Chapter 1

Plato versus Aristotle

A. Plato

1. The Socratic background[1]

Plato's impetus to philosophize came from his association with Socrates, and Socrates was preoccupied with questions of ethics, so this was where Plato began.

A point which had impressed Socrates was that we all used the notions of goodness and beauty and virtue – and again of the particular virtues such as courage, wisdom, justice, piety, and so on – but could not *explain* them. Faced with a question such as 'what really is goodness?', or 'what really is justice?', we soon found ourselves unable to answer. Most ordinary people would begin by seeing no real problem, and would quite confidently offer a first-off response, but Socrates would then argue very convincingly that this response could not be right. So they would then try various other answers, but again Socrates would show that they too proved unsatisfactory when properly examined. So he was led to conclude that actually we did not know what we were talking about. As he said, according to the speech that Plato gives him in his *Apology* (21b–23b), if there is any way in which he is wiser than other people, it is just that others are not aware of their lack of knowledge. On the questions that concern him he is alone in knowing that he does *not* know what the answers are.

It may be disputed whether and to what extent this is a fair portrait of the actual Socrates, but we need not enter that dispute. At any rate it is

[1] This section involves several conjectures on my part. I have attempted to justify them in my (1986, pp. 94–101).

Plato's view of Socrates, as is clear from Plato's early writings, and that is what matters to us.

Now Socrates was not the only one to have noticed that there were such problems with the traditional Greek ethics, and other thinkers at the time had gone on to offer their own solutions, which were often of a subjectivist – or one might even say nihilist – tendency. For example, it was claimed that justice is simply a matter of obeying the law, and since laws are different from one city to another, so too justice is different from one city to another. Moreover, laws are simply human inventions, so justice too is simply a human invention; it exists 'by convention' and not 'by nature'. The same applies to all of morality. And the more cynical went on: because morality is merely a human convention, there is no reason to take it seriously, and the simple truth of the matter is just that 'might is right'.[2]

Socrates did not agree at all. He was firmly convinced that this explanation did not work, and that morality must be in some way 'objective'. Plato concurred, and on this point he never changed his mind. He always held that when we talk about goodness and justice and so on then *there is* something that we are talking about, and it exists quite independently of any human conventions. So those who say that might is right are simply mistaken, for that is not the truth about what rightness is. But there is a truth, and our problem is to find it. However, this at once gives rise to a further question: why does it seem to be so very difficult to reach a satisfying view on such questions? And I believe that Plato's first step towards a solution was the thought that the difficulty arises because there are no clear *examples* available to us in this world.

Normally the meaning of a word is given (at least partly) by examples, for surely you could not know what the word 'red' means if you had never seen any red objects. But Plato came to think that the meaning of *these* words cannot be explained in the same way. One problem is that what is the right thing to do in one context may also be the wrong thing to do in another. Another problem is that we *dispute* about alleged examples of rightness, in a way in which we never seriously dispute whether certain things are red or one foot long, or whatever. In such cases we have procedures which are generally accepted and which will settle any disputed questions, but there are no such generally accepted procedures for

[2] For some examples of such opinions see, e.g., the views attributed to Callicles in Plato's *Gorgias* (482c–486c), and to Thrasymachus in Plato's *Republic* (336b–339a). See also the theory proposed by Antiphon in his fragment 44A (D/K).

determining whether something is right or wrong. (To mention some modern examples, consider abortion, euthanasia and capital punishment.) Plato also gives other reasons – different reasons at different places in his early writings – for holding that, in the cases which concerned him, there are no unambiguous examples available to us. I shall not elaborate on these, but simply observe that this situation evidently leads to a puzzle: how *do* we understand what these words mean, and how could our understanding be improved?

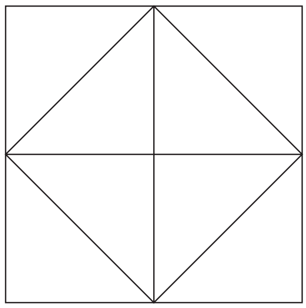
It was this question, I believe, that first led Plato to a serious interest in mathematics, for it seemed to provide a hopeful analogy. In mathematics too there are no clear examples, available to our perception, of the objects being considered (e.g. the numbers), and yet mathematical knowledge is clearly possible. So perhaps the same might apply to ethics?

2. The theory of recollection

(i) Meno *80b–86d*

In the first half of Plato's dialogue *Meno*, the topic has been 'what is virtue?' The respondent Meno has offered a number of answers to this question, and Socrates has (apparently) shown that each of them is inadequate. Naturally, Meno is frustrated, and he asks how one could set about to seek for an answer to such a question, and how – even if one did happen to stumble upon the answer – one could recognize it as the right answer. In broad terms his question is: how is it possible ever to enquire into a topic such as this? In answer Socrates turns to an example from mathematics, which he hopes will show that one should not despair. He puts forward the theory that all (genuine) knowledge is really 'recollection', and offers to demonstrate this by a simple lesson in geometry.

He summons one of Meno's slave-boys, who has had no education in geometry, and poses this question: if we begin with a square of a given area (in this case of four square feet), how shall we find a square that has double that area? As is usual with a Socratic enquiry, the slave first offers an answer that is obviously wrong, and then thinks a little and proposes another which is also quite clearly wrong. He is then stumped. So (as *we* would say) Socrates then takes him through a simple proof to show that, if we start with any square, then the square on its diagonal is twice the area of the one that we began with. To prove this, Socrates himself draws a crucial figure: start with the given square, then add to it three more squares that are equal to it, to give a larger square that is four times the given area. Then draw in the diagonals of the squares as shown:



It is now easy to argue that the central square, formed by the four diagonals, is twice the area of the one that we began with. As Plato describes the lesson, at each step of the proof Socrates merely *asks* the boy some question, and the boy provides the correct answer, as if this at least is a point that he already knows. (For example: does the diagonal of a square bisect that square into two equal halves? – Yes. And what do we get if we add four of those halves together? And so on.) The moral that we are invited to draw is that the boy already possessed the knowledge of each individual step in the argument, and so Socrates' questioning has merely brought back to his mind some result that he really knew all along. Here Plato is evidently exaggerating his case, for although our slave-boy might have known beforehand all the premises to the proof, there is obviously no reason to think that, before Socrates questioned him, he had ever put those premises together in the right way to see what conclusion followed from them. But this criticism is of no real importance to the example.

We may press home the force of Plato's illustration in several ways that Plato himself fails to emphasize. First, it is obvious to all readers that *Socrates* must know the answer to the problem before he starts, and that that is how he knows what figure to draw and what questions to ask about it. But this cannot be essential, for it is obvious that whoever first discovered this geometrical theorem did not have such help from someone else who already knew it. As Plato might have said, one can ask *oneself* the right questions, for mathematical discovery is certainly possible. Second, although Plato insists that the slave-boy was just 'drawing knowledge out of himself', he does not insist as he might have done that this knowledge cannot be explained as due simply to his perception of the diagram that Socrates has drawn in the sand. At least three reasons might be given for this: (i) the diagram was no doubt somewhat inaccurate, as diagrams always are, but that does not prevent us from grasping the proof; (ii) since it is indeed a *proof* that we grasp, we can see on reflection that the result will hold also for all other squares, whatever their size, and not just for the one drawn here; (iii) moreover, we see that this is a *necessary* truth, and that there *could not* be any exceptions to it, but no one diagram could reveal that.

So one moral that Plato certainly wishes us to draw is that mathematics uses *proof*, and that a proof is seen to be correct *a priori*, by using (as he might say) the 'eye of the mind', and not that of the body. Another moral that he probably wishes us to draw is that in this example, as in other cases of mathematical proof, the premises are also known *a priori* (e.g. one knows *a priori* that the diagonal of a square bisects it into two equal halves). If so, then mathematics is a wholly *a priori* study, nowhere relying on our perceptions. (But we shall see that if he did think this when he wrote the *Meno* – and I guess that he did – then later in the *Republic* he will change his mind.) A final moral, which he certainly draws explicitly to our attention, is that since we 'draw this knowledge out of ourselves' (i.e. not relying on perception), we must have been born with it. It is, in later language, 'innate'. From this he infers further that the soul (or mind, i.e. *psychē*) must have existed before we were born into this world, and so is immortal. But the *Meno* is itself rather evasive on how a previous existence might explain this supposedly innate knowledge, and for this I move on to my next passage.

