chapter one

Functions in Economics

SECTION 1.1: Introduction

We can express economic analysis more precisely when we use mathematics. The approach may not always be appropriate, because economics deals with people and sometimes we may prefer to give a verbal description of their behaviour. The application of mathematics, however, has allowed economic theory to advance and provides the basis for computer models of the economy that have been developed. As you

progress in economics you will find mathematics is used in various ways in textbooks and in journal articles.

In introductory economics you study the relationship between various cost curves such as average variable cost, total cost and marginal cost. Mathematics makes relationships explicit and tells us that if one of the cost curves has a particular shape, each of the others has another specific shape. The positions at which the average and marginal curves cross one another can also be exactly determined. In measuring elasticity of demand there can be ambiguity, but mathematics gives us a precise measure in the form of point elasticity, and this resolves the difficulty. Economic models include those representing supply and demand in a market, and models of the economy used in macroeconomics. Each model is expressed as a system of equations. We can investigate how it works and solve the equations to find equilibrium values for the system.

Mathematics is used for modelling financial processes, and it enables us to show how optimization subject to constraints can be achieved. In ways such as these it is useful to businesses. It is therefore a career-relevant subject and a good grade is useful for impressing potential employers. In this chapter you learn how to handle the kinds of functional relationships needed for economic and financial modelling.

The MathEcon chapter 1 screens in the section titled Introductory Mathematics suggest ways of using the screens, give you a dynamic demonstration of plotting points on a graph and highlight important algebraic results. The questions and quizzes challenge your understanding of particular concepts, so be sure to try them. Useful reference material is provided by the Summary of the Basics screen, which lets you access the main rules or methods for each of the previous screens.

The last section of this chapter gives you ideas for using Microsoft Excel to plot graphs and carry out calculations. A computer spreadsheet such as Excel can help you present many kinds of quantitative information. Since it relieves the tedium of computations it makes it possible for you to investigate more aspects of a topic, thus deepening your understanding. The ability to use Excel is a transferable skill and to help you develop it there are example worksheets included on the CD.

To see examples of how mathematics is useful in both microeconomics and macroeconomics, work through the MathEcon section 2 screen titled Economic Problems – Basic Mathematics. Another screen you may like to try is Objectives and Actions: Looking Ahead. This lets you make choices to try to achieve a particular objective. The algebra underlying the model is revealed by the Advanced button.

SECTION 1.2: Coordinates and Graphs

A horizontal and a vertical line form the axes of a graph. Each is marked with a measurement scale. The intervals shown on each scale are chosen for convenience and are not necessarily the same on both axes. The horizontal axis is called, in general, the x axis and the vertical axis the y axis. In a particular example we may give other names to the axes to indicate what is being measured on them. Where the axes intersect, x and y both take the value 0. This is called the origin and is denoted 0. As we move to the right along the horizontal axis the values of x get larger (and y remains at 0). When we move up the vertical axis y increases (and x stays at 0).

> x axis: the horizontal line along which values of x are measured. Values along the axis increase from left to right.
> y axis: the vertical line up which values of y are measured. Values on the axis increase from bottom to top.
> origin: the point at which the axes intersect, where x and y are both 0.

The pair of values $x = 7$, $y = 2$ can be represented by a point plotted 7 units to the right of the origin and 2 units up from it. This is shown as point A in figure 1.1. A point can be described by its coordinates, which in general are written (x,y). That is, the horizontal, or x distance from the origin is always given first, followed by the vertical, or y distance. The coordinates of point A are $(7,2)$.

To see how we plot points, use the MathEcon screen Coordinates and Graphs. For an example of graph plotting in economics, see The Demand Curve of an Individual screen.
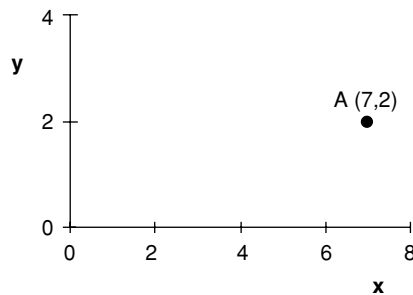
Figure 1.1   Point A has coordinates (7,2)

> coordinates: a pair of numbers (x,y) that represent the position of a point. The first number is the horizontal distance of the point from the origin, the second number is the vertical distance.

PLOTTING NEGATIVE VALUES

A graph like that shown in figure 1.1 lets us plot only positive values of x and y. It is said to comprise the positive quadrant. As economists we sometimes have to deal with negative values. For example, as the quantity sold increases marginal revenue may become negative, and investment may be negative if stocks are run down. To

depict negative values we use a graph such as that shown in figure 1.2 where the x and y axes extend in both directions from the origin, dividing the area into four quadrants. As we move leftwards along the x axis the values of x get smaller. To the left of the origin x is negative and as we move further left still it becomes more negative and smaller. So, for example, -6 is a smaller number than -2 and occurs to the left of it on the x axis. On the y axis negative numbers occur below the origin.

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positive quadrant: the area above the x axis and to the right of the y axis where both x and y take positive values.

1

Plot the points A = (3,0) B = (−5,4) C = (2,−4) D = (−2,−3)

The points are located as shown in figure 1.2.
A (3,0) is 3 units to the right of the origin and on the x axis.
B (−5,4) is 5 units to the left of the origin and 4 units up.
C (2,−4) is 2 to the right of the origin and 4 below the x axis.
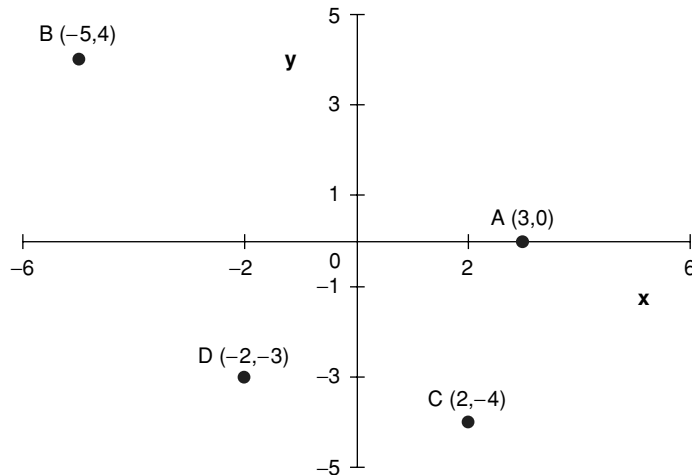D (−2,−3) is 2 to the left of the origin and 3 down from the x axis.

Figure 1.2

<table>
<tr><td>1.1</td><td>Plot the points A = (6,2) B = (0,5) C = (–7,3) D = (–4,–2)
E = (3,–6)</td></tr>
</table>

SECTION 1.3: Variables and Functions

A variable takes on different values, perhaps at different times, for different people or in different places. So that we can analyse the relationship between variables, we identify each of them by a letter or symbol. If there are just two variables the values they take can easily be plotted as points on a graph and the variable names x and y are often used, corresponding to the horizontal and vertical axes respectively. We shall use these as our basic variable names, but other letters can also be used and in economics examples we shall choose names as appropriate.

> variable: a quantity represented by a symbol that can take different possible values.
> constant: a quantity whose value is fixed, even if we do not know its numerical amount.

A constant remains fixed while we study a relationship. It could be the proportion of income that is saved, or the level of utility that is achieved on a particular indifference curve. Sometimes we may know the numerical value of a constant, but if we do not or if we want to obtain general results we may use a letter or symbol to represent the constant. Letters commonly used to represent constants are: a, b, c, k.

If one variable, y, changes in a systematic way as another variable, x, changes we say y is a function of x. The mathematical notation for this is

$$y = f(x)$$

where the letter f is used to denote a function. The brackets used in specifying this functional relationship do not indicate multiplication. The variable inside them is the one whose values we need to know to determine the values of the other variable, y, on the left-hand side. Letters other than f can be used to denote a function. If we have more than one functional relationship we can indicate they are different by using different letters, such as g or h. For instance, we may write

$$y = g(x)$$

which we again read as 'y is a function of x'.

An example of a function in economics is that total cost is a function of output. We may choose Q to represent output and write TC as a single variable name standing for total cost. The function can then be expressed as TC = f(Q). We are using Q in place of variable x and TC in place of y.

> function: a systematic relationship between pairs of values of the variables, written y = f(x).

SUBSTITUTION OF X VALUES

A function gives us a general rule for obtaining values of y from values of x. An example is

$$y = 4x + 5$$

The expression $4x$ means $4 \times x$. It is conventional to omit the multiplication sign. To evaluate the function for a particular value of x, multiply that x value by 4 and then add 5 to find the corresponding value of y. If $x = 6$

$$y = (4 \times 6) + 5 = 29$$

These x and y values give us the point (6,29) on the graph of the function. Now let us find the y value when $x = 0$. Substituting for x we obtain $y = (4 \times 0) + 5 = 5$, since multiplying by 0 gives 0.

See section 1.4 on multiplying by 0.

Substituting different values of x gives different points on the graph. Since the function tells us how to obtain y from any x value, y is said to be dependent on x, and x is known as the independent variable. Notice that on a graph the independent variable is plotted on the horizontal axis, and the dependent variable on the vertical axis.

Remember...

- If y is a function of x, y = f(x).
- A function is a rule telling us how to obtain y values from x values.
- x is known as the independent variable, y as the dependent variable.
- The independent variable is plotted on the horizontal axis, the dependent variable on the vertical axis.

LINEAR FUNCTIONS

If the relationship between x and y takes the form

$$y = 6x$$

when we substitute various x values to obtain the corresponding y values we find that all the pairs of x and y values are points lying on a straight line. Listed below are some x values with the corresponding y values:

x	0	5	10
y	0	30	60

The points (0,0), (5,30) and (10,60) all lie on a straight line as shown in figure 1.3. Notice that the line passes through the origin. Each y value is 6 times the x value, and we say that y is proportional to x.
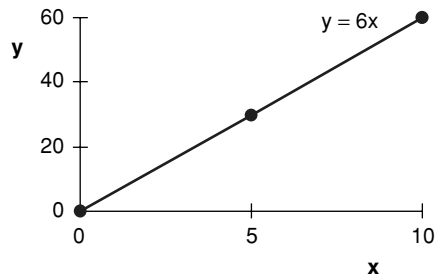

Figure 1.3   A linear function, $y = 6x$

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proportional relationship: each y value is the same amount times the corresponding x value, so all points lie on a straight line through the origin.
linear function: a relationship in which all the pairs of values form points on a straight line.

In general, a function of the form $y = bx$ represents a straight line passing through the origin. Since y is always b times x, the relationship between the variables is said to be a proportional one.

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shift: a vertical movement upwards or downwards of a line or curve.

Adding a constant to a function shifts the function vertically upwards by the amount of the constant. The function

$$y = 6x + 20$$

has y values that are 20 more than those of the previous function at every value of x. Using the same x values as before we obtain the points: (0,20), (5,50), (10,80). As figure 1.4 shows, the function $y = 6x + 20$ forms a straight line with the same slope as the previous line, but cutting the y axis at 20. This is called the intercept.
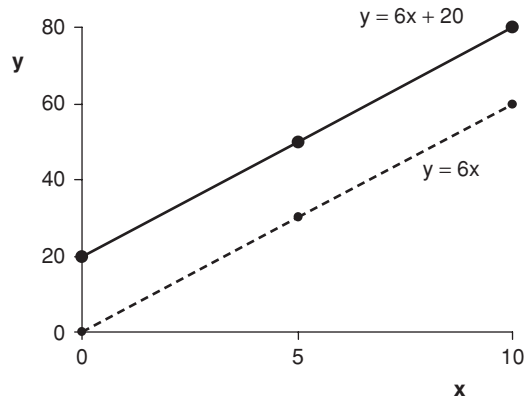
Figure 1.4   Adding a constant shifts the linear function up

intercept: the value at which a function cuts the y axis.

We study linear functions further in section 2.6 Linear Equations.

It is useful to recognize linear functions. If a function has just a term in x and, perhaps, a constant it is linear. A linear function has the general form $y = a + bx$. You can plot the graph of a linear function by finding just two points. Connecting these points with a straight line gives you the graph, and you can extend it beyond the points if you wish.

A function with just a term in x and (perhaps) a constant is a linear function. It has the general form

$$y = a + bx$$

Remember...

POWER FUNCTIONS

> power: an index indicating the number of times that the item to which it is applied is multiplied by itself.

Many of the functions economists use to model relationships are power functions in which the variable x appears raised to a power. Simple examples are

$$y = x^2 \qquad \text{(a)}$$

$$z = 7x^2 \qquad \text{(b)}$$

In function (a), the number superscripted to the right of x is called a power. When something is raised to a power we have to multiply that item by itself the number of times shown by the power. In this case, then, $x^2 = x \times x$, and the rule given by function (a) is that we must multiply x by itself to find the value of y. Similarly, function (b) can be written $z = 7 \times x \times x$. To evaluate it for a particular value of x we find x^2 (often called x squared) and then multiply it by 7 to obtain z. If $x = 5$ we have

$$y = 5^2 = 5 \times 5 = 25 \qquad \text{(a)}$$

$$z = 7 \times 5^2 = 7 \times 25 = 175 \qquad \text{(b)}$$

Functions often have more than one term and one of them may be a constant, for example

$$y = 140 + 7x^2 - 2x^3$$

In this function the highest power of x that appears is 3. The superscripted symbol x^3 is called x cubed and a power function in which x^3 is the highest power of x is called a cubic function. A function such as

$$y = 25x^2 + 74$$

has x^2 as the highest power of x and is called a quadratic function.

