio B is never less than the value of portfolio A at the expiration of the options, the current value of portfolio B must be at least as large as the current value of portfolio A. If this were not the case, arbitrage would be possible. Thus,  $p_1 + p_3 - 2 p_2 \ge 0$  and  $p_2 \le 0.5 \times (p_1 + p_3)$ .