(ii) Phaedo *72e–77d*

The theme throughout Plato's dialogue *Phaedo* is the immortality of the soul. Several arguments for this are proposed, and one of them begins by referring back to the *Meno*'s recollection theory. But it then goes on to offer a rather different argument for this theory. Whereas the *Meno* had invoked recollection to explain how we can come to see the *truths* of mathematics (i.e. by means of proof), in the *Phaedo* its role is to explain how we grasp the *concepts* involved. Although the example is taken from mathematics in each case, still the application that is ultimately desired is to ethics, i.e. to such concepts as goodness and justice and so on.

The example chosen here is the concept of equality, and the overall structure of the argument is completely clear. The claim is that we do understand this concept, but that our understanding cannot be explained as due to the examples of equality perceived in this world, for there are no unambiguous examples. That is, whatever in this world is correctly called 'equal' is *also* correctly called 'unequal'. But of course the words 'equal' and 'unequal' do not mean the same as one another, so their difference in meaning must be explained in another way. The suggestion is that it should be explained by invoking our 'experience', in 'another world', of genuine and unambiguous examples. This we somehow bring with us when we are born into the world of perceptible things, and what we perceive here can trigger our recollection of it, but cannot by itself provide the understanding. The same is supposed to apply to all those other concepts that Plato is finding problematic.

Unfortunately Plato's discussion in the *Phaedo* does not make it clear just what 'defect' infects all perceptible examples of equality, though he does claim that we all recognize that there always is such a 'defect'. On one interpretation his point is just that no two perceptible things, e.g. sticks or stones, are ever *perfectly* equal, say in length or in weight or whatever. I think myself that this interpretation is highly improbable, for why might Plato have believed such a thing? Two sticks – e.g. matchsticks – can certainly *look* perfectly equal in length, and though *we* might expect that a microscope would reveal to us that the equality was not exact, we must recall that there were no microscopes in Plato's day. (In any case, would not the example of two things that *look* perfectly equal, e.g. in length, be enough to provide our understanding of the concept of perfect equality, at least in length?) A different interpretation of Plato's thought, which I find very much more plausible, is that although two perceptible things may be (or appear) perfectly equal in *one* respect, still they will also be unequal in *another*. One might apply such an idea in this way: even if two sticks do seem to be exactly the same length, the same shape, the same weight, the same colour, and so on, still they will not be in the same place as one another, and in that respect they are bound to be unequal. The intended contrast will then be with the objects of pure mathematics, which simply do not have places. For example, the two 'ones' which are mentioned in the equation '$2 = 1 + 1$' are really equal to one another in absolutely *every* way.[3] I am inclined to think myself that Plato's real thought was even more surprising than this suggests, but I do not need to explore that suggestion here.[4] At any rate, the main point is clearly this: the only examples of equality

[3] Plato did think that this equation mentions two 'ones', or as we might say 'two units' which together compose the number 2, as I show in a moment.

[4] I have defended my preferred interpretation in my (1986, chapter 4).

in this world are 'defective', because they are also examples of inequality, so they cannot explain what we do in fact understand.

It is clear that we are expected to generalize: there are no satisfactory examples in this world of *any* of the things that mathematics is really about. For example, arithmetic is about pluralities of units, units which really are equal to one another in all ways, and are in no way divisible into parts, but there are no such things in this world (*Republic*, 525d–526a; cf. *Philebus* 56d–e). Or again, geometry is about *perfect* squares, circles, and so on, which are bounded by lines of no thickness at all, and the things that we can perceive in this world are at best rough approximations (*Republic*, 510d–511a). Mathematics, then, is not about this world at all, but about what can metaphorically be called 'another world'. So our understanding of it can be explained only by positing something like an 'experience' of that 'other world', which on this theory will be something that happens *before* our birth into this world.

I should add one more detail to this theory. The recollection that ordinary people are supposed to have of that 'other world' is in most cases only a *dim* recollection. That is why most of us cannot *say* what goodness is, or what justice is, or even what equality is. We do have some understanding of these concepts, for we can use them well enough in our ordinary thought and talk, but it is not the full understanding that would enable us to 'give an account', i.e. to frame explicit definitions of them. So the philosopher's task is to turn his back on sense-perception, and to search within himself, trying to bring out clearly the knowledge that is in some sense latent within him. For that is the only way in which real understanding is to be gained. We know, from the case of mathematics, that this *can* be done. This gives us reason to hope that it can also be done for ethics too, for – as Plato sees them – the cases are essentially similar: there are no unambiguous examples in this world, but we do have some (inarticulate) understanding, and only recollection could explain that.

3. Platonism in mathematics

Henceforth I set aside Plato's views on ethics. What is nowadays regarded as 'Platonism' in the philosophy of mathematics has two main claims. The first is ontological: mathematics is about real objects, which must be regarded as genuinely existing, even though (in the metaphor) they do not exist 'in this world'. This metaphor of 'two worlds' need not be taken too seriously. An alternative way of drawing the distinction, which is also present in Plato's own presentation, is that the objects of mathematics are not 'perceptible objects', but 'intelligible objects'. This need not be taken to mean that they exist 'in a different place', but perhaps that they exist 'in a different way'. The main claim is just that they do exist, but are not objects that we perceive by sight or by touch or by any other such sense. The second claim is epistemological: we do know quite a lot about these objects, for mathematical knowledge genuinely is *knowledge*, but this knowledge is not based upon perception. In our jargon, it is *a priori* knowledge. This second claim about epistemology is quite naturally thought of as a consequence of the first claim about ontology, but – as I shall explain at the end of this chapter – there is no real entailment here. Similarly the first claim about ontology may quite naturally be thought of as a consequence of the second, but again there is no real entailment. However, what is traditionally called 'Platonism' embraces both of them.

Platonism is still with us today, and its central problem is always to see how the two claims just stated can be reconciled with one another. For if mathematics concerns objects which exist not here but 'in another world', there is surely a difficulty in seeing how it can be that we know so much about them, and are continually discovering more. To say simply that this knowledge is '*a priori*' is merely to give it a name, but not to explain how it can happen. As we have seen, in the *Meno* and the *Phaedo* Plato does have an account of how this knowledge arises: it is due to 'recollection' of what we once upon a time experienced, when we ourselves were in that 'other world'. This was never a convincing theory, partly because it takes very literally the metaphor of 'two worlds', but also because the explanation proposed quickly evaporates. If we in our present embodied state cannot even conceive of what it would be like to 'experience' (say) the number 2 itself, or the number 200 itself, how can we credit the idea that we did once have such an 'experience', when in a previous disembodied state, and now recollect it? Other philosophers have held views which have some similarity to Plato's theory of recollection, e.g. Descartes' insistence upon some ideas being innate, but I do not think that anyone else has ever endorsed his theory.

Indeed, it seems that quite soon after the *Meno* and the *Phaedo* Plato himself came to abandon the theory. At any rate, he does not mention it in the account of what genuine knowledge is – a discussion that occupies much of the central books of his *Republic*, which was written quite soon after. Nor does it recur in his later discussion of knowledge in the

Theaetetus.[5] Moreover, if he did quite soon abandon it, that would explain why Aristotle never mentions it in any of his numerous criticisms of Plato's theories. But, so far as one can see, Plato never proposed an alternative theory of how mathematical knowledge is possible, and this was left as a problem for his successors.

4. Retractions: the Divided Line in *Republic* VI (509d–511e)

On many subjects Plato's views changed as time went by, and this certainly applies to his views on the nature of mathematics. I give just one example, namely the simile of the Divided Line that we find in book VI of the *Republic*. Unfortunately it is not entirely clear just how this simile is to be interpreted, so I shall merely sketch the two main lines of interpretation. But I remark at the outset that my preference is for the second.