> quadratic function: a function in which the highest power of x is 2. There may also be a term in x and a constant, but no other terms.
> cubic function: a function in which the highest power of x is 3. There may also be terms in x^2, x and a constant, but no other terms.

SKETCHING FUNCTIONS

To help us visualize the nature of the relationship between two variables it is often useful to sketch a graph of the function. This does not have to be a precise graph. We just want to see its general shape. The method is to choose some x values, substitute them in the function to find the corresponding y values, and then plot the graph. Note that in economics it is often only positive values of x that are meaningful. Usually you need to plot a number of points to see the shape of the function, but for a linear function two points are sufficient. The steps in the process are listed below.

> In chapter 6 we will study a method for identifying maximum and minimum values.

To sketch a function   **Remember...**

- Decide what to plot on the x axis and on the y axis.
- List some possible and meaningful x values, choosing easy ones such as 0, 1, 10.
- Find the y values corresponding to each, and list them alongside.
- Look for points where an axis is crossed ($x = 0$ or $y = 0$).
- Look for maximum and minimum values at which the graph turns downwards or upwards.
- If you are not sure of the correct shape, try one or two more x values.
- Connect the points with a smooth curve.

> For information on multiplying negative numbers see section 1.4 Basic Rules of Algebra.

The functions that we use to represent economic relationships are usually single valued functions. When we substitute a value of x in the function we normally obtain a unique value of y that corresponds to it. You should be aware, however, that multivalued functions exist. These have more than one y value corresponding to one x value. For the function $y^2 = x$ there are two y values corresponding to every x value. If x equals 9, say, y is a number which when multiplied by itself gives 9. This implies either $y = 3$ or $y = -3$, since $3 \times 3 = 9$ and $(-3) \times (-3) = 9$.

> Now use the MathEcon screen Variables and Functions to consolidate your understanding. Answer the question it poses, then continue and click the question button to check that you understand the terms constant, variable and function.

USING EXCEL

You can use a computer spreadsheet such as Excel to calculate the values of functions and plot their graphs. You enter the formula for the function once and then copy it, so that calculating a large number of values of the

function becomes easy. You can inspect those values for maximum and minimum values, and can plot them on a graph. The file Function.xls contains examples for you to use, including a worksheet called Power function that will plot graphs for you. Further information about this is provided in section 1.13.

Sketch and briefly describe the following functions for positive values of x

$$y = 2x^3 - 50x \qquad\qquad\text{(a)}$$

$$y = 14 \qquad\qquad\text{(b)}$$

2

(a) This is a cubic function, so we need to choose a number of x values, say from 0 to 10, and substitute them in the function. You may find it helpful to evaluate each term separately and then to subtract the second from the first. The sketch we obtain is shown in figure 1.5. The curve passes through the origin, is negative for values of x between 0 and 5, crosses the x axis at 5, and is positive for larger x values. If you plot less points your sketch will be less accurate, but will probably still be adequate for many purposes. Notice, how-ever, that using only the x values 0, 5 and 10 would fail to reveal that part of the curve lies below the x axis.

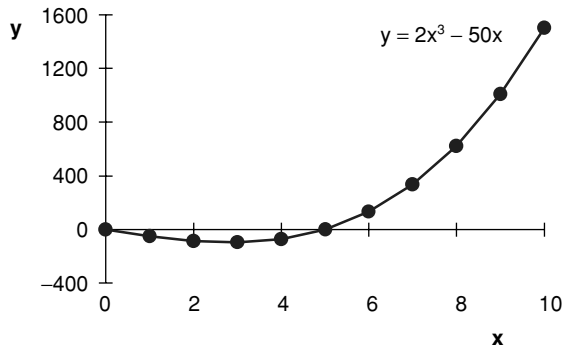| x	0	1	2	3	4	5	6	7	8	9	10
$2x^3$	0	2	16	54	128	250	432	686	1024	1458	2000
$50x$	0	50	100	150	200	250	300	350	400	450	500
$y = 2x^3 - 50x$											
$y =$	0	−48	−84	−96	−72	0	132	336	624	1008	1500

Figure 1.5

(b) The function $y = 14$ says y is a constant. Whatever the value of x, y has the value 14. The graph in figure 1.6 shows a horizontal straight line.
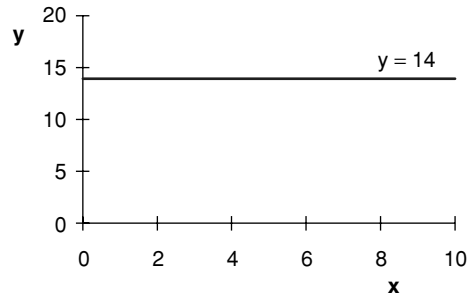
Figure 1.6

1.2   Sketch graphs of the functions for values of x between 0 and 10

(a) $y = 0.5x$ (b) $y = 0.5x + 6$ (c) $y = x^2$ (d) $y = 3x^2$

Which of them are linear? Which is a proportional relationship? What is the effect of adding a constant term?

1.3   Draw the line $y = x$ using equal scales on the horizontal and vertical axes. What angle does the line make with the horizontal axis?

SECTION 1.4: Basic Rules of Algebra

In evaluating algebraic or arithmetic statements certain rules have to be observed about the order in which various operations take place. This means you need to look carefully at an expression before you evaluate it and decide what to calculate first. It is no more difficult to calculate the right answer than the wrong one – you just have to take care in applying the rules about precedence. In an economics example the correct answer should correspond to what would be the diagrammatic solution to the problem, whereas a wrong answer can be completely meaningless – negative price and quantity, for example.

As an illustration, let us find the value of y from the following equation when $x = 3$:

$$y = 10 + 6x^2$$

- We first substitute the value 3 for x and square it, giving 9
- then multiply this by 6, obtaining 54
- and finally add this result to the value 10 giving the answer $y = 64$.

Notice that since $x^2 = x \times x$ it is different from $2x$ (except when $x = 2$). As to the order of the calculation, we have to pick out the term involving exponentiation (raising to a power) and do this first, followed by multiplication and after that addition. Brackets can be used to alter, or clarify, the order in which operations are to take place. Our previous statement is changed if brackets are inserted as shown and x remains 3:

$$y = (10 + 6)x^2$$

A number or symbol immediately before or after brackets implies that everything inside the brackets is to be multiplied by this item. Hence the brackets indicate that what they enclose is to be evaluated first and the result multiplied by the value outside. The above expression could be written

$$y = (10 + 6) \times x^2$$

Evaluating this, we do the bit in brackets first and so add $10 + 6$, giving 16. As before, x^2 is 9 but this is now multiplied by 16 giving a value for the expression of 144. Notice that in writing algebraic statements the multiplication sign is often omitted, or sometimes replaced by a dot.

The reciprocal of x is $1/x$. Instead of dividing by a value we can multiply by the reciprocal of that value. For example, $(y + 11) \div x = (y + 11) \times 1/x$.

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exponentiation: raising to a power.
reciprocal of a value: is 1 divided by that value.

An expression in brackets immediately preceded or followed by a value implies that the whole expression in the brackets is to be multiplied by that value.

Remember...

The order of algebraic operations is

1. If there are brackets, do what is inside the brackets first
2. Exponentiation
3. Multiplication and division
4. Addition and subtraction

You may like to remember the acronym BEDMAS, meaning brackets, exponentiation, division, multiplication, addition, subtraction.

Remember...

In working with economic models you should use brackets if you substitute an expression for a variable. This helps ensure you keep the calculation correct. In the worked example below notice how the brackets help us get the correct negative sign on the Y term in the answer.

3

If exports, X, and imports, Z, are given by $X = 450$ and $Z = 70 + 0.1Y$ respectively, write an expression for net exports, $NX = X - Z$ in terms of income, Y.

$$NX = X - Z$$

Putting brackets around the expression for Z as we substitute gives

$$NX = 450 - (70 + 0.1Y)$$

and multiplying out we obtain

$$NX = 450 - 70 - 0.1Y = 380 - 0.1Y$$

CALCULATORS AND COMPUTERS

Most modern calculators use algebraic logic, but remember you do have to put in the multiplication signs. Try entering each of the calculations in the above subsection in your calculator in the order in which it is written and check for the correct answer.

Computer spreadsheet programs also allow you to enter arithmetic statements in a format similar to that in which they are written. The computer does need to recognize that you are giving it a value to evaluate. In Microsoft Excel this is done by starting the expression with an equals sign. Signs to indicate multiplication and exponentiation are also needed, and for these the * and ^ are used. Further information and examples on using Excel for calculations are provided in section 1.13.

ORDER WITHIN AN EXPRESSION

Except when the rules about precedence indicate otherwise, we usually evaluate algebraic expressions working from left to right. Sometimes, however, you can make a calculation easier by choosing which part of it you do first. If you do this you must be sure your order of operations is legitimate.

Addition or multiplication of numbers can occur in any order. Take as an example $19 + 27 + 1$. This is the same as $19 + 1 + 27$, which is quicker to add in your head and gives 47.

A multiplicative example is $5 \times 17 \times 2 = 5 \times 2 \times 17$. It is easier first to multiply 5 by 2 than by 17, and so the answer of 170 is easy to work out in your head with the second arrangement.

Brackets may sometimes be used in sums or products to group terms, perhaps because of their economic significance, but the value of the result is not affected. For example

$$(19 + 1) + 27 = 19 + (1 + 27) = 47$$

and

$$(5 \times 2) \times 17 = 5 \times (2 \times 17) = 170$$

In subtraction and division the order of the terms is important. The value following a minus sign is subtracted from the value that precedes it. When a division is specified the first number is to be divided by the second. Interchanging the terms of a subtraction or division would alter the value of the expression and so is not permissible. For example

$$8 - 6 = 2, \text{ while } 6 - 8 = -2$$

and

$$8 \div 4 = 2, \text{ while } 4 \div 8 = 1/2$$

The MathEcon screen Basic Rules of Algebra describes these properties further. Try the quiz for some practise in evaluating a complex expression.

NEGATIVE NUMBERS

We can regard the x axis as a number scale. The values get bigger as we move to the right and smaller as we move to the left. Addition moves you to the right along the scale and subtraction moves you to the left, if the value being added or subtracted is positive. As you evaluate a complex expression you may have to add or subtract a negative number. The negative sign reverses the direction in which you move. Removing brackets around negative numbers, a plus and a minus sign together become a minus, while two minus signs together become a plus. Hence

$$7 + (-3) = 7 - 3 = 4 \tag{a}$$

Adding a negative number is the same as subtracting a positive number and you move to the left on the number scale. However,

$$7 - (-3) = 7 + 3 = 10 \tag{b}$$

Subtracting a negative number is the same as an addition and you move to the right on the number scale.

When you evaluate an expression that begins with a negative number you start at a position on the x axis to the left of the origin. If you add a positive number you move to the right, while a subtraction moves you to the left. Remember you can reorder the terms in an addition if you wish. Examples are

$$-5 + 6 = 6 - 5 = 1 \tag{c}$$

$$-5 - 6 = -(5 + 6) = -11 \tag{d}$$

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Factors and brackets are
explained further in the next
but two subsection.
</div>

Both terms in (d) have a negative sign so we may rewrite the expression using brackets as shown. Notice that the result is more negative than the number we started with.

In multiplications involving one or more negative numbers we bring the signs to the front of the terms we are multiplying. Again one plus and one minus sign together become a minus, while two minus signs become a plus. Division is similar to multiplication and the same rules about signs apply. For example

$$2 \times (-5) = -10 \tag{e}$$

$$(-2) \times (-5) = -(-10) = 10 \tag{f}$$

$$10 \div (-2) = -10 \div 2 = -5 \tag{g}$$

$$(-10) \div (-2) = -(-10 \div 2) = 5 \tag{h}$$

Take care in applying these rules to squared terms. Notice that -5^2 is -25 because exponentiation is done first and the negative of the result is taken, whereas $(-5)^2$ is 25 because the brackets instruct us to multiply -5 by itself, so there are two negative signs which yield a positive value.

When two signs come together   **Remember...**

$$- + (\text{or} + -) \text{ gives } -$$

$$- - \text{ gives } +$$

CALCULATORS AND NEGATIVE SIGNS

If you enter two arithmetic signs consecutively into a calculator, for example, $\times$ followed by $-$, it often assumes the second is a correction for the first. Some calculators have a $(-)$ sign for entering a minus sign that precedes a number. If yours does not, then for correct calculator evaluation of expressions (a), (b) and (e) to (h) you should enter the brackets as shown. If an expression starts with a minus sign as does (c), you may find that if you begin by entering -5 into your calculator the minus sign is not shown. You may, however, get the correct result. To get the calculator to show you it has a negative sign at the start of a calculation you can put the term in brackets. Alternatively, if you have a $+/-$ key you can enter a positive number and press the $+/-$ key to change the sign. Try out your calculator on expressions (a) to (h) so you are sure of how to enter expressions that include negative signs.

MULTIPLICATION AND DIVISION INVOLVING 1 AND 0

When we multiply or divide by 1 the expression is unchanged, whereas if we multiply or divide by -1 the sign of the expression changes. Hence

$$39 \times 1 = 39, \ 23 \times (-1) = -23, \ 92/1 = 92, \ 17/(-1) = -17$$

In algebraic expressions, if we have $2 \times x$ we write $2x$, but if we have $1 \times x$ we simply write x. It is important to know that the 1 is implied, although it is not stated. Also, because multiplication by 1 does not change anything, any expression can be considered as being multiplied by 1. This concept has an important parallel in matrix algebra.

4   Remove the brackets from the expression $y = -(6x^3 - 15x^2 + x - 1)$

Each term inside the brackets is multiplied by -1, so we have
$$y = -6x^3 + 15x^2 - x + 1$$

When we multiply by 0, the answer is 0. For example, consider different multiples of 8

$$3 \times 8 = 24 = 8 + 8 + 8$$

$$2 \times 8 = 16 = 8 + 8$$

$$1 \times 8 = 8 \text{ (there is just one 8 with nothing to add to it)}$$

$$0 \times 8 = 0 \text{ (there are no 8's, and therefore nothing)}$$

Division divides a value into parts, but if there is nothing to begin with the result of division is nothing, for example

$$0 \div 4 = 0$$

Division of 0 gives the answer 0.

Division by 0, however, gives a quite different result, namely one that is infinitely large if it is positive, or infinitely small if it is negative. Check to see what your calculator gives for $14 \div 0$. You should find it does not give you a numerical answer. We sometimes want to exclude the possibility of division by 0 and so, for example, we may define a function for a specified set of values, excluding any that would imply division by 0. To see that division by 0 gives a very large number which may have a negative sign (making it infinitely small), consider dividing 5 by successively smaller numbers:

$$5 \div 1 = 5$$

$$5 \div \frac{1}{2} = 10$$

$$5 \div \frac{1}{10} = 50$$

$$5 \div \frac{1}{100} = 500$$

$$5 \div \frac{1}{1000} = 5000$$

As the number we are dividing by gets closer and closer to 0 in the list above, the result gets larger and larger. It seems that division by 0 gives an infinitely large positive number. But if each of the divisors above were negative, they would still be getting closer and closer to 0, but would be approaching it from the left of zero on the number scale. Each answer would be the negative of the one shown and so division by 0 can give a result of infinitely large magnitude but with a negative sign, so that the value is infinitely small.

   The only general statement we can make about the result of dividing 0 by 0, $0 \div 0$, is to say it is undefined. In a particular case it may be possible to study what happens as both the numerator and the denominator get very small.

$1 \times x = x$, $(-1) \times x = -x$ **Remember...**

- Any value multiplied by 0 is 0.
- 0 divided by any value except 0 is 0.
- Division by 0 gives an infinitely large number which may be positive or negative.
- Be wary of division by 0.

An example of where you may encounter division by zero in studying economics is shown on the MathEcon screen Deriving the Short-Run Average Fixed Cost (AFC) and Average Variable Cost (AVC) Curves. Total fixed cost (FC) is a constant and average fixed cost is found by dividing it by the level of output, Q. The values of Q that we consider are positive values starting from 0. If $Q = 0$, then AFC = FC/Q is infinite. Labour employed, L, determines the values of output, Q, and variable cost, VC. When $L = 0$, both Q and VC are zero. The formula AVC = VC/Q therefore gives $0 \div 0$. This result is meaningless. What we can do instead is use a small fractional value for L. If $L = 0.001$, $Q = 0.35$, VC = 20 and AVC = 20/0.35 = 57.14. If you want to check this, the production function from which the values of Q are calculated is shown on the screen Short-Run Production Functions: A Numerical Example.

FACTORS AND MULTIPLYING OUT BRACKETS

To help us evaluate an expression we may want to take out a common factor from each of several terms, or on other occasions we may wish to perform the opposite operation of multiplying out the brackets. With a bit of practise you will find you can recognize a pattern in the best way to approach a particular type of problem. For example, suppose

$$y = 6x - 3x^2$$

Remembering that this is a shorthand way of writing

$$y = (6 \times x) - (3 \times x \times x)$$

we see that each of the terms on the right-hand side can be exactly divided by $3 \times x$, because $(6 \times x)/(3 \times x) = 2$ and $(-3 \times x \times x)/(3 \times x) = -x$. The amount that we can divide by, $3 \times x$, is called a common factor. We can rewrite the original expression using brackets that contain the terms after they have been divided by the common factor, and with the common factor outside, multiplying the whole. This is called factorizing the expression and gives

$$y = 3 \times x \times (2 - x)$$

It is usual to write this without the multiplication signs as

$$y = 3x(2 - x)$$

---

factorizing: writing an expression as a product that when multiplied out gives the original expression.

Alternatively, it may be useful to multiply out brackets and so remove them. If one expression is written immediately next to an expression in brackets, the implication is that these are multiplied together. To multiply out, we multiply each term in the brackets by the expression outside the brackets. Consider, for example

$$3x(2 - x)$$

To multiply out the brackets we multiply 2 by $3x$, then multiply $-x$ by $3x$ giving

$$3x(2 - x) = 6x - 3x^2$$

When two brackets are multiplied together, to remove them we multiply each term in the second bracket by each term in the first bracket. It is then usual to simplify the result by collecting terms where possible. For example

$$(a - b)(-c + d) = -ac + ad + bc - bd$$

> Remember to use the rules about signs discussed in the Negative Numbers subsection.

Another example of multiplying out brackets is

$$(x - 7)(4 - 3x) = 4x - 3x^2 - 28 + 21x$$
$$= -3x^2 + 25x - 28$$

Here the simplification involves adding the two terms in x. Notice that since factorization is the reverse process to multiplying out brackets, had you been given the right-hand side of the expression you could have factorized it to obtain the left-hand side, but the

You can see another example on the MathEcon screen Basic Rules of Algebra under the heading Combined Multiplication and Addition.

factors might not have been immediately obvious. Factorizing a quadratic expression involves some intelligent guesswork. You have to look for two expressions that multiply together to give you the one you started with. You should also be aware that not every quadratic expression factorizes to a product of expressions that contain integer values. The following standard results of multiplying out brackets are helpful:

$$(a + b)^2 = a^2 + 2ab + b^2$$

$$(a - b)^2 = a^2 - 2ab + b^2$$

$$(a + b)(a - b) = a^2 - b^2$$

You are asked to show these results in practice problem 1.7 below.

An expression in brackets written immediately next to another expression implies that the expressions are multiplied together.

Remember...

Multiplying out brackets

One pair: multiply each of the terms in brackets by the term outside.

Two pairs: multiply each term in the second bracket by each term in the first bracket.

Factorizing: look for a common factor, or for expressions that multiply together to give the original expression.

ACCURACY

When your calculations give non-integer results it is often appropriate to round your answer. To give an answer correct to two decimal places you round it up if the value in the third decimal place is 5 or over. If such an answer is to be used in a further calculation, however, to maintain as much accuracy as possible you should retain the value in your calculator and continue the calculation. This is especially important when answers to different parts of a calculation are to be multiplied together, since any inaccuracies would be compounded. In evaluating

$$\left(100 \times \frac{5}{9}\right)\left(36 \times \frac{2}{7}\right)$$

you can input this into a calculator in one step as $(500/9) \times (72/7)$ and round up to get an answer of 571.43. But notice that if you do the calculation in steps, rounding up after each one (i.e. $500/9 = 55.56$; $72/7 = 10.29$; then 55.56×10.29), you get the inaccurate result of 571.71 because each of the intermediate results has been rounded up.

1.4 Evaluate each expression without using a calculator and then check your answer using a calculator

(a) $35 - 2x^2$ when $x = 4$ and when $x = 5$

(b) $(25 - 23)x^2$ when $x = 4$ and when $x = -4$

(c) $\dfrac{10}{5} - 4$ (d) $\dfrac{10}{5 - 4}$

(e) $\dfrac{10}{4} - 5$ (f) $2 + (3 \times 5)$

(g) $-6 \times (3 - 7)$ (h) $32 - 2 \div 10$

(i) $-12 - 8 \times 3$ (j) $-15 - (-9)$

(k) $\sqrt{25}$

1.5 If a consumer spends two-thirds of any increase in income and her income increases by \$100, what is the increase in her spending?

1.6 Factorize

(a) $-10x - 45x^2$ (b) $143x - 52$

(c) $5x^2 + 5x - 20xy$

1.7 (a) Multiply (or divide) out the brackets and simplify as far as possible

    (i) $8x(10 - 7x)$ (ii) $(2x + 5)(9 - 3x)$

    (iii) $(11 - x)(12 - 4x)$ (iv) $24 + 0.8(x - 675)$

    (v) $\dfrac{1600 - 12x}{0.8}$

(b) Simplify

    (i) $232x - 2x^2 - 100 - 150x + 0.36x^2$

    (ii) $\dfrac{100 + 150x - 36x^2}{x}$

    (iii) $x - 0.8x$ (iv) $4(x - 0.75x)$

(c) Subtract $150 + 70x - x^2 + 0.5x^3$ from $270x - 3x^2$ and simplify your answer

(d) Evaluate $30 + 18x - 0.6x^2$ when $x = 15$

(e) By writing the square of a term in brackets as the product of terms, show the following results:

(i) $(a + b)^2 = a^2 + 2ab + b^2$ (ii) $(a - b)^2 = a^2 - 2ab + b^2$

(iii) $(a + b)(a - b) = a^2 - b^2$

1.8 (a) Check the following

(i) $(x - 7)$ and $(x + 5)$ are factors of $x^2 - 2x - 35$

(ii) $(3x + 1)$ and $(x + 8)$ are factors of $3x^2 + 25x + 8$

(iii) $(2x + 9)$ and $(x - 1)$ are factors of $2x^2 + 7x - 9$

(b) Factorize

(i) $x^2 + 10x + 21$ (ii) $3x^2 + 14x - 5$

(iii) $2x^2 + 8x - 10$ (iv) $3x^2 + 26x + 55$

SECTION 1.5: Fractions and Sharing

Sometimes the functions economists use involve fractions. For example, $\frac{1}{4}$ of people's income may be taken by the government in income tax, and $\frac{5}{7}$ of disposable income may be spent on consumption. Sometimes an optimal situation is identified when one ratio equals another, as shown by the following example. When the ratio of the prices of two goods equals the ratio of their marginal utilities, spending is optimally allocated between the two goods.

fraction: a part of a whole.
ratio: one quantity divided by another quantity.

In numerical calculations you may choose if you prefer to write fractions as decimals, for example, $\frac{1}{4} = 0.25$. But for demonstrating various algebraic results you need to know the basic rules of working with fractions. These rules are listed below.

A fraction is a part of a whole. For example, if a household spends $\frac{1}{5}$ of its total weekly expenditure on housing, the share of housing in the household's total weekly

Figure 1.7

expenditure is ⅕. We can find the amount spent by multiplying the share, which is a fraction, by the total amount. If the household's total weekly expenditure is $250, the amount it spends on housing is one fifth of that amount.

Amount spent on housing = share of housing × total weekly expenditure

$$= \frac{1}{5} \times 250 = \$50$$

numerator: the value on the top of a fraction.
denominator: the value on the bottom of a fraction.

The top line of a fraction is called the numerator and the bottom line is called the denominator. The denominator shows how many parts the total amount is considered

as being divided into, and the numerator shows the number of those parts that belong to the item we are considering. In the housing expenditure example, total weekly expenditure is considered as being divided into five parts (5 is the denominator) and one of these parts is spent on housing (1 is the numerator).

A fraction can also be written as a ratio of algebraic symbols. For example, if h = amount spent on housing and x = total weekly expenditure, the share of housing = h/x. The rules for working with fractions are first demonstrated numerically, but you then need to practise the same rules with fractions written in algebraic symbols using the examples that follow.

CANCELLING

When working with fractions we can divide both the numerator and the denominator by the same amount and the fraction is unchanged. If we have the fraction $^{30}\!/_{40}$ we can simplify it by dividing both top and bottom by 10.

$$\frac{30}{40} = \frac{(30 \div 10)}{(40 \div 10)} = \frac{3}{4}$$

10 is said to be a factor of both the numerator and the denominator, and can be cancelled.

When cancelling, if the whole of the numerator (or denominator) is cancelled out it becomes 1. For example, with

$$\frac{14 \times 6}{56 \times 9}$$

dividing by 14 on the top and bottom gives

$$\frac{(14 \div 14) \times 6}{(56 \div 14) \times 9} = \frac{1 \times 6}{4 \times 9} = \frac{6}{36}$$

Dividing the numerator and denominator by 6 simplifies the fraction further

$$\frac{6 \div 6}{36 \div 6} = \frac{1}{6}$$

When a fraction includes algebraic symbols, if the same symbol occurs as a multiplier in both the numerator and the denominator of a fraction, these symbols can be cancelled. For example

$$\frac{20y^2}{5y} = \frac{4(y \times y)}{y} = 4y$$

since the y in the denominator cancels with one of the y's in the numerator.

COMMON DENOMINATOR

Economists often want to compare two fractions to see if they are equal or, if not, which is bigger. They also want to add and subtract fractions. For all of these operations we need to express our fractions as having the same, or common, denominator. For example, to find which is the bigger of $\frac{3}{7}$ and $\frac{9}{20}$ we multiply both numerator and denominator of each fraction by the denominator of the other. This is the reverse operation to cancelling and leaves the value of the fraction unchanged.

$$\frac{3}{7} = \frac{20 \times 3}{20 \times 7} = \frac{60}{140}, \text{ while } \frac{9}{20} = \frac{7 \times 9}{7 \times 20} = \frac{63}{140}$$

Now that both fractions have the same denominator, 140, we can see immediately that the second is bigger because it has a numerator of 63 which is bigger than that of 60 which the first fraction has. Using the symbol > to indicate that the first expression is greater than the second, we write $\frac{63}{140} > \frac{60}{140}$, and so $\frac{9}{20} > \frac{3}{7}$. This is an example of an inequality expression, where the inequality sign indicates which of the two values it is separating is the greater.

> sign: the greater than sign indicates that the value on its left is greater than the value on its right.
> < sign: the less than sign indicates that the value on its left is less than the value on its right.