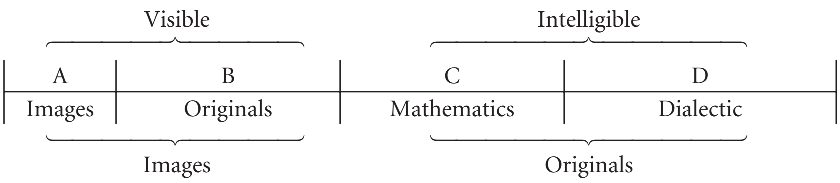
At this stage in the *Republic* the topic is what we have come to call Plato's 'theory of forms'.[6] The theory has been introduced in the *Phaedo*, where it is these so-called 'forms' that we are supposed to have encountered in a previous existence, and which we now dimly recollect. I think it is clear that in the *Phaedo* Plato's view of these supposed forms is muddled. On the one hand they are regarded as properties, common to the many perceptible things that are said to participate in them, and on the other hand they are also taken to be perfect examples of those properties, which perceptible things imperfectly resemble. Thus there is a single form of beauty, which all beautiful things participate in, but it is itself an object that is supremely beautiful, and which is imitated by other things that are beautiful, but are always less beautiful than it is. Similarly there is a form of justice that is itself perfectly just, a form of largeness that is itself perfectly large, and so on. This is the theory that is already familiar to us before we come to the *Republic*.

Our simile is introduced as a contrast between things that are visible and things that are intelligible, and the initial idea is the familiar point that the former are images or copies of the latter. But then Plato adds that

5 The theory recurs at *Phaedrus* 249b–c, but I am inclined to think that it is there regarded as one of the 'poetical embellishments' that Socrates later apologizes for at *Phaedrus* 257a, i.e. as something that is not to be taken too seriously.

6 The word 'form' in this context is a more or less conventional translation of Plato's words '*idea*' and '*eidos*'.

this relation also holds *within* each realm, e.g. as some visible things are images (shadows or reflections) of others. To represent this we are to consider a line, divided into two unequal parts, with each of those parts then subdivided in the same ratio. The one part represents the visible, and the other the intelligible, and the subdivisions are apparently described like this:



The stipulation is that A is to B (in length) as C is to D, and also as A + B is to C + D. (It follows that B is the same length as C, though Plato does not draw attention to that point.) The main problem of interpretation is obvious from the labels that I have attached to this diagram. Plato certainly introduces A + B and C + D as representing *objects* that are either visible or intelligible. He also seems to describe the sections A and B as each representing *objects* (namely: in A there are shadows and reflections, e.g. in water or in polished metal; in B there are the material objects which cause such images, e.g. animals and plants and furniture and so on). But, when he comes to describe the relationship between C and D, his contrast seems to be between different *methods of enquiry*, i.e. the method used in mathematics and the method that he calls dialectic. The first line of interpretation supposes that what Plato really has in mind all along is distinct kinds of *objects*, so (despite initial appearances) we must supply different kinds of object for sections C and D. The second supposes that Plato is really thinking throughout of different *methods*, so (despite initial appearances) we must supply different methods for sections A and B. I begin with the first.

There are different versions of this line of interpretation, but the best seems to me to be one that draws on information provided by Aristotle, though the point is not clearly stated anywhere in Plato's own writings.[7] Aristotle tells us that Plato distinguished between the forms proper and the objects of mathematics. Both are taken to be intelligible objects rather than perceptible ones, but the difference is that there is only one of each

7 As recent and distinguished proponents of this interpretation I mention Wedberg (1955, appendix) and Burnyeat (2000).

proper form (e.g. the form of circularity) whereas mathematics demands many perfect examples of each (e.g. many perfect circles). So the objects of mathematics are to be viewed as 'intermediate' between forms and perceptible things: they are like forms in being eternal, unchangeable, and objects of thought rather than perception; but they are like perceptible things insofar as there are many of each kind (Aristotle, *Metaphysics* A, 987^b14–18). If we grant this doctrine, it will be entirely reasonable to suppose that section C of the divided line represents these 'intermediate' objects of mathematics, while section D represents the genuine forms. But the problem with this interpretation is whether the doctrine should be granted.

One must accept that there is good reason for holding such a view, and that Aristotle's claim that Plato held it cannot seriously be questioned.[8] But one certainly doubts whether Plato had already reached this view at the time when he was writing the *Republic*, and I myself think that this is very unlikely. For, if he had done, why should he never state it, or even hint at it in any recognizable way?[9] Why should he tell us that what a geometer is really concerned with is 'the square itself' or 'the diagonal itself', when this is his standard vocabulary for speaking of *forms* (such as 'the beautiful itself', 'the just itself')? If he did at this stage hold the theory of intermediates, would you not expect him here to use plural expressions, such as 'squares themselves'? There is also a more general point in the background here. The theory which Aristotle reports surely shows the need to *distinguish* between a form, as a universal property, and any (perfect) instances of that property that there may be. But I do not believe that Plato had seen this need at the time when he was writing the *Republic*, for the confusion is clearly present at 597c–d of that work.[10] For these reasons I am sceptical of this first line of interpretation. Let us come to the second.

This second approach is to see the simile as really concerned throughout with methods of enquiry, and it is easy to see how to apply this line of thought. The sections of the line should be taken to represent:

[8] The claim is repeated in chapter 1 of his *Metaphysics*, book M, where he also describes how other members of Plato's Academy reacted to this idea. He cannot just be making it up.

[9] *Republic* 525d–526a can certainly be seen as implying that in mathematics there are *many* 'number ones'. But I doubt whether Plato had absorbed this implication.

[10] It seems probable that Plato *later* recognized this as a confusion, for the dialogue *Parmenides* constructs several arguments which rely upon it and which quite obviously have unacceptable conclusions. (The best known is the argument that, since Aristotle, has been called the argument of 'The Third Man'. This is given at *Parmenides* 132a, cf. 132d–e.)

A: The indirect study of ordinary visible objects, *via* their images (e.g. shadows or reflections).
B: The direct study of such objects in the usual way.
C: The indirect study of intelligible forms, *via* their visible images (e.g. geometrical diagrams). This is the method of mathematics.
D: The direct study of such objects, using (let us say) pure reason and nothing else. This is the method of dialectic (i.e. of philosophy).

On this interpretation we do not have to suppose that Plato intends any ontological distinction between the objects of mathematics and those of dialectic. Visible images are more obviously relevant in the one case than in the other, but it is open to us to hold that the method of dialectic *could* be applicable to mathematical forms, and that the method of mathematics could be used in any study of forms. Moreover, there are some hints that this is what Plato intends. But before I come to this I must be more specific on what Plato takes the method of mathematics to be.

He proposes *two* features as characteristic of this method. The first is the one that we have mentioned already, namely that it uses visible diagrams. The second is that it proceeds by deduction from 'hypotheses', which are taken to be evident to all, and for which no justifications are given. Moreover, he apparently sees a close connection between these two features, saying that the one necessitates the other.[11] The usual explanation of this point is that the reason why these 'hypotheses' are thought to be evident is just that they seem obviously to fit the visible diagrams. (Plato's text never quite says this, but 511a comes close to it, and no better connection suggests itself.) Just what these 'hypotheses' are, in the case of mathematics, is not clearly explained,[12] but their overall role is obvious: they are the premises from which mathematical proofs start, and Plato has now recognized that there must be such premises.

Unfortunately we do not know enough about the state of mathematics in Plato's day to be able to say what he must have been thinking of. This is because Euclid's well-known *Elements*, which was written between 50 and

[11] At *Republic* 510b the thought appears to be that the method is forced to use hypotheses because it employs visible images, but at 511a the connection appears to be the other way round. The latter suggestion is I think the better.

[12] 510c gestures towards some examples, but they appear to be examples of the *subject-matter* of such hypotheses, and we are left to guess at what *propositions* about these subjects are intended. (The text is: 'hypothesising the odd and the even, and the [geometrical] figures, and three kinds of angle'.) See next note.

100 years later, was so clearly an improvement on what had come before it that it eclipsed all earlier work. We do know that there had been earlier 'Elements', and presumably they were known to Plato, but they have been lost, and we cannot say how closely their style resembled what we now find in Euclid. (Euclid distinguishes the premises into definitions, common notions, and postulates. Both of the last two we would class as axioms.) I think myself that it is quite a possible conjecture that earlier 'Elements' did not admit to any starting points that we would call axioms, but only to what we would classify as definitions. So I think it is quite possible that the 'hypotheses' which Plato is thinking of were mainly, and perhaps entirely, what we would call definitions.[13] But in any case what he has now come to see is that mathematical proofs do have starting points, and that these are not justified within mathematics itself, but simply assumed. For that reason he now says that they do not count as *known* (in the proper sense of the word), and hence what is deduced from them is not known either. In the *Meno* mathematics had certainly been viewed as an example of knowledge, but now in the *Republic* it is denied that status. This is a notable change of view.

It may not be quite such a clear-cut change as at first appears, for at 511d there is a strong hint that one *could* apply the dialectical method to the hypotheses of mathematics, thereby removing their merely hypothetical status. If so, then the implication is that mathematics *could* become proper knowledge, even though as presently pursued (i.e. at the time when Plato was writing the *Republic*) it is not. In the other direction I remark that something like the method of mathematics, i.e. a method which (for the time being) simply accepts certain hypotheses without further justification, can evidently be employed in many areas, including an enquiry into the nature of the moral forms. In fact *Meno* 86e–87c attempts to do just that in its enquiry into what virtue is (though the attempt does not succeed (*Meno* 86e–99c)); and *Republic* 437a invokes a 'hypothesis', which is left without further justification, and which plays an important role in its analysis of what justice is. In broad terms, Plato thinks of the method of mathematics as one that starts by assuming some hypotheses and then goes 'downwards' from them (i.e. by deduction), whereas the method of philosophy (i.e. dialectic) is to go 'upwards' from the initial

[13] This would explain why the elucidation offered at 510c (previous note) simply mentions certain concepts – i.e. concepts to be defined? – and does not give any propositions about them.

hypotheses, finding reasons for them (when they are true), until eventually they are shown to follow from an 'unhypothetical first principle'. While the 'downward' method is something which we have no difficulty in understanding, it is not easy to say quite how the 'upward' method is supposed to work, but I cannot here pursue that problem.[14]

At any rate, the upshot is this. Plato himself did not always remain quite the 'Platonist' about mathematics that I described in the previous section. This is because he came to think that mathematics (as presently pursued) begins from unjustified assumptions – or from assumptions that are 'justified' only in the wrong way, i.e. by appeal to visible diagrams – and that this means that it is not after all an example of the best kind of knowledge. Perhaps he also thought that this defect could in principle be remedied, but at any rate he has certainly pointed to a problem with the usual 'Platonic' epistemology: proofs start from premises, and it is not clear how we know that those premises are true. As for the ontology, he remains always a 'Platonist' in that respect. Either in the *Republic*, or (as I think more probable) at a later date, he became clearer about just what the objects of mathematics are, namely not the forms themselves but perfect examples thereof. But in any case they remain distinct from the ordinary perceptible objects of this world, accessible only to thought rather than perception, for they have a 'perfection' which is not to be found in this world. That is the main reason why, in the *Republic*, he lays down for the aspiring philosopher a lengthy and arduous preliminary training in mathematics: it is because this subject directs our mental gaze away from ordinary material things and towards what is 'higher'. (It may be that he *also* thought that the philosopher's 'upward path' towards an 'unhypothetical first principle' would start from reflection on the hypotheses of mathematics. But that is a mere speculation.[15])

In brief, the *Republic* indicates at least a hesitation over the epistemology, but no serious shift in the ontology. By way of contrast, let us now turn to Aristotle, whose views on both these issues were very different.

[14] A classic discussion of Plato's 'method of hypothesis', which assembles all the relevant evidence, is Robinson (1953, chapters 6–13). I have offered a few observations myself in my (1986, chapter 8).

[15] Book VII outlines five areas of mathematics, which are all to be studied, until their 'kinship' with one another is seen (531d). Is that perhaps because such a study will allow one to begin on the project of explaining the several initial hypotheses, by seeing how each may be viewed as an instance of some more general truth that explains them all?

B. Aristotle[16]

5. The overall position

In several places Aristotle gives us an outline sketch of his position on the nature of mathematics, and it is very clear that he rejects the Platonic account.[17]

First, he thinks that Plato was quite wrong to 'separate' the objects of knowledge from the ordinary objects of this world. He will agree with Plato that there are things which can be called 'forms', and that knowledge (properly speaking) is always knowledge of such forms, but he claims against Plato that these forms have no existence apart from their instances in this world. For example, there is a form of man, and there is a form of circle, but these forms exist only *in* actual men and actual circular objects (such as the top of a round table). He firmly denies the Platonic idea that there are, as it were, 'two worlds', one containing perceptible objects and the other imperceptible but intelligible objects. There is only the one world, and it is that world that mathematics is about.

It may *look* as if in geometry we are concerned with a special kind of object, as we say what properties 'the square' or 'the circle' possess, but actually we are speaking in very general and abstract terms of the properties which all ordinary square or circular objects have in common, simply in virtue of being square or circular. When engaged in a geometrical investigation, one does not think of the geometrical figures being studied as things which have weight or temperature or mobility and so on. But that is because it is only the geometrical properties of an object that are here in question. Of course, any ordinary object will have many other properties too, and the truths of geometry are truths about ordinary objects, but their other properties are here ignored as irrelevant. Aristotle presumably intends a similar account to apply to arithmetic. It may *look* as if we are concerned with some rather special objects called 'numbers', which have none of the properties of ordinary perceptible things. But actually we are just speaking at a very general and abstract level of what one might call 'embodied numbers', e.g. the number of cows in the field, or the number of coins on the table, and so on. These are things

[16] Much of this discussion is taken from the fuller treatment in my (2009b).
[17] The principal passages are *Metaphysics* M.3 and *Physics* II.2, 193^b22–194^a12. But see also *De Anima* III.7, 431^b12–17 and *Metaphysics* K.3, 1061^a28–$^b4$.

that we can perceive. That is the broad outline: the truths of mathematics are truths about perfectly ordinary objects, but truths at a high level of generality. Compared with Plato, it seems like a breath of fresh air.

On the topic of epistemology Aristotle similarly claims that our knowledge of such truths is again perfectly ordinary empirical knowledge, based upon perception in much the same way as all other scientific knowledge is based upon perception. He quite naturally supposes that if mathematics need not be understood as concerned with a special and 'separate' kind of object, then equally our mathematical knowledge need not be credited to a special and rather peculiar faculty for 'a priori' knowledge. So both the Platonic ontology and the Platonic epistemology are to be rejected.

That is the broad outline of his position. Unfortunately we do not find very much by way of argument for it. Certainly, Aristotle very frequently states objections to Plato's general theory of forms, construed as objects which enjoy a separate existence of their own. But that is not the end of the argument. For, as we have already noted (p. XX), Plato eventually came to distinguish between the forms themselves and the objects studied in mathematics, regarding these latter as 'intermediate' between forms proper and perceptible things. Granted this distinction, one might for the sake of argument concede to Aristotle that he has good reason for rejecting the Platonic view of forms, but insist that the question of the separate existence of the objects of mathematics is not thereby settled. For these objects are not supposed to be forms, but to be perfect examples. And mathematics might still need perfect examples, which exist 'separately' from all the imperfect examples in this world, even if the same does not apply to the forms themselves.