Common denominators are also needed in working with algebraic expressions and we find them in the same way. For example, to put the fractions $3y/5x$ and $7x/y$ on a common denominator we multiply each fraction by the denominator of the other. This gives

$$\frac{3y}{5x} = \frac{3y \times y}{5x \times y} = \frac{3y^2}{5xy} \text{ and}$$

$$\frac{7x}{y} = \frac{7x \times 5x}{y \times 5x} = \frac{35x^2}{5xy}$$

Notice that unless we know the numerical values of x and y we cannot immediately say which of these two fractions is bigger.

ADDITION AND SUBTRACTION OF FRACTIONS

If fractions have the same denominator we can immediately add them or subtract them. We simply add or subtract the numerators and place the result on the common denominator. For example

$$\frac{3}{7} + \frac{1}{7} = \frac{3+1}{7} = \frac{4}{7} \text{ and}$$

$$\frac{9}{11} - \frac{2}{11} = \frac{9-2}{11} = \frac{7}{11}$$

If the denominators are not the same we must find a common denominator for the fractions before adding or subtracting them. For example, to find $^1/_4 + {}^2/_3$ we multiply both numerator and denominator of the first fraction by 3, and of the second fraction by 4. Both fractions then have denominators of 12 and we can add them.

$$\frac{1}{4} + \frac{2}{3} = \frac{3}{12} + \frac{8}{12} = \frac{3+8}{12} = \frac{11}{12}$$

To perform the subtraction $^{24}/_{35} - {}^3/_{10}$ we could use 350 as a common denominator using our basic rule of using the denominator of each fraction to multiply both numerator and denominator of the other. But if we notice that the first denominator is 7×5 and the second is 2×5 we see that the multiplier of 5 is already common to both denominators and we need only multiply by the other term in each denominator. Thus we use 2 as the multiplier for the first fraction and 7 for the second giving us 70 as the denominator. This is said to be the lowest common denominator.

$$\frac{24}{35} - \frac{3}{10} = \frac{48}{70} - \frac{21}{70} = \frac{48-21}{70} = \frac{27}{70}$$

---

lowest common denominator: the lowest value that is exactly divisible by all the denominators to which it refers.

An algebraic example is

$$\frac{2}{x+2} + \frac{5}{x+1}$$

Notice that although 2 appears both in the numerator and denominator of the first fraction it does not multiply the whole of the denominator and hence cannot be cancelled. Choosing the product of both denominators as the common denominator we have

$$\frac{2(x+1) + 5(x+2)}{(x+2)(x+1)}$$

and multiplying out the numerator gives

$$\frac{(2x+2+5x+10)}{(x+2)(x+1)}$$

Collecting terms in the numerator we have

$$\frac{7x+12}{(x+2)(x+1)}$$

MULTIPLICATION AND DIVISION OF FRACTIONS

To multiply two fractions we multiply the numerators and the denominators. For example

$$\frac{1}{2} \times \frac{3}{5} = \frac{1 \times 3}{2 \times 5} = \frac{3}{10}$$

To divide one fraction by another we turn the divisor upside down and multiply by it. You can check that this works by seeing that the reverse operation of multiplication gets you back to the value you started with. For example

$$\frac{5}{7} \div \frac{3}{4} = \frac{5}{7} \times \frac{4}{3}$$

$$= \frac{5 \times 4}{7 \times 3} = \frac{20}{21}$$

Check by multiplying the answer by the value you divided by and cancelling:

$$\frac{20}{21} \times \frac{3}{4} = \frac{20 \times 1}{7 \times 4}$$

$$= \frac{5 \times 1}{7 \times 1} = \frac{5}{7}$$

Algebraic fractions can be multiplied and divided in just the same way. You may then be able to cancel terms, but always check carefully that whole terms are equal before you cancel them. For example

$$\frac{x + 4}{x} \times \frac{x^2}{2} = \frac{(x + 4) \times x^2}{x \times 2} = \frac{(x + 4)x}{2}$$

cancelling the x in the denominator with one of the x's in the numerator x^2. Note that although there is a 4 in the numerator it is added to x and therefore it is not possible to cancel the 2 in the denominator with it.

Use the MathEcon screen Fractions and Sharing for more examples using the rules of fractions. An economic application is choosing what fraction of your income you want to spend on consumption. The screen Objectives and Actions: Looking Ahead lets you take part in a simulation where you choose what fraction or multiple of your income you consume in different time periods.

- Amount of an item = fractional share of item × total amount.   **Remember...**
- A fraction is: numerator/denominator.
- Cancelling is dividing both numerator and denominator by the same amount.
- To add or subtract fractions first write them with a common denominator and then add or subtract the numerators.
- Fractions are multiplied by multiplying together the numerators and also the denominators.
- To divide by a fraction turn it upside down and multiply by it.

5

Evaluate the following without using a calculator

(a) $\dfrac{9}{11} + \dfrac{1}{2}$ (b) $\dfrac{5}{12} \times \dfrac{4}{7}$ (c) $\dfrac{1}{3} \div 4$ (d) $\dfrac{7}{10} - \dfrac{5}{16}$ (e) $\dfrac{8}{15} \div \dfrac{12}{25}$

(f) In a population of 28 million people aged between 20 and 60, $^3\!/_4$ are working or looking for work. How many people in the age group are economically active?

(a)
$$\frac{9}{11} + \frac{1}{2} = \frac{18 + 11}{22} = \frac{29}{22} \text{ or } 1\tfrac{7}{22}$$

(b) Cancelling the 4's, since $12 = 4 \times 3$
$$\frac{5}{12} \times \frac{4}{7} = \frac{5}{3} \times \frac{1}{7} = \frac{5}{21}$$

(c) Dividing by 4 is the same as multiplying by $^1\!/_4$
$$\frac{1}{3} \div 4 = \frac{1}{3} \times \frac{1}{4} = \frac{1}{12}$$

(d) You could use 160 as the denominator, but the lowest common denominator is 80
$$\frac{7}{10} - \frac{5}{16} = \frac{56 - 25}{80} = \frac{31}{80}$$

(e) We turn the fraction we are dividing by upside down and multiply by it
$$\frac{8}{15} \div \frac{12}{25} = \frac{8}{15} \times \frac{25}{12}$$

Cancelling 4's and 5's gives
$$\frac{2}{3} \times \frac{5}{3} = \frac{10}{9} \text{ or } 1\tfrac{1}{9}$$

(f) The economically active population in the age group is those who are working or looking for work. The number economically active $= {}^3\!/_4 \times 28 = 21$ million people.

6

Simplify

(a) $\dfrac{3}{x^2} \times \dfrac{x}{9}$ (b) $\dfrac{5x}{2(x + 1)} + \dfrac{3x}{4(x - 1)}$

(a) Here we can cancel 3's and also x's since $x^2 = x \times x$:
$$\frac{3}{x^2} \times \frac{x}{9} = \frac{1}{x} \times \frac{1}{3} = \frac{1}{3x}$$

(b) Putting both terms on $4(x + 1)(x - 1)$ as the lowest common denominator gives $[10x(x - 1) + 3x(x + 1)]/[4(x + 1)(x - 1)]$. Multiplying out the brackets in the numerator we find
$$\frac{10x^2 - 10x + 3x^2 + 3x}{4(x + 1)(x - 1)} = \frac{13x^2 - 7x}{4(x + 1)(x - 1)}$$

1.9 Evaluate the following without using a calculator

(a) $\dfrac{3}{4} - \dfrac{1}{2}$ (b) $12 \times \dfrac{1}{3}$

(c) $\dfrac{1}{6} \times \dfrac{5}{7}$ (d) $\dfrac{3}{10} + \dfrac{7}{15}$

(e) $\dfrac{8}{9} \div 2$ (f) $\dfrac{24}{35} \times \dfrac{7}{8} \div \dfrac{3}{5}$

(g) $\dfrac{11}{12} - \dfrac{7}{18}$

1.10 If your marginal utility in consuming good X is 5 and your marginal utility in consuming good Y is 9 when you are buying X and Y at prices of 30 and 45 respectively, is the ratio of your marginal utilities equal to the ratio of the prices?

1.11 Simplify

(a) $\dfrac{x^2}{4} \times \dfrac{3}{x}$ (b) $\dfrac{15M}{P^2} \times P \div \dfrac{15M}{P}$

(c) $\dfrac{5x^2 - 3x}{x(1 + 3x)}$ (d) $\dfrac{8x}{12x^2 - 16x}$

(e) $\dfrac{7}{x + 2} + \dfrac{5x}{2x - 1}$

SECTION 1.6: Powers and Indices

A power or index applied to a value shows the number of times the value is to be multiplied by itself. For example

$$x^3 = x \cdot x \cdot x$$

and

$$x^5 = x \cdot x \cdot x \cdot x \cdot x$$

index or power: a superscript showing the number of times the value to which it is applied is to be multiplied by itself.

Notice that $x^1 = x$.

When we multiply together expressions comprising the same value raised to a power, we add the indices and raise the value to that new power. An example will show you that this rule works. Multiplying x^3 by x^5 gives three x's and then five x's all multiplied together. So we have eight x's multiplied together, which by definition is x^8.

$$x^3 . x^5 = (x . x . x)(x . x . x . x . x) = x^8$$

The rule gives the same result:

$$x^3 . x^5 = x^{3+5} = x^8$$

To divide two expressions where each is the same variable raised to some power, we subtract the powers to find the power of the variable in the answer. Again you can see by an example that the rule works. Dividing x^5 by x^3 we have five x's divided by three x's. Each of the bottom x's cancels with a top x, leaving two x's in the numerator, so the result is x^2.

$$\frac{x^5}{x^3} = \frac{x . x . x . x . x}{x . x . x} = x . x = x^2$$

Using the rule we obtain the same answer:

$$\frac{x^5}{x^3} = x^{5-3} = x^2$$

In such divisions, if the larger of the two values is the divisor, cancelling gives us a numerator of 1 and the denominator comprises the value raised to an appropriate power. Using the rule about subtracting the powers when we divide we obtain a negative index for our variable, so there are two alternative ways of writing the result. For example

$$\frac{x^3}{x^5} = \frac{x . x . x}{x . x . x . x . x} = \frac{1}{x . x} = \frac{1}{x^2}$$

and

$$\frac{x^3}{x^5} = x^{3-5} = x^{-2}$$

> In chapter 5 we will find that it is important for us to be able to switch between these two alternative formats.

We can now deduce the value of x^0. Using the rule that when we divide we subtract the indices, we see that we get x^0 whenever x to a particular power is divided by x to the same power. Looking at it another way, we see that the numerator and denominator are the same and cancelling them gives us the value 1. Hence $x^0 = 1$. For example

$$\frac{x^4}{x^4} = x^{4-4} = x^0$$

and

$$\frac{x^4}{x^4} = \frac{x . x . x . x}{x . x . x . x} = 1$$

We can also give an interpretation to fractional powers. Consider what happens when we multiply an expression with a fractional power by itself as many times as the denominator of the fraction. For example, multiply together two of $x^{1/2}$

$$(x^{1/2})(x^{1/2}) = x^{1/2+1/2} = x^1 = x$$

But the number that when multiplied by itself gives x is the square root of x, $\sqrt{x}$. This implies that

$$x^{1/2} = \sqrt{x}$$

Similarly, $x^{1/3}$ is the cube root of x since

$$(x^{1/3})(x^{1/3})(x^{1/3}) = x^{1/3+1/3+1/3} = x^1 = x$$

and in general $x^{1/n}$ is the nth root of x.

Any expression can be raised to a power. We use brackets to enclose the expression and write the power outside the brackets. The power shows us how many times the expression is multiplied by itself. If we write this out in full we can then multiply out. Consider for example $(5x)^3$. Writing this as a product of the terms and then multiplying out gives

$$(5x)^3 = (5x)(5x)(5x) = 5^3x^3 = 125x^3$$

It follows that when a product is raised to a power, each term in the product is raised to the power. The general result is written

$$(ax)^n = a^n x^n$$

Now consider an expression with an index which is then all raised to a power, for example $(x^3)^3$. We can rewrite this and multiply out, collecting the terms. We obtain

$$(x^3)^3 = (x^3)(x^3)(x^3) = x^9$$

Notice that

$$x^9 = x^{3\times3}$$

When an expression to a power is raised to a power we combine the powers by multiplying them. In general

$$(x^m)^n = x^{mn}$$

To multiply, add the indices; to divide, subtract the indices.

Remember...

$$x^{-n} = \frac{1}{x^n}$$

$$x^0 = 1$$

$$x^{1/2} = \sqrt{x}$$

$$(ax)^n = a^n x^n$$

$$(x^m)^n = x^{mn}$$

When expressions that contain powers are multiplied or divided by one another we use the above rules to simplify them. Notice that the rules allow us to combine indices for the same symbol that appears more than once in an expression. We cannot combine indices for different variables, so for example there is no way of simplifying x^3/y^2.

7

Simplify $w = 2x^{1/2}y^{1/3}/x^{-1/2}y^{2/3}$

$$w = 2x^{1/2}y^{1/3}(x^{1/2}y^{-2/3}) = 2x^{1/2+1/2}y^{1/3-2/3} = 2xy^{-1/3} = \frac{2x}{y^{1/3}}$$

POWERS AND YOUR CALCULATOR

Most scientific calculators have a button marked x^y which is used to raise a number to a power. Enter the number, press the x^y button, enter the power to which the number is to be raised and press = to obtain your answer. If the power is negative, precede it by (−) or enter the positive value and press the +/− button followed by the = key. Alternatively, you can put negative or fractional powers in brackets. Try using your calculator to evaluate some expressions involving negative and fractional powers to check that you understand them. For example

$$64^{1/2} = 64^{0.5} = 8 \quad \text{and} \quad 10^{-2} = \frac{1}{10^2} = 0.01$$

SCIENTIFIC NOTATION

If the result of a calculation is very large or very small, calculators and computers usually display it using scientific notation. This takes the form of a number followed by an exponent, telling you that the number is to be multiplied by 10 to that power. For example, if you enter 3000^3 in your calculator using the x^y button you may get the answer 2.7^{10}. This is to be interpreted as 2.7×10^{10}. What this means is that the decimal point in the number shown is to be moved to the right the number of places given in the exponent, so that the value is 27,000,000,000. Another example is $1 \div 970$, where your calculator may give the answer $1.030927835^{-0.3}$. To find the value we move the decimal point three places to the left, which gives 0.001030927835. The reason for the use of scientific notation is that it allows the calculator to show more significant figures in the answer than would otherwise be possible.

exponent: a superscripted number representing a power.

1.12 Simplify

(a) $x^7 . x^8$

(b) $x^{1/3} . x^{1/4}$

(c) $\dfrac{x^9}{x^4}$

(d) $x^9 . x^{-4}$

(e) $\dfrac{x^{3/4}}{x^{1/4}}$

(f) $(x^4)^2$

(g) $(2x)^4$

(h) $\dfrac{15x^7}{3x^2 . 5x^5}$

(i) $\dfrac{12x^{1/4}y^{-1/2}}{x^{-3/4}y^{1/2}}$

(j) $\dfrac{x^{a-1}}{x^a}$

1.13 Show that

$$\left(\frac{1}{1+r}\right)^t = (1+r)^{-t}$$

1.14 Evaluate

(a) 5^2

(b) 2^5

(c) 9^{-2}

(d) 8^0

(e) $64^{1/2}$

(f) $27^{1/3}$

(g) $(3^2)^4$

(h) $82 - 3^4$

(i) $82 + (-3)^4$

(j) $\left(\dfrac{x}{5}\right)^2$ when $x = 35$

(k) $(x^2 + 40)^{1/2}$ when $x = 9$

SECTION 1.7: Functions of More Than One Variable

In the relationships studied by economists the dependent variable is often thought to depend on a number of other variables. For example, the utility a person obtains may depend on the quantities of several goods that he or she consumes. Another example is that the output a firm produces may depend on the amounts it uses of each of a number of inputs.

> multivariate function: the dependent variable, y, is a function of more than one independent variable.

For the case where the dependent variable, which we shall continue to call y, depends on two variables x and z we express the function

$$y = f(x,z)$$

which we read as y is a function of x and z. In this function there are two independent variables x and z. If we have possible values for x and z we may substitute them to obtain the corresponding value of y. Since there are two independent variables, we may fix one of them, say x, at a particular value and change the other variable, z. This lets us investigate how y changes as z changes. The approach corresponds to comparative statics analysis in economics where economists investigate the effect of changing one variable while other things remain unchanged. We can, of course, also investigate the effects on y of changing x while z is held constant.

Remember...

- If $y = f(x,z)$ y is a function of the two variables x and z.
- We substitute values for x and z to find the value of the function.
- If we hold one variable constant and investigate the effect on y of changing the other, this is a form of comparative statics analysis.

We shall see an example of substituting values for two independent variables in section 1.12 where we investigate the quantities of output produced by a firm with a specific production function and employing two different factors of production, labour and capital. A more general approach to describing the effects of changing one of the independent variables but not the other will be discussed in chapter 8.

To depict the relationship between three variables graphically using an axis for each requires a three-dimensional graph, but there are also ways of presenting the information as a two-dimensional graph. Economists usually use one of the two-dimensional options as we see in the example in section 1.12.

SECTION 1.8: Economic Variables and Functions

We now investigate and plot various economic functions. The analysis in sections 1.9 to 1.12 corresponds to analysis presented in any elementary microeconomics textbook. This book, however, emphasizes mathematical relationships and shows the shapes of

curves that correspond to particular functions. We shall find the rules of algebra helpful in calculating the values to plot, and also in using definitions of the relationships between variables to link one function to another.

Notice that we usually replace the general variable names *x* and *y* with names chosen to suit our variables and we label the axes of our diagrams to correspond. Suitable scales for the axes depend on the particular model, but for some economic variables – such as price, quantity, cost and labour employed – only positive values are meaningful. Other variables, such as profit, can take negative values.

It is a general principle of economic modelling to choose a functional form that is as simple as possible while representing the appropriate form of relationship. You will therefore find that linear relationships are used in various contexts and that where a curve is needed to depict the relationship a quadratic or cubic function may be used. The analysis presented in sections 1.9 to 1.11 is concerned with two variable relationships. A multivariate relationship is examined in section 1.12, focusing on particular aspects that can be represented on two-dimensional graphs.

You can see examples of functional relationships in the MathEcon screens titled Factors Affecting Demand, where you investigate the demand for peaches, and Short-Run Production Functions: A Numerical Example, where you plot a curve representing the short-run production function. The screen Economic Problems – Basic Mathematics has four different economic problems for you to practise.

SECTION 1.9: Total and Average Revenue

When a firm sells a quantity, Q, of goods each at price P, its total revenue, TR, is the price that is paid multiplied by the quantity sold and so

$$TR = P \cdot Q$$

Average revenue, AR, is the revenue received by the firm per unit of output sold. This is its total revenue divided by the quantity sold. Hence,

$$AR = TR \div Q = P$$

substituting the above expression for TR.

The average revenue curve shows the average revenue or price at which different quantities are sold. It therefore shows the prices that people will pay to obtain various quantities of output and so it is also known as the demand curve.

$$TR = P \cdot Q$$

$$AR = \frac{TR}{Q}$$

Remember...

A market demand curve is assumed to be downward sloping. Different prices are associated with different quantities being sold and more is sold at lower prices. There will also be an associated downward sloping marginal revenue, MR, curve but we postpone consideration of its exact relationship with TR and AR until chapter 5.

If average revenue is given by

$$P = 72 - 3Q$$

sketch this function and also, on a separate graph, the total revenue function.

8

The average revenue function has P on the vertical axis and Q on the horizontal axis. The general form of a linear function is $y = a + bx$. Comparing our average revenue function we see that it takes this linear form with $y = P$, $a = 72$, $b = -3$ and $x = Q$. We therefore need find only two points on our function to sketch the line, and can then extend it as required. For simplicity we choose $Q = 0$ and $Q = 10$. The corresponding P values are listed, the two points are plotted and the line is then extended to the horizontal axis as shown in figure 1.8.

Chosen values of Q: Q 0   10
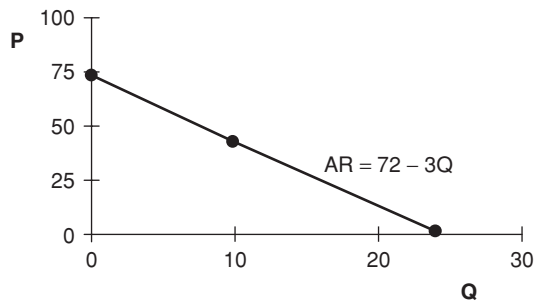Substituting in $P = 72 - 3Q$: P 72   42

Figure 1.8

We next find an expression for TR.

$$TR = P \cdot Q = (72 - 3Q)Q = 72Q - 3Q^2$$

As before, the horizontal axis is called Q. Revenue values are again being plotted on the vertical axis, but they are now values of TR, which in general is much larger than AR and so a different scale is appropriate. The function is a quadratic one, so we must find a number of points. Choosing some values of Q, say the even numbers between 0 and 16 and also 24, we calculate the value of $72Q$ and of $3Q^2$ and subtract the second from the first to find TR as shown in the table. The graph in figure 1.9 shows a curve which at first rises relatively steeply, then flattens out and reaches a maximum at $Q = 12$, after which it falls. Notice that the curve is symmetric. Its shape to the right of its maximum value is the mirror image of that to the left. The values of TR at 14 and 16 are the same as those at 10 and 8. Notice that a downward sloping linear demand curve implies a total revenue curve which has an inverted U shape.

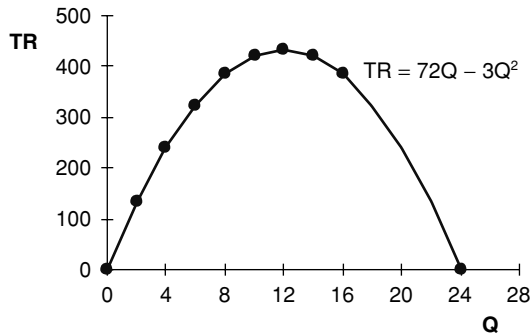| Q	0	2	4	6	8	10	12	14	16	24
72Q	0	144	288	432	576	720	864	1008	1152	1728
$3Q^2$	0	12	48	108	192	300	432	588	768	1728
TR	0	132	240	324	384	420	432	420	384	0

Figure 1.9

symmetric: the shape of one half of the curve is the mirror image of the other half.

Some firms may sell all their output at the same price. This is a feature of firms operating under the market structure known as perfect competition. These firms face a horizontal demand curve and have a total revenue function which is an upward sloping straight line passing through the origin.

There is a description of how to use Excel to plot the above demand and total revenue functions in section 1.13, and the worksheets used are included in the Function.xls file on the CD.

1.15 A firm in perfect competition sells its output at a price of 12. Plot its total revenue function, $TR = 12Q$.

1.16 Sketch the market average revenue function given by

$$AR = 25 - 5Q$$
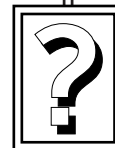
SECTION 1.10: Total and Average Cost

A firm's total cost of production, TC, depends on its output, Q. The TC function may include a constant term, which represents fixed costs, FC. The part of total cost that varies with Q is called variable cost, VC. We have, then, that TC = FC + VC. Average cost per unit of output is found by dividing by Q. We can find average total cost, denoted AC, which is given by AC = TC ÷ Q, together with average variable cost AVC = VC ÷ Q and average fixed cost AFC = FC ÷ Q. The relationship between marginal cost, MC, and the other cost curves is defined in chapter 5 of this book.

Various cost curve relationships are defined on the MathEcon screen Short-Run Cost Definition.

Remember...

FC is the constant term in TC.

VC = TC − FC

AC = TC$/Q$

AVC = VC$/Q$

AFC = FC$/Q$

9

For a firm with total cost given by

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

identify its AC, FC, VC, AVC and AFC functions. List some values of TC, AC and AFC, correct to the nearest integer. Sketch the total cost function and, on a separate graph, the AC and AFC functions.

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

$$AC = TC/Q = 120/Q + 45 - Q + 0.4Q^2$$

FC = 120 (the constant term in TC)

$$VC = TC - FC = 45Q - Q^2 + 0.4Q^3$$

$$AVC = VC/Q = 45 - Q + 0.4Q^2$$

$$AFC = FC/Q = 120/Q$$

Some possible values for Q and for each of the terms in the total cost function are shown in the table. The corresponding TC, AC and AFC values are calculated and are plotted in figures 1.10 and 1.11. Notice that when $Q = 0$ the first terms in AC and in AFC involve dividing by zero. To avoid the problem of an infinite result, the smallest value of Q for which AC and AFC are calculated is 0.3.

Q	0	0.3	1	3	5	8	10	12	15
$45Q$	0	13.5	45	135	225	360	450	540	675
Q^2	0	0.09	1	9	25	64	100	144	225
$0.4Q^3$	0	0.0108	0.4	10.8	50	204.8	400	691.2	1350

Correct to the nearest integer

TC	120	133	164	257	370	621	870	1207	1920
AC		445	164	86	74	78	87	101	128
AFC		400	120	40	24	15	12	10	8

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

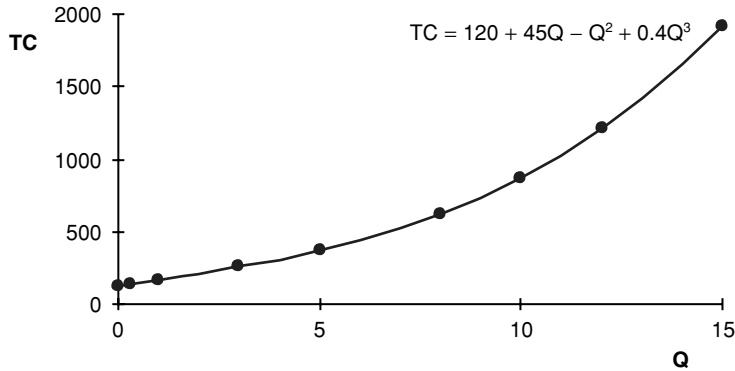**Figure 1.10**

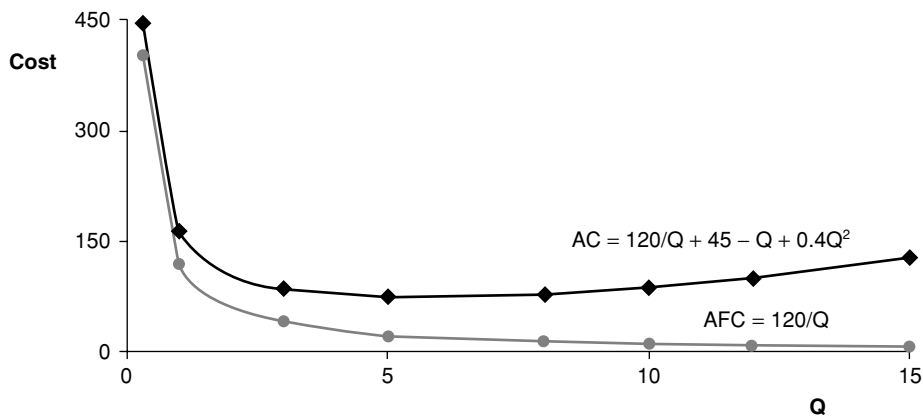$$AC = 120/Q + 45 - Q + 0.4Q^2$$

$$AFC = 120/Q$$

Figure 1.11 Average Total Cost and Average Fixed Cost

In figure 1.11 notice that the average total cost curve at first falls as output rises, but later the curve rises again. By contrast, average fixed cost is always declining as output increases.