There is only one place where Aristotle seriously addresses this question, namely in chapter 2 of book M of the *Metaphysics*, and his arguments there are less than compelling. I here pass over all the details, noting only this one general point. The two main arguments that Aristotle gives, at $1076^{b}11$–39 and $1076^{b}39$–$1077^{a}14$, aim to show that if we must assume the existence of those intermediates that Plato desires, then we must also assume the existence of many other intermediates too, which will lead to an incredible and quite needless duplication of entities. But he never tells us *why* Plato thought that these intermediates were in fact needed, nor how his reasons should be countered. As I have said, the usual explanation is this: Plato held that mathematics was about *perfect* examples, and so – since mathematics is true – there must *be* perfect examples. But no examples that we can perceive are perfect, so there must somewhere be imperceptible examples, available to the intellect but not to perception. Let us assume that this is indeed Plato's argument for supposing that his 'intermediates' are *needed*. Then the chief weakness in the counter-arguments that Aristotle presents in *Metaphysics* M.2 is that they have nothing to say about what is wrong with this Platonic argument. They simply do not address the opposition's case. Nor is there anywhere else in his writings where this line of thought is explicitly considered. So I now turn to consider what we might say on Aristotle's behalf.

6. Idealizations

It is a vexing feature of Aristotle's discussion that we cannot even be sure of whether he himself did or did not accept the Platonic premise that there are no perfect examples in this world. His main discussions are quite silent on this point, and although there are a couple of asides elsewhere they are not to be trusted.[18] What he *should* have done is to accept the premise for geometry but deny it for arithmetic, so let us take each of these separately.

Arithmetic

We, who have been taught by Frege, can clearly see that Plato was mistaken when he claimed that this subject introduces idealizations. The source of his error is that he takes it for granted that, when numbers are applied to ordinary perceptible objects, they are applied 'directly'; i.e. that it is the object itself that is said to have this or that number. But Frege made it quite clear that this is not so. In his language, a 'statement of number' makes an assertion about a *concept*, not an object, i.e. it says how many objects fall under that concept. (An alternative view, which for present purposes we need not distinguish, is that numbers apply not to physical objects but to sets of those objects, and they tell us how many members the set has.) To illustrate, one may ask (say) how many cows there are

[18] In the preliminary discussion of problems in *Metaphysics* B, we find the claim that perceptible lines are never perfectly straight or perfectly circular (997^b35–998^a6). But in the context there is no reason to suppose that Aristotle is himself endorsing this claim, rather than mentioning it as a point that might appeal to his Platonist opponent. On the other side, a stray passage in *De Anima* I.1 apparently claims that a material straight edge really does touch a material sphere at just one point (403^a12–16). But I am very suspicious of this passage, for as it stands it makes no sensible contribution to its context. I have discussed the passage in an appendix to my (2009b).

in this field, and then one is asking of the *concept* 'a cow in this field' how many objects fall under it. The answer will (in most cases) be entirely unambiguous, say sixteen. There is nothing 'imperfect' in this application of the number. One may say that here we take as our 'units' the cows in the field, but there is no implication that a cow is an indivisible object, or that the cows are 'equal' to one another in any respect beyond all being cows in this field. Nor is it implied that the matter in question could not be counted under some other concept – i.e. taking something else as the 'unit' – say pairs of cows or kilograms of cow.

Aristotle has grasped this point. He frequently compares counting to measuring, with the idea that in each case one *chooses* something as the 'unit', which is then *treated* for that purpose as indivisible. (E.g.: 'The measure must always be something that is the same for all [the things measured], for example if the measure is a horse then horses [are being measured], and if a man then men' (*Metaphysics* N.1, 1088^{a}8–9).[19] Notice that my supplement 'measured' would in each case be very naturally replaced by 'counted'.) What is somewhat surprising is that he never presents this as a criticism of Plato. He certainly argues against the conclusions that Plato was led to, and he points to a number of difficulties in the view that a number is 'really' a plurality of 'perfect units' which enjoy a 'separate' existence. (This is the theme of most of chapters 6–8 of book M, to 1083^{b}23. The arguments are often cogent, but I shall not discuss them here.[20]) But he does not seem to have asked just what it was that led Plato astray, so we get no diagnosis of the opponent's errors. Worse, there are hints in the positive account which he does give – and which I come to shortly – which leave us wondering just what he himself is proposing as an alternative to Plato's picture.

Geometry

It is fair to say that geometry 'idealizes', in that it concerns what has to be true of *perfect* squares, circles, and so on. But the first thing to say is that

[19] I translate the mss. reading. Ross prefers to emend to '. . . if horses [are being measured] then the measure is a horse, and if men then a man'. But in either case the main idea is the same. Other passages of the *Metaphysics* which clearly show a good understanding of how numbers are applied in practice are: Δ.6, 1016^{b}17–24; I.1, 1052^{b}15–17, 1053^{a}24–30, 1054^{a}4–9; M.7, 1082^{b}16–19; N.1, 1087^{b}33–1088^{a}14. Cf. also 1052^{b}31–1053^{a}2; 1092^{b}19–20; *Physics* IV.12, 220^{a}19–22.

[20] I shall also leave undiscussed Aristotle's arguments, which occupy most of book N, against the Platonic idea that the numbers (and other things) are somehow 'generated' from 'the one' and 'the indefinite dyad'.

what geometry claims about perfect circles may very well be true even if there are no perfect circles at all, for the claims may be construed hypothetically: *if* there are any perfect circles, then such-and-such will be true of them (e.g. they can touch a perfectly straight line at just one point and no more). One might ask how geometry can be so useful in practice if there are no such entities as it speaks of, but (a) this is a question for the Platonist too (since 'in practice' means 'in this perceptible world'), and anyway (b) the question is quite easy to answer.

We are nowadays familiar with a wide range of scientific theories which may be said to 'idealize'. Consider, for example, the theory of how an 'ideal gas' would behave – e.g. it would obey Boyle's law precisely[21] – and this theory of 'ideal' gases is extremely helpful in understanding the behaviour of actual gases, even though no actual gas is an ideal gas. This is because the ideal theory simplifies the actual situation by ignoring certain features which make only a small difference in practice. (In this case the ideal theory ignores the actual size of the molecules of the gas, and any attractive or repulsive force that those molecules exert upon one another.) But no one nowadays could suppose that because this theory is helpful in practice there must really *be* 'ideal gases' somewhere, if not in this world then in another; that reaction would plainly be absurd. Something similar may be said of the idealizations in geometry. For example, a carpenter who wishes to make a square table will use the geometric theory of perfect squares in order to work out how to proceed. He will know that in practice he cannot actually produce a *perfectly* straight edge, but he can produce one that is very nearly straight, and that is good enough. It obviously explains why the geometric theory of perfect squares is in practice a very effective guide for him. We may infer that geometry may perfectly well be viewed as a study of the spatial features – shape, size, relative position, and so on – of ordinary perceptible things. As ordinarily pursued, especially at an elementary level, it does no doubt involve some idealization of these features, but that is no good reason for saying that it is not really concerned with this kind of thing at all, but with some other 'ideal objects' that are not even in principle accessible to perception. All this, however, is on the assumption that geometry may be construed hypothetically: it tells us that *if* there are perfect squares, perfect circles, and so on, then they must have such-and-such

[21] For a body of gas maintained at the same temperature, where 'P' stands for the pressure that it exerts on its container, and 'V' for its volume, Boyle's law states that $PV = k$, for some constant k.

properties. That is helpful, because it implies that the approximate squares and circles which we perceive will have those properties approximately. But, one may ask, does not geometry (as ordinarily pursued) assert outright that *there are* perfect circles? That would distinguish it from the theory of ideal gases, and is a question which I must come back to.

7. Complications

Within the overall sketch of his position that Aristotle gives us in *Metaphysics* M.3, there are two brief remarks which certainly introduce a complication. At 1078^{a}2–5 he says, somewhat unexpectedly, that mathematics is not a study of what is perceptible, even if what it studies does happen to be perceptible. More important is 1078^{a}17–25, where he says that the mathematician *posits* something as separate, though it is not really separate. He adds that this leads to no falsehood, apparently because the mathematician does not take the separateness as one of his premises. A similar theme is elaborated at greater length in the outline given in *Physics* II.2, at 193^{b}31–5. There again we hear that the mathematician, since he is not concerned with features accidental to his study, does *separate* what he is concerned with, for it can be separated 'in thought', even if not in fact. And again we are told that this leads to no falsehood. On the contrary, Aristotle seems to hold that such a fictional 'separation' is distinctly helpful, both in mathematics and in other subjects too (1078^{a}21–31).

Our texts do not tell us what kind of thing a mathematical object is conceived as being, when it is conceived as 'separate'. I think myself that the most likely answer is that it is conceived as the Platonist would conceive it, i.e. as existing in its own 'separate world', intelligible and not perceptible. Further, if – as seems probable – Aristotle concedes that perceptible objects do not *perfectly* exemplify the properties treated in elementary geometry, then it will presumably be this mental 'separation' that smoothes out the actual imperfections. But it must be admitted that this is pure speculation, and cannot be supported from anything in our texts.[22]

[22] I believe that Aristotle holds that when a geometrical figure is conceived as separate, it is also conceived as made of what he calls 'intelligible matter' (*Metaphysics* Z.10, 1036^{a}1–12; Z.11, 1036^{b}32–1037^{a}5; H.6, 1045^{a}33–6). This is what allows us to think of a plurality of separate figures – e.g. circles – that are all exactly similar to one another. For what distinguishes them is that each is made of a different 'intelligible matter'.

I note further that in *Metaphysics* M.3, and for the most part in his programmatic discussions elsewhere, Aristotle is mainly thinking of geometry. But he clearly believes that something very similar applies to arithmetic too, so presumably we are to take it that the mathematician also conceives of the numbers as separate entities, though that too is a fiction. And if in the geometrical case it is the separation in thought that also allows for idealization, then we should perhaps think that when numbers are conceived as separate then they too are conceived in an idealized way. One supposes that this will again be a Platonic way, in which a number is conceived as made up of units which each have their own separate existence, which are perfectly equal to one another in every way, and which are in no way divisible. As I have said, much of *Metaphysics* M.6–8 argues (very successfully) that numbers cannot *really* be like this, but perhaps Aristotle means to concede that that is how mathematicians do in practice think of them.

That is one way in which the outline sketch that we began with becomes complicated. Another is that the objects of mathematics are said to exist *potentially* rather than actually.[23] One presumes that his thought here is that these objects, when considered as existing separately, can be said to exist potentially because it is possible for them to exist actually, i.e. to exist in actual physical objects. Thus a circle exists actually in a circular table-top, and the number 7 exists actually wherever there are (say) 7 cows. Then the idea will be that some rather complex geometrical figures, e.g. a regular eicosahedron, may not actually exist anywhere in the physical world, but this figure still has a potential existence because it could do so. The same would apply to a very large number, too large to be exemplified. But there may be a further complication to be added here, namely the idea that a mathematical object is brought into actual existence not only by being physically exemplified but also just by being thought of. At any rate, at *Metaphysics* Θ.9, 1051^{a}21–3, Aristotle notes that geometers will often prove some result by 'constructing' lines additional to those already given, and he comments that this construction makes actual what was previously only potential. Moreover, he must be taking it to be the construction *in thought* that matters, for he adds 'and the explanation is that thinking is actuality'. In any case, the main point is that Aristotle is conceding that, in

[23] *Metaphysics* M.3, 1078^{a}28–31 says that they exist not actually but 'in the way of matter'. Aristotle does constantly think of matter as existing potentially and not actually, and I do not believe that he intends the comparison with matter to extend any further than this.

a sense, 'there are' many more mathematical objects than are either actually embodied or actually thought of. He wishes to say that 'there are' such things in the sense that they have a potential existence if not an actual one. I shall consider in my next section whether this is an adequate solution to what must, for Aristotle, be a serious problem.

Meanwhile, let us sum up the position so far. Aristotle holds that mathematics is the study of certain properties which perfectly ordinary perceptible objects possess, and in that way its objects are just ordinary perceptible objects. But the study proceeds at a high level of generality, paying no heed to all the non-mathematical properties of these objects. It is therefore a convenient fiction to suppose that it concerns some special and peculiar objects which have *no* properties other than the mathematical ones. There are not really any such objects, but it does no harm to imagine that there are, and at the same time both to 'smoothe out' the small geometrical irregularities which actual physical objects are likely to display, and to expand the account by including geometrical and arithmetical properties that may not be exemplified at all. That is permissible because we are still concerned with objects that have at least a potential existence, if not an actual one. But still, the foundation of the subject must be the actual physical bodies and their actual geometrical and arithmetical properties. For that is where our understanding must begin.

This outline sketch leaves many questions unanswered, and one can only guess at the answers that Aristotle might have given. For example, I would *expect* him to say that a simple statement of pure arithmetic, such as '$7 + 5 = 12$', should not be interpreted as referring to some puzzling entities called 'the numbers themselves', but as generalizing over ordinary things in some such way as this: if there are 7 cows in one field, and 5 in another, then there are 12 in both fields taken together; and the same holds not only for cows and fields but also for everything else too. In fact he does not actually say this, or anything like it; he is completely silent on the meaning of arithmetical equations. Again, I would *expect* him to say that we find out that $7 + 5 = 12$ by the ordinary procedure of counting cows in fields, and other such familiar objects. But in fact he never does explicitly address the question of how we come to know such truths of simple arithmetic, and he never does respond to the Platonic claim that the knowledge must be *a priori*.

Here too all that we have are some very general and programmatic pronouncements. In the well-known final chapter of the *Posterior Analytics* (i.e. II.19) he claims that *all* knowledge stems from experience. Perception is of particulars, but memory allows one to retain many particular cases in one's mind, and this gives one understanding of universals. This is put forward as an account of how one grasps 'by induction' the first principles of any science, and the similar account in *Metaphysics* A.1 makes it clear that mathematics is not an exception (981^{a}1–3, ^{b}20–5). But it is quite clear that this says far too little. Indeed, Aristotle claims that we must somehow come to see that these first principles are *necessary* truths, but has no explanation of how we could ever do this. Elsewhere we find the different idea that what Aristotle calls 'dialectic' also has a part to play in the discovery of first principles, but again the discussion stays at a very superficial level, and Aristotle really has nothing useful to say about how it could do so. One can only conclude that he must think that our knowledge of mathematics (like our knowledge of everything else) is empirical and not *a priori*, but he has not addressed the problem in any detail.[24]

As we shall see more fully in later chapters, there are many objections which this kind of empiricism has to meet. But here I shall mention only one, because it raises a problem that Aristotle himself did see and did discuss, namely the use that is made in mathematics of the notion of infinity. Even the elementary arithmetic and geometry that Aristotle was familiar with often invoke infinities. But how could this be, if they are based upon perception? For surely we do not perceive infinities?

8. Problems with infinity

Aristotle's treatment of infinity is in chapters 4–8 of *Physics* III. After introducing the subject in chapter 4, the first positive claim for which he argues (in chapter 5) is that there is not and cannot be any body that is infinitely large. This is because he actually believes something stronger, namely that the universe is a finite sphere, which (he assumes) cannot either expand or contract over time, so the size of the universe is a maximum size that cannot ever be exceeded. In his view, there is absolutely nothing outside this universe, not even empty space, so there is a definite limit even to the possible sizes of things. I shall not rehearse his arguments, which – unsurprisingly – carry no conviction for one who has been brought up to

[24] Does he really think that absolutely *all* knowledge is based upon perception, e.g. including our knowledge of what follows from what (as first codified in his own system of syllogisms)? All that one can say is that he never draws attention to any exception.

believe in the Newtonian infinity of space. I merely note that this is his view.