10

Electricity users pay a $15 standing charge each quarter plus $0.10 for each unit of electricity used. Draw a graph showing the total cost per quarter, y, for various possible amounts of electricity used, x. Write an expression for y in terms of x. Write also an expression for the average cost per unit used. How would you describe average cost if only a very small number of units of electricity are used?

If x units of electricity are used the cost for these units is $0.1x$. To this amount we must add the standing charge of $15, so the total cost in dollars is $0.1x + 15$. Some possible values of x and the corresponding y values are shown in the table and the relationship is plotted in figure 1.12. The function can be written as

$$y = 0.1x + 15$$

$$AC = y/x = 0.1 + 15/x$$

If only a very small number of units are used the average cost is very high, because the standing charge of $15 is shared over only the very small number of units.

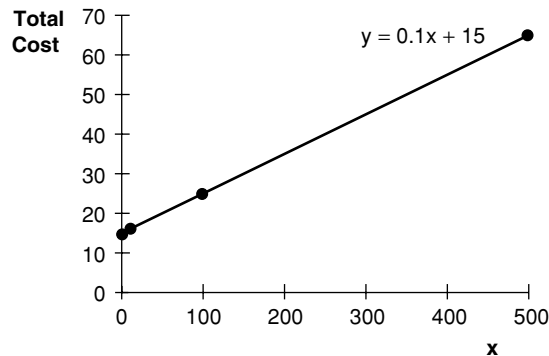| x	0	1	10	100	500
y	15	15.1	16	25	65

Figure 1.12

1.17 Sketch the average cost curve

$$AC = 9 - Q + 0.5Q^2$$

1.18 A photocopier costs $180 per month to rent, plus $0.05 for each copy produced. Draw a graph showing the total monthly cost, y, for the number of copies made, x (x from 0 to 10,000). Write an expression for total cost in terms of x.

1.19 Sketch the total cost function

$$TC = 300 + 40Q - 10Q^2 + Q^3$$

and write expressions for AC, FC, VC and AVC.

Section 1.11: Profit

Profit is the excess of a firm's total revenue, TR, over its total cost, TC, and so we calculate it by subtracting TC from TR. Using the symbol π as the variable name for profit, we write

$$\pi = TR - TC$$

Note that π is used here simply as a convenient variable name. There is no connection with the constant π used, for example, in measuring the area of a circle. Maximization of profits is usually assumed to be the objective of a firm. In chapter 6 we find how to identify the output at which this is achieved.

You can see examples of profit functions in MathEcon. The screen titled Profit Maximization is for a firm with a horizontal demand curve, while the Profit Maximization by a Monopolist screen shows a firm with a downward sloping demand curve.

$$\text{Profit} = \pi = TR - TC$$

Remember...

11

A firm has the total cost function

$$TC = 120 + 45Q - Q^2 + 0.4Q^3$$

and faces a demand curve given by

$$P = 240 - 20Q$$

What is its profit function?

$$TR = P \cdot Q = 240Q - 20Q^2$$

$$\pi = TR - TC$$

Since TC comprises several terms we enclose it in brackets as we substitute

$$= 240Q - 20Q^2 - (120 + 45Q - Q^2 + 0.4Q^3)$$

Taking the minus sign through the brackets and applying it to each term in turn gives

$$= 240Q - 20Q^2 - 120 - 45Q + Q^2 - 0.4Q^3$$

and collecting like terms we find

$$\pi = -120 + 195Q - 19Q^2 - 0.4Q^3$$

> **1.20**   If the firm in practice problem 1.19 faces the demand curve
>
> $$P = 100 - 0.5Q$$
>
> find an expression for the firm's profit function and sketch the curve.

Section 1.12: Production Functions, Isoquants and the Average Product of Labour

The long-run production function shows that a firm's output, Q, depends on the amounts of factors it employs (always assuming that whatever factors are employed are used efficiently). If a production process involves the use of labour, L, and capital, K, we write $Q = f(L, K)$. This is an example of a multivariate function. The dependent variable, Q, is a function of two independent variables, L and K.

One way of representing this relationship on a two-dimensional graph is to use the vertical axis for the dependent variable, Q, and to choose one of the independent variables to plot on the horizontal axis. This implies we temporarily fix the other independent variable at some particular value. If we choose to fix the value of K and plot L on the horizontal axis we are in fact plotting a short-run production function corresponding to the selected value of K. If K changes we obtain a new curve. The long-run production function may be represented by a series of short-run curves each corresponding to a particular level of K, as shown later in figure 1.13.

An alternative graphical approach shown in figure 1.14 is to use one axis for each of the independent variables, L and K. We then find various combinations of these inputs which give the same values of Q and connect them with a curve. This curve, all points on which have equal output, is called an isoquant. Other isoquants can be found in a similar way so that an isoquant map is obtained.

The average product of labour is used as a measure of labour productivity. Defined as APL $= Q \div L$, it is plotted on the vertical axis against L on the horizontal axis, as shown in figure 1.15.

> You can see an example of plotting production functions on the MathEcon screen 1.3 Short-Run Production Functions: A Numerical Example. If you would like to try the method in Excel, the worksheet used to calculate the values used in figure 1.13 is available from the Production Function tab in the Function.xls file for you to explore.

A production function shows the quantity of output obtained from specific quantities of inputs, assuming they are used efficiently.

Remember...

- In the short run the quantity of capital is fixed.
- In the long run both labour and capital are variable.
- Plot Q on the vertical axis against L on the horizontal axis for a short-run production function.
- Plot K against L and connect points that generate equal output for an isoquant map.
- Average Product of Labour (APL) = $Q \div L$

An isoquant connects points at which the same quantity of output is produced using different combinations of inputs.

12

A firm has the production function $Q = 25(L \cdot K)^2 - 0.4(L \cdot K)^3$. If $K = 1$, find the values of Q for $L = 2, 3, 4, 6, 12, 14$ and 16. Sketch this short-run production function putting L and Q on the axes of your graph. Next suppose the value of K is increased to 2. On the same graph sketch the new short-run production function for the same values of L. Add one further production function to your sketch, corresponding to $K = 3$, using the same L values again.

Now sketch another representation of this production function as an isoquant map. Plot L and K on the axes and look for combinations of L and K amongst the values you have calculated which give the same value of Q. Such points lie on the same isoquant.

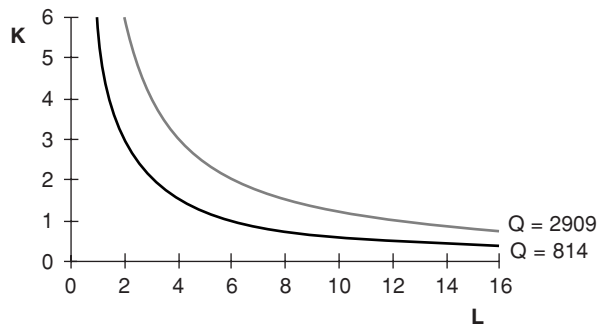For the short-run production function with $K = 3$, find and plot the average product of labour function.

The table lists values of L and K and shows the values of Q obtained by substituting each pair of values into the production function $Q = 25(L \cdot K)^2 - 0.4(L \cdot K)^3$. The Q values listed in the row for which $K = 1$ are plotted to give the first short-run production function shown in figure 1.13, and the other production functions are obtained by plotting each of the other rows.

Three shaded cells in the table each contain the Q value 814 (rounded up to the nearest integer) and the other three shaded cells each contain a nearest integer value of 2909. Plotting the pairs of L and K values for which $Q = 814$ and joining them up gives the first isoquant mapped in figure 1.14, and the second isoquant is obtained from the pairs of L and K for which $Q = 2909$.

K \ L	2	3	4	6	12	14	16
1	96.8	214.2	374.4	813.6	2908.8	3802.4	4761.6
2	374.4	813.6	1395.2	2908.8	8870.4	10,819.2	12,492.8
3	813.6	1733.4	2908.8	5767.2	13,737.6	14,464.8	13,363.2
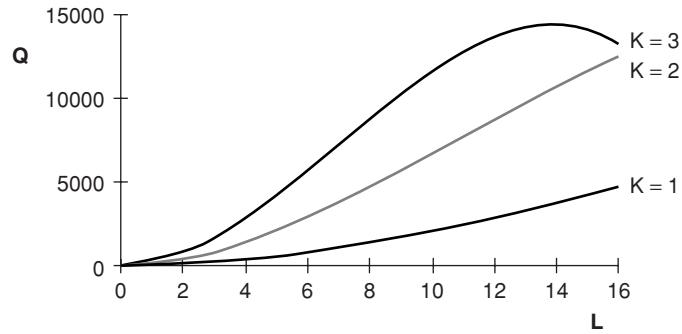
Figure 1.13 Production function for different values of K

Figure 1.14 Isoquant map

For $K = 3$ we have $Q = 25(3L)^2 - 0.4(3L)^3 = 225L^2 - 10.8L^3$.

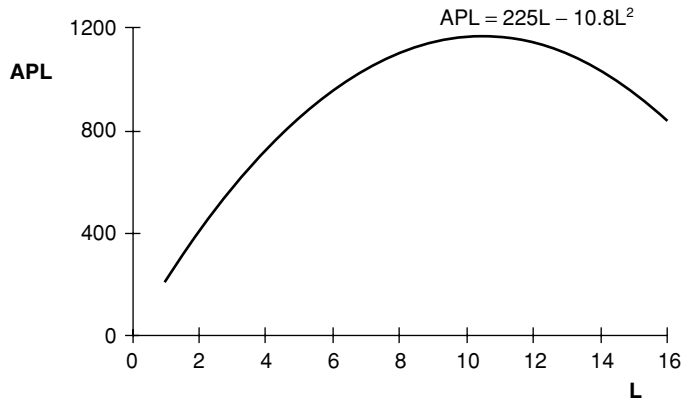$$\text{APL} = Q/L = 225L - 10.8L^2$$

This is plotted in figure 1.15.

$$APL = 225L - 10.8L^2$$

Figure 1.15 The average product of labour function

1.21 A firm has the production function $Q = L^{\frac{1}{2}} \cdot K^{\frac{1}{5}}$. Sketch its short-run production function if $K = 1$, using values for L of 1, 4, 16, 64, 144. If instead $K = 32$, what is the value of Q when $L = 4$? What point on the production function for $K = 1$ gives the same output? Using another graph, sketch the isoquant that passes through these two points. Also find and plot the average product of labour when $K = 1$.

SECTION 1.13: Functions in Excel

You can use Excel to plot graphs and do calculations for you. While getting Excel to plot functions for you is not a substitute for being able to plot them by hand, it is a quick way of producing smart looking graphs, and it does enable you to easily compare the shapes of curves that correspond to different functions. This section sets out some suggestions about how to do this. Often there is more than one way of doing something in Excel. The methods described here may therefore not be the only way of obtaining a particular result, but it is hoped they may provide useful guidelines from which you may develop your own approach. The worksheets described below are included in the file entitled Function.xls on the CD. You may find it helpful to load them for reference as you are creating your own worksheets. For help on a particular topic, use the Help provided in Excel and search for the topic you require.

The worksheet called Power function included in Function.xls and accessed by the tab near the bottom of the screen is designed to let you explore how the shape of a function changes if you change the coefficients of its variables. It is displayed in figure
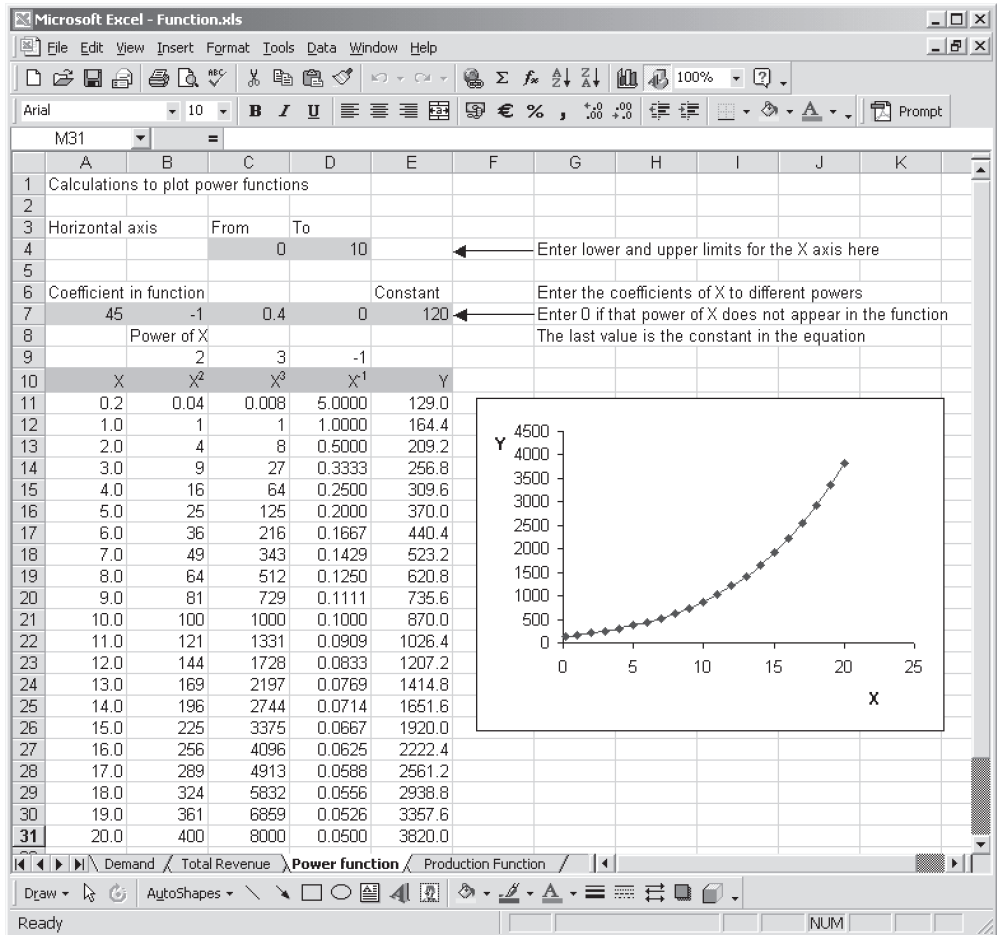
The table shown in the Excel worksheet:

	A	B	C	D	E	F
1	Calculations to plot power functions					
2						
3	Horizontal axis		From	To		
4			0	10		Enter lower and upper limits for the X axis here
5						
6	Coefficient in function				Constant	Enter the coefficients of X to different powers
7	45	-1	0.4	0	120	Enter 0 if that power of X does not appear in the function
8		Power of X				The last value is the constant in the equation
9		2	3	-1		
10	X	X^2	X^3	X^{-1}	Y	
11	0.2	0.04	0.008	5.0000	129.0	
12	1.0	1	1	1.0000	164.4	
13	2.0	4	8	0.5000	209.2	
14	3.0	9	27	0.3333	256.8	
15	4.0	16	64	0.2500	309.6	
16	5.0	25	125	0.2000	370.0	
17	6.0	36	216	0.1667	440.4	
18	7.0	49	343	0.1429	523.2	
19	8.0	64	512	0.1250	620.8	
20	9.0	81	729	0.1111	735.6	
21	10.0	100	1000	0.1000	870.0	
22	11.0	121	1331	0.0909	1026.4	
23	12.0	144	1728	0.0833	1207.2	
24	13.0	169	2197	0.0769	1414.8	
25	14.0	196	2744	0.0714	1651.6	
26	15.0	225	3375	0.0667	1920.0	
27	16.0	256	4096	0.0625	2222.4	
28	17.0	289	4913	0.0588	2561.2	
29	18.0	324	5832	0.0556	2938.8	
30	19.0	361	6859	0.0526	3357.6	
31	20.0	400	8000	0.0500	3820.0	

Figure 1.16   Power function worksheet

1.16, set up to show the graph of the Total Cost function plotted earlier in figure 1.10. It is ready for you to use to plot any power function that includes terms in X, X^2, X^3, X^{-1} and a constant.

You enter the appropriate coefficient for each term in the highlighted boxes at the top of each column. If there is a particular term, say X^{-1}, that does not appear in the function you wish to plot, you simply enter a 0 value for its coefficient. Try changing some of the coefficients to see how the graph changes.

You may also like to change the range on the horizontal axis for which values are displayed. You can enter the approximate range of values you want to plot on the horizontal axis in highlighted cells near the top of the screen. A formula then calculates values to plot on the horizontal axis, choosing 20 integer values in or slightly beyond the range you select. If you choose a scale that involves plotting a value for $Q = 0$, the formula chooses to use 0.2 instead to avoid the possible problem of dividing by 0. You

can use this worksheet to display the graph for any continuous power function involving powers from −1 to 3. A graph that includes a term in X^{-1} has a discontinuity at $x = 0$. Because of this you should only plot such a graph either for a positive or a negative range of x values.

PLOTTING FUNCTIONS IN EXCEL

To plot a function in Excel you need two columns of values, the left-hand one containing the values to plot on the horizontal axis and the right-hand one comprising the values of the function to be plotted on the vertical axis. You can then use the Chart Wizard to plot the graph. You choose values for the left-hand column in just the same way as you would mark out a scale on the axis if you were drawing the graph by hand, but you can get Excel to fill in the values for you. To obtain the right-hand column of values you enter a formula that calculates the first value of the function and copy it down the column. We shall plot a linear function, the demand curve $AR = 72 - 3Q$ shown in figure 1.8 and available from the demand tab of Function.xls.

THE FILL HANDLE

You could plot a linear function using just two points, but it can be useful to find the values at different points on the line and so we use a more general approach. For the horizontal axis you need to choose a set of values that are not too far apart and in an appropriate range. Just type in the first two or three of them (0, 2, 4 in figure 1.17)
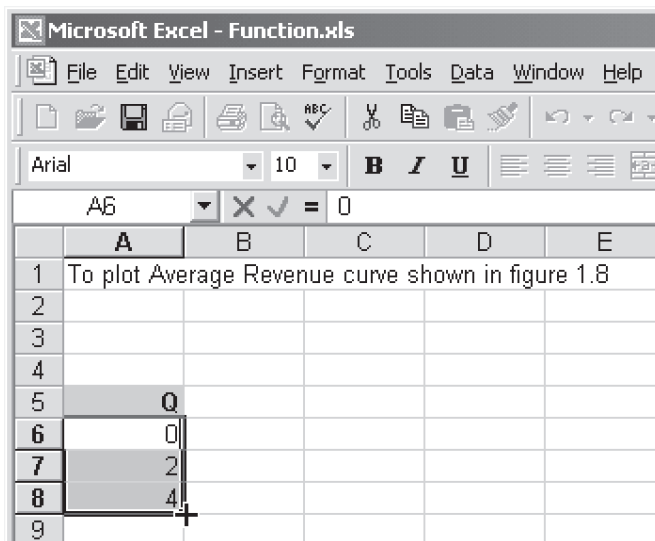
Figure 1.17 The Fill Handle

and then fill in as many values as you want using the Fill Handle. To do this, first select the values you have entered with your mouse and release the mouse button. Next, move your mouse gently at the bottom right-hand corner of the selected cells until you see the thin black cross as shown in figure 1.17. This is the Fill Handle. Click on it and drag down the column to fill in more values of the sequence. In figure 1.18 the values have been filled in steps of 2 to 24, forming a column of values of Q.

EXCEL FORMULAE

For each of the values of Q we need to calculate the corresponding value of AR = $72 - 3Q$, and to do this we need to enter a formula. Although you can type numbers into Excel formulae you make much better use of Excel's capabilities if you enter values into cells where they are displayed and use cell references in the formulae you enter. A cell reference, such as A6, tells Excel to use in its calculation the value that it finds in the cell identified by the column letter and row number. One advantage of this approach is that you can easily do 'what if' analysis. Changing the value in a cell automatically recalculates all the values calculated from it and so lets you see immediately the effect of the change. Appropriate use of cell references also means that a
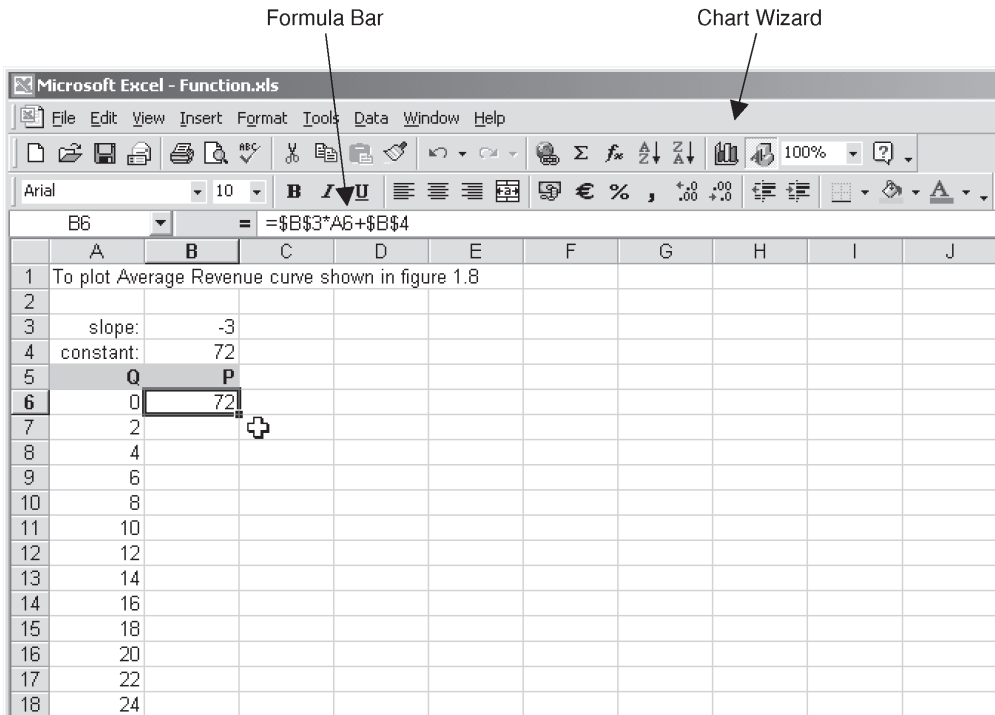
Figure 1.18 The formula entered in B6 displays in the Formula Bar

formula that you enter in one cell can be copied to other cells to complete, say, a column of calculations.

Enter the constant and slope of the line in appropriate cells (B3 and B4 in figure 1.18). Now select the cell where the first value is to be calculated, B6 in figure 1.18. We need a formula to pick up the slope value of −3 from B3, multiply it by the appropriate value of Q, which is the value to the left of the cell where we are entering the formula, and add to this the constant of 72 from B4. So that it will copy, the formula we use is =$B$3*A6+$B$4 which contains both absolute and relative cell addresses. An absolute cell address is indicated by dollar signs in front of the column letter and row number, in this case $B$3. Such an address will not change as the formula is copied, so the calculation will always use the value in the specified cell. A cell address without dollar signs, for example A6, is called a relative address. This means that the calculation will use the value in the cell that is in a particular position with reference to the current cell, here, one cell to its left. This form of address is used to pick up the value of Q so that when we copy the formula down the column a different value of Q is used in each row. Figure 1.18 shows the worksheet once the formula has been entered. Notice that when you select the cell the formula displays in the Formula Bar near the top of the screen.

Formulae in Excel:

Remember...

- Are entered in the cell where you want the result to be displayed
- Start with an equals sign
- Must not contain spaces

Operators:

- Are ( ) brackets, + add, − subtract, * multiply, / divide, ^ raise to the power of
- Relative cell addresses (e.g. A6) change as the formula is copied
- Absolute cell addresses (e.g. $B$3) remain fixed as the formula is copied

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As you type a formula you can put a cell address into it by clicking on the appropriate cell. This puts the relative version of the address in the formula. To make it absolute, after clicking on the cell you should press function key F4 so that, for example, B3 changes to $B$3. Function key F4 works as a toggle key. If you press it two or three times you get addresses that are part absolute and part relative (two different versions) and if you press it a fourth time you return to a relative address.

To complete the column of values of P you copy the formula in B6 down the column. You can use the usual Edit, Copy, Edit, Paste or you can again make use of the Fill Handle and drag the formula down the column. The two columns of values of Q and P can be seen in figure 1.19.
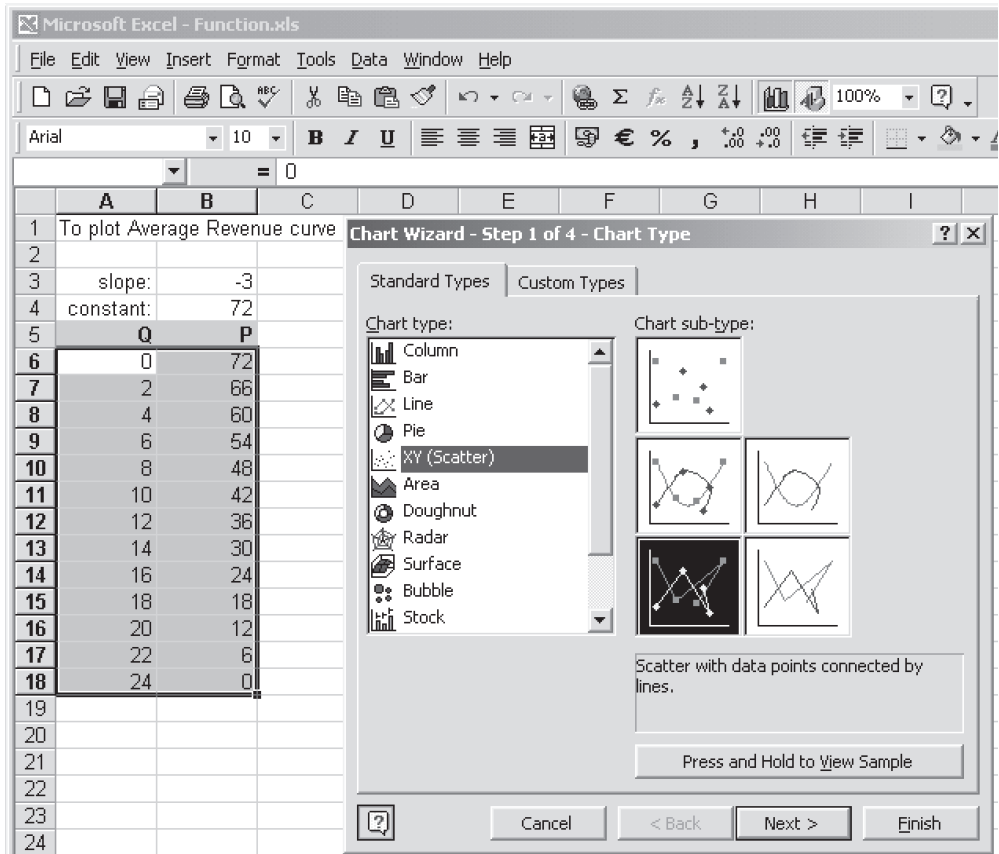
Figure 1.19   Using Chart Wizard XY scatter selected

USING CHART WIZARD

To plot a graph in Excel, you start by selecting the two columns of values that are to be plotted. The left-hand one goes on the horizontal axis and the right-hand one on the vertical axis. Click the Chart Wizard button. You must select the XY (Scatter) type of chart to plot values on the X axis as shown in figure 1.19. All the other types of chart available in Excel simply plot labels on the horizontal axis. Of the subtypes of chart offered, you can choose to connect the points with lines since you are plotting a line (if you were plotting a curve you would want to connect the points with curves) and you may choose to have the data points plotted as well if you wish. Clicking Next gives you a preview of your graph and confirms that Excel has correctly interpreted that the data form two columns (rather than rows). Clicking Next again brings you to the chart options. You can enter a title and axis titles. You can also de-select Major gridlines, and Show legend, since we are only plotting one data series. You can then

click Finish and the chart is placed in your worksheet. If you don't want the plot area shaded, right click the mouse over it and choose 'Format Plot Area'. Under 'Patterns Border' select 'None', then to the right under 'Area' select 'None' and click OK. This removes the shading. To alter the direction in which the vertical axis title is plotted, right click over it and choose 'Format Axis Title'. Choose the Alignment tab, move the text orientation pointer until it is horizontal and click OK. You can move the axis titles by clicking on them and dragging. Also, if you click on a blank area of the chart to select the whole chart you can position it in the worksheet as you wish. Once the chart is finished, you can move your mouse cursor over it to display the coordinates of each point, as shown in figure 1.20.