In consequence he must deny one of the usual postulates of ordinary Euclidean geometry, namely that a straight line can be extended in either direction to any desired distance (Euclid, postulate 2). For in his view there could not be any straight line that is longer than the diameter of the universe. It follows that he cannot accept the Euclidean definition of parallel lines (Euclid, definition 23), as lines in the same plane which will never meet, however far extended. But parallelism can easily be defined in other ways, and of course one can apply Euclidean geometry to a finite space, as in effect Aristotle says himself. At 207^{b}27–34 he claims that his position 'does not deprive the mathematicians of their study', since they do not really need an infinite length, nor even the permission always to extend a finite length. His idea is that whatever may be proved on this assumption could instead be proved by considering a smaller but similar figure, and then arguing that what holds for the smaller figure (which *is* small enough to be extended as desired) must also hold for the larger original, if the two are exactly similar.[25] So his denial of an infinite length is indeed harmless from the mathematician's point of view, but his other claims are less straightforward.

At the start of chapter 6, which opens his positive account of infinity, Aristotle mentions three serious reasons for supposing that there is such a thing:

> If there is, unqualifiedly, no infinite, it is clear that many impossible things result. For there will be a beginning and an end of time, and magnitudes will not be divisible into magnitudes, and number will not be infinite (206^{a}9–12).

His position is that there must be *some* sense in which these things can be said to be infinite, even if it is not 'unqualifiedly'. I must here set aside his views on time, with the excuse that this is a question in physics or metaphysics, rather than in mathematics, but the infinite divisibility of

[25] As was in effect discovered by the English mathematician John Wallis (1616–1703), and known to Gerolamo Saccheri in his book *Euclides ab omni naevo vindicatus* (1733), the assumption that, for any figure, there is a similar but smaller figure of any size you please, is characteristic of a Euclidean space, and could replace Euclid's parallel postulate. So as it happens Aristotle's response is relying on Euclidean geometry. (I take the information from Heath, 1925, pp. 210–12.)

geometrical magnitudes, and the infinity of the numbers, are central to our concerns.

Aristotle quite frequently says that his main claim is that all infinity is always potential, which apparently implies that it cannot ever be actual. But in fact this misdescribes his real position, and further elucidation is certainly required. The central assumption that he makes, which is an assumption that he never argues for, and never even states in a clear way, is that an infinite totality could exist only as the result of an infinite process being completed. Moreover he (quite understandably) believes that an infinite process – i.e. a process that has no end – cannot ever be completed. So there can never be a (completed) infinite totality, though there may perfectly well be an (ongoing) infinite process. No doubt *most* processes will stop at some time, though they may be said to be potentially infinite because they *could* always have been continued further. That is the usual case. There are some processes which never will stop – in Aristotle's view the process of one day succeeding another is an example – and these are processes which are actually infinite. But there cannot be a process which both stops and is unending, which is to say that no infinite process can ever be completed, and hence that there cannot be a time at which there exists an infinite totality.

Aristotle applies this general view to the supposed infinite divisibility of a geometrical object, such as a line. There could (in theory) be an unending process of dividing a finite line into parts. To cite Zeno's well known example, one may take half of a line, and then half of what remains, and then half of what still remains, and so on for ever. But Aristotle holds that these parts, and the points that would divide them from one another, do not *actually* exist until the divisions are actually made. This is because, if they do exist, then one who moves over a finite distance must have *completed* an infinite series of smaller movements, each half as long as its predecessor, which he regards as impossible. So his idea is that one who simply moves in a continuous way over a certain distance does not count as 'actualizing' any point on that distance. To do so he would have to pause at the point, or stick in a marker, or just to count the point as he passes it. The general idea, I think, is that to 'actualize' a point one must *do* something, at or to that point, which singles it out from all neighbouring points. And the (rather plausible) thought is that no infinite series of such *doings* could be completed in a finite time.[26] Consequently a finite line never will

[26] I have argued in my (1972/3) that although this thought is 'plausible' still it is false.

contain infinitely many actual points (or actual parts), but we can still say that its divisibility is 'potentially infinite', on the ground that, however many divisions have been made so far, another is always possible.

This position is compatible with the basic assumptions of Greek geometry, because in Greek geometrical practice points, lines, planes, and solids were all taken as equally basic entities. Besides, a common view was that the most basic kind of entity is the solid, since planes may be regarded as the surfaces of solids, lines as the boundaries of planes, and points as the limits of lines. On this view points are the *least* basic of geometrical entities, and we do not have to suppose that infinitely many of them are needed simply in order to 'construct' lines, planes, and so on, from them. (Indeed, Aristotle opens book VI of his *Physics* with an argument which aims to show that lines *cannot* be made up of nothing but points.)

Where one might expect a tension is over the existential postulates of geometry, for do not these assume that *there are* points, lines, planes, and so on, even when there is nothing that has marked them out? But, on reflection, this is not obvious. At any rate, it is not one of Euclid's demands. As it happens, Euclid makes no explicit claim about the existence of points (though he should have done), but he is quite definite about lines. His first three postulates are:[27]

Let the following be postulated:

1 to draw a straight line from any point to any point;
2 to produce a finite straight line continuously in a straight line;
3 to describe a circle with any centre and distance.

I say no more about the second, for I have earlier remarked that it is not essential, but what of the first and third? Are they not straightforward claims to existence? Well, that is certainly not how Euclid himself presents them. He may readily be interpreted as claiming not that these lines do (already) exist, but that they can (if we wish) be brought into existence by being drawn. (His proofs often require them to *be* drawn.) It is often said of Greek geometry that it is 'constructive' in the sense that it does not assume the existence of any figure that cannot be constructed (with ruler and compasses). In that case Aristotle need have no quarrel with it.[28] For such a construction

[27] I quote Heath's translation, in his (1925, pp. 195–9).

[28] Plato does have a quarrel. He complains that lines do not have to be drawn in order to exist (*Republic* VII, 527a–b).

must be describable, and Aristotle would seem to be entitled to assume that it does not exist until it has been described (in thought, or on paper). On that view, there will never be a time when there are more than finitely many geometrical figures in existence, so his account of infinity may still stand. But although it is commonly held that Greek geometry was 'constructive', in the sense roughly indicated here, I do not think that the same is ever said about arithmetic. This is a more serious problem for him.

As I have already noted, he has earlier acknowledged that everyone believes that there are infinitely many numbers. He has suggested that this is because the numbers do not give out 'in our thought' (203^b22–5), but apparently he is committed to saying that they do give out in fact. For if a number exists only when there is a collection of physical objects that has that number, then there cannot (according to him) be infinitely many of them. This is because he holds that the universe is finite in extent, and that physical objects never are infinitely divided, from which it must follow that every actual collection is finite. The same conclusion holds if we add, as Aristotle might desire, that simply thinking of a particular number is enough to 'actualize' it. For still at any one time there will be only finitely many numbers that have actually been thought of. There is no doubt a potential infinity, but Aristotle seems to be committed to the view that, even in this case, there is no actual infinity.

To confirm this point we may note a line of argument that Aristotle sometimes uses against his Platonist opponent. The Platonist thinks that all the numbers have an actual existence, in separation from the perceptible things of this world. Aristotle assumes that in this case the Platonist must accept that there is such a thing as the number of all the numbers. If so, then this number must (on plausible assumptions) be an infinite number. But Aristotle thinks that it is impossible for there to be such a thing as an infinite number, (a) because, since a number can always be reached by counting, this would mean that one could count to infinity, and (b) because every number must be either odd or even, but an infinite number would not be either.[29] Now Aristotle presents these thoughts as creating a difficulty for the Platonic conception of numbers as existing separately, but we may certainly ask whether they would also apply to numbers construed in Aristotle's way, i.e. as existing only *in* (collections of) independently existing items. So far as one can see, the points made would apply equally in either case, and hence there equally cannot be infinitely many Aristotelian

29 For (a) see *Physics* III.5, 207^b7–10; for (b) see *Metaphysics* M.8, 1083^b37–1084^a4.

numbers. So there is here no exception to his overall position. The series of numbers is *potentially* infinite, but only potentially. That is to say that at no time will there *actually* be more than finitely many of them.

One cannot be content with this conclusion. One asks, for example, how many numbers there are *today*, and surely no answer is possible. The conception is that more may be realized tomorrow, but that there must be some definite (and finite) answer to how many there are now, and every such answer is ridiculous. It is of course true that the numbers do not give out 'in our thought', but it is very difficult to take seriously the idea that they do give out 'in fact'.