Now that you have a graph of a linear demand curve, it is easy to see how it changes if either the slope or the constant of the line changes. Try typing different values in cells B3 and B4 and watch the effects. Of course, if you type a positive value for the slope you will get an upward sloping line, which may represent supply rather than demand, so you would have to change the title of the graph!
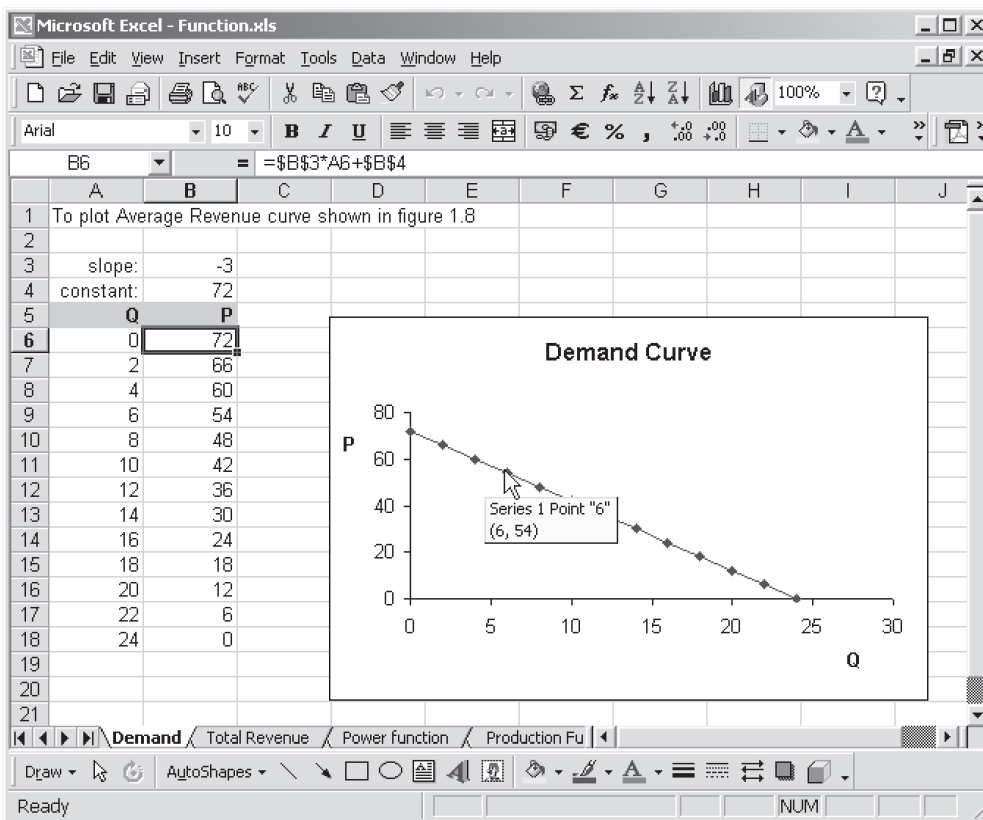
**Figure 1.20**   The Average Revenue chart, showing a point's coordinates
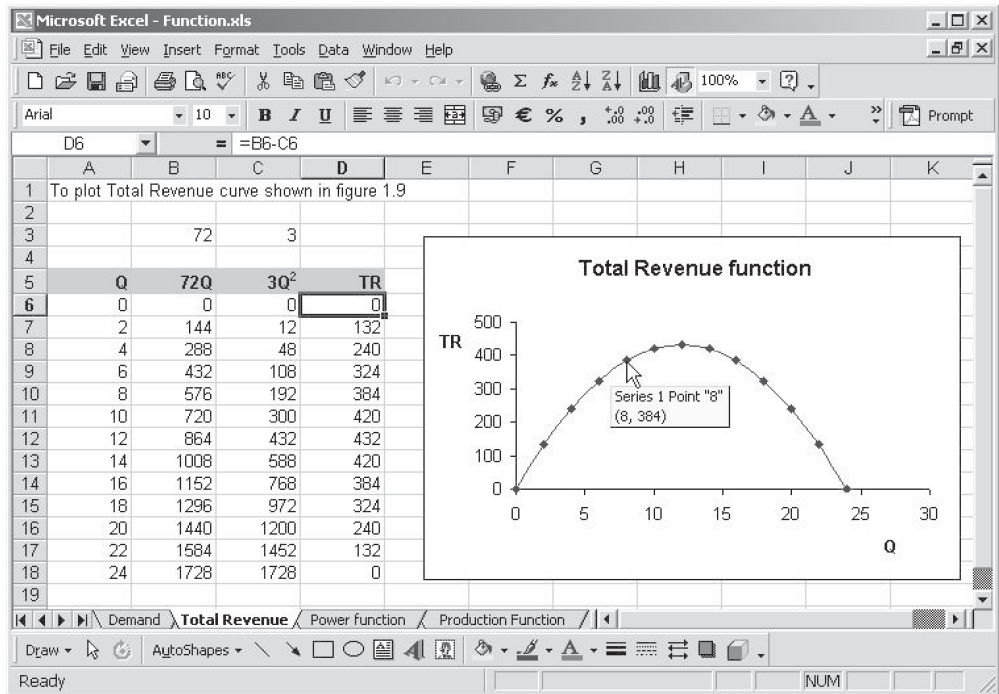
Figure 1.21 The Total Revenue chart, showing a point's coordinates

You may now like to try plotting other economics functions for yourself. There are two more examples for you in the Function.xls workbook so you can see how different formulae are used. Access the different worksheets using the tabs near the bottom of the screen. The total revenue function $TR = 72Q - 3Q^2$ is shown in figure 1.21. Total revenue has been found in stages, calculating each of its terms separately and finding the value of TR in column D. This lets you see exactly what you are calculating and may help you to avoid mistakes. The graph plots the values in column D against those in column A. To select the data in these columns, select those in column A then press and hold Control while selecting the values in column D. Once both columns are selected, release the Control button and you are ready to click the Chart Wizard button as before.

Chapter 1: Answers to Practice Problems

1.1
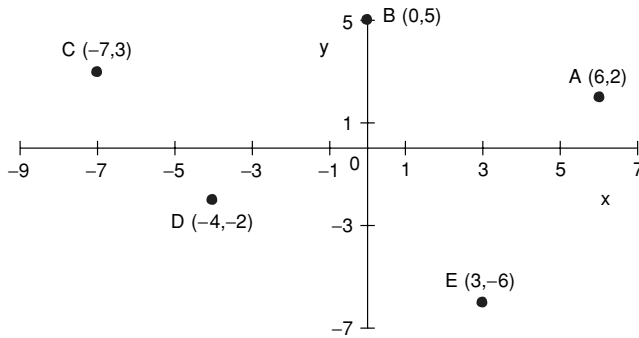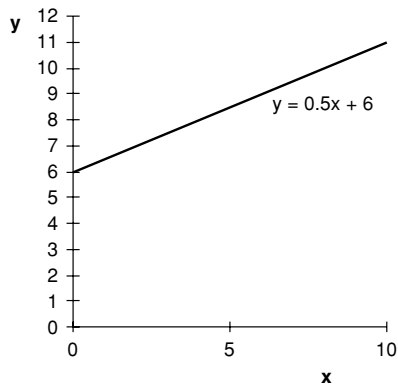

Figure 1.22

1.2 (a) Linear function, proportional relationship between x and y. Whatever value x takes, y is half of it. Choose two or more values of x, e.g. $x = 0$, $y = 0$; $x = 10$, $y = 5$.
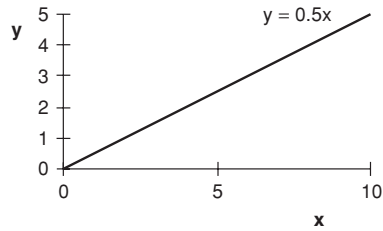
Figure 1.23

(b) Linear, non-proportional relationship. Plotting on the same scale as figure 1.23, the lines are parallel but this one is shifted up by the amount of the constant and cuts the y axis at 6.

Figure 1.24

(c) Quadratic function. Choose several values of x, say

x	0	1	2	3	4	5	6	7	8	9	10
y	0	1	4	9	16	25	36	49	64	81	100
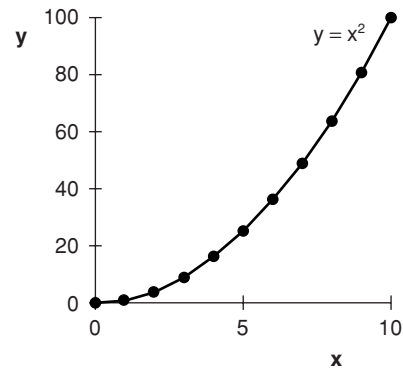
Figure 1.25

(d) This quadratic function also passes through the origin.
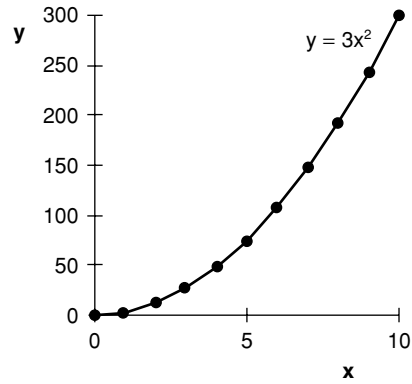
Figure 1.26

1.3   45° angle
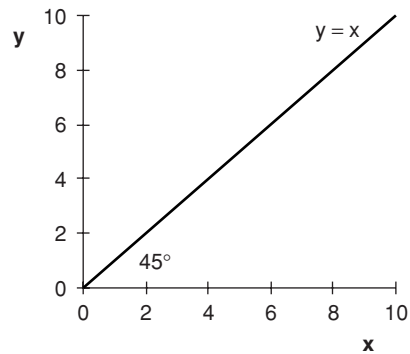
Figure 1.27

1.4 (a) 3, −15   (b) 32, 32

     (c) −2   (d) 10

     (e) −2.5   (f) 17

     (g) 24   (h) 31.8

     (i) −36   (j) −6

     (k) +5 or −5 (does your calculator tell you this?)

1.5 $66.67

1.6 (a) $-5x(2 + 9x)$ (b) $13(11x − 4)$

     (c) $5x(x + 1 − 4y)$

1.7 (a) (i) $80x − 56x^2$

        (ii) $-6x^2 + 3x + 45$

        (iii) $132 − 56x + 4x^2$

        (iv) $24 + 0.8x − 540 = 0.8x − 516$

        (v) $2000 − 15x$

    (b) (i) $82x − 1.64x^2 − 100$

        (ii) $100/x + 150 − 36x$

        (iii) $0.2x$

        (iv) x

    (c) $-150 + 200x − 2x^2 − 0.5x^3$

    (d) $30 + 18(15) − 0.6(15^2) = 165$

    (e) (i) $(a + b)(a + b) = a^2 + 2ab + b^2$

        (ii) $(a − b)(a − b) = a^2 − 2ab + b^2$

        (iii) $(a + b)(a − b) = a^2 − ab + ab − b^2 = a^2 − b^2$

1.8 (a) (i)   Yes, product gives $x^2 + 5x − 7x − 35$

        (ii)   Yes, product gives $3x^2 + 24x + x + 8$

        (iii) Yes, product gives $2x^2 − 2x + 9x − 9$

    (b) (i) $(x + 3)(x + 7)$

        (ii) $(x + 5)(3x − 1)$

        (iii) $(x − 1)(2x + 10) = 2(x − 1)(x + 5)$

        (iv) $(3x + 11)(x + 5)$

1.9   (a) $\frac{1}{4}$ (b) 4 (c) $\frac{5}{42}$ (d) $\frac{23}{30}$ (e) $\frac{4}{9}$ (f) 1 (g) $\frac{19}{36}$

1.10 Ratio of marginal utilities = $MU_x/MU_y = \frac{