Could Aristotle evade this criticism? Well, perhaps he could, but only by applying to arithmetic a line of thought that he must accept for geometry, although he never candidly admits it. Geometry is an idealizing subject. It *assumes* the existence of all kinds of perfect figures, and surely most of them are not actually exemplified in the physical world. In that way it is a fiction. But obviously it is a very useful fiction, and its practical applications are immensely valuable. I think that Aristotle should say the same about arithmetic. It too 'idealizes' by assuming that every number really does have a successor. That is its way of 'smoothing out' the imperfections of the world that we actually inhabit. But at the same time he might hope to argue that this 'fictional', 'idealized', 'smoothed out' theory is something which, in practice, we cannot do without. We shall have more on this theme in what follows.

I conclude with a question. In opposition to Plato, Aristotle is aiming for an empirical theory of mathematics, which does not accept the real existence of any objects that are not empirically available. There are many problems for such theories, and we shall find more as we go on. But one of the most central, and the most difficult, is that which Aristotle himself noticed when he first put forward an empirical theory. How can it explain infinity? If we cannot answer this, then must we be forced back to Platonism?

C. Prospects

The basic feature of a Platonic ontology is just that it posits a special kind of objects, nowadays usually described as abstract objects, to be what mathematics is about. These objects are thought of as existing both independently of any physical instances or exemplifications, and independently of all human thought. The basic feature of a Platonic epistemology is that it insists that our knowledge of mathematics is a special kind of knowledge, called *a priori* knowledge, i.e. knowledge which is (in principle) independent of what we can learn from our perceptual experience of the physical world. In each case, this is a very brief and very general characterization of an overall approach which can take different forms in different philosophers. I have here given some description of Plato's own version, partly because that is of some historical interest, and partly because one finds even in Plato himself a hesitation over both the ontology and the epistemology.

So far as the ontology is concerned, Plato began with the idea that mathematics was about what he called forms, but then changed to the idea that it was about 'intermediates', i.e. perfect examples of those forms. Subsequent philosophers have said instead that it is about universals, e.g. such things as properties and relations and functions and so on, all Platonically construed. A more common version these days is that it is about sets, but (in its purest version) about sets which nowhere involve ordinary perceptible objects in their composition. (These are the so-called 'pure sets'; I shall introduce them in chapter 5, section 4.) All of these are different versions of 'the Platonic ontology'.

As for the epistemology, we have seen that Plato himself began with a theory to explain *a priori* knowledge, namely that it is a matter of 'recollecting' a previous existence that was not on this earth, but apparently he quite soon abandoned this theory. Since Plato there have been other theories aiming to explain how *a priori* knowledge is possible, which we shall come to in due course. So far as Plato himself is concerned, we have seen that he at least saw a difficulty with this claim to *a priori* knowledge. For although it is true that mathematics is full of proofs, and that our ability to construct and follow a proof is often held to be *a priori*, still Plato noticed that mathematical proofs do begin from axioms. So there is still the problem of how (if at all) we know these axioms to be true, and later Platonizing philosophers have given very different accounts of this.

The basic feature of an Aristotelian ontology is essentially negative: we do not need to posit such special and peculiar objects as the Platonist desires, so we should not do so. The general idea is that mathematics can perfectly well be construed as a study of entirely ordinary objects, even if the initial appearance suggests otherwise. To make a convincing case for this, one must of course say just how the basic propositions of mathematics can be construed as propositions concerning ordinary objects, and this is a task which Aristotle himself does not really address. I imagine that he took it to be quite obvious in the case of geometry, and that he assumed without much further consideration that the case of arithmetic would be similar. In more modern developments geometry plays no important role (for reasons that I give in chapter 3, section 1), but arithmetic has become central. How are we to explain, in broadly Aristotelian terms, the apparent reference to such things as numbers? I would say that the first worthwhile attempt at this question was by Bertrand Russell (which I discuss in chapter 5, section 2), and this attempt certainly showed that the problem was by no means simple.

As for the Aristotelian epistemology, one may say that its basic idea is just that the method of mathematics is no different in principle from the methods of what we call 'the natural sciences'. Everyone will agree that knowledge gained in this way counts as empirical knowledge, though there will be disagreement on just what entitles it to be knowledge, and on how in detail it is achieved. Nowadays one would say that there are basically two sides to it, one being the observation of those happenings that we take to be observable, and the other being the construction of theories to explain and predict these observations. It is fair to say that Aristotle himself had no good account of the second, and that his successors have had to pay more attention to this. But the empirical approach that one associates with him is certainly still with us today.

I remark that it is quite *natural* for the Platonic ontology and epistemology to go together with one another, and similarly the Aristotelian ontology and epistemology. To put it crudely: if the objects of mathematics are really just ordinary perceivable objects, then it is natural to suppose that it is perception that tells us about them, and vice versa. On the other hand, if mathematical objects are objects of a special kind, available to thought but not to perception, then it is natural to suppose that the thought in question must be a special kind of thought. However, these links are not inevitable. For example, it is possible to combine the Aristotelian ontology with a Platonic epistemology, and Bertrand Russell's approach might be described in this way. For he aims to offer a reductive analysis of mathematical statements, which reveals them as generalizations about ordinary objects rather than singular statements about a special kind of object called a number. But at the same time he holds that these generalizations are not ordinary empirical generalizations but truths of logic, and that logic is known *a priori*. That combination of views is quite easy to understand (though – as Russell discovered – not too easy to maintain. I shall discuss it in chapter 5, section 2.). In the other direction it is possible to combine the Aristotelian claim that all knowledge is empirical with the Platonic claim that mathematics is about objects of a very special kind, namely abstract objects which are not the sort of thing that could have a location in space. The idea here is that such objects are assumed by our scientific theories, and that if we do not accept this assumption then we shall be deprived of all the usual scientific explanations of familiar physical phenomena. (This line of thought is mainly due to Quine and Putnam. I shall discuss it in chapter 9, section 3.)

To sum up: there are many different versions of Platonism, and of the empiricism that Aristotle proposed. There are also approaches which combine some aspects of the one with some aspects of the other. We shall find all these themes constantly recurring in the thinking of the twentieth century. At the same time we shall also find a quite different theme which has not yet emerged at all, and which I introduce in the next chapter. In broad outline this is the idea that the objects which mathematics is about exist only in minds and nowhere else.

Suggestions for further reading

Plato

The passages cited from Plato's *Meno* and *Phaedo* are relevant, and comparatively straightforward. There are many translations of Plato's dialogues that are easily available, and any of them will do perfectly well. The problems with what the *Republic* has to say about mathematics have been much debated by Plato scholars, and for present purposes this debate might reasonably be bypassed, since it is of little relevance to contemporary philosophy. But for those who wish to pursue the topic I recommend reading more of the *Republic*, say from 506b to 534d, and beginning with the following commentaries: Robinson (1953, chapters 10–11); Malcolm (1962); Cross and Woozley (1964, chapters 9–10). As I have noted, Wedberg (1955, Appendix), is a classic discussion opposed to these. For something more recent I suggest Mueller (1992) and Burnyeat (2000).

Aristotle

For chapters 2–3 of his *Metaphysics*, book M, I recommend Annas (1976), which contains a convenient translation (with notes), and an introductory essay on the philosophy of mathematics in both Plato and Aristotle. The several problems with Aristotle's overall empiricism are best postponed until we reach a more detailed example of this kind of approach, such as that of J.S. Mill (which I discuss in chapter 3, sections 1–2). But Aristotle's treatment of infinity, in chapters 4–8 of book III of his *Physics*, is special to him, and deserves consideration. There is a convenient translation (with notes) in Hussey (1983). For further reading on this particular topic I recommend Bostock (1972/3) and Lear (1979/80). For a more general appraisal of Aristotle's position I suggest beginning with Mueller (1970) and Lear (1982). I have discussed Aristotle's philosophy of mathematics in greater detail in my (2009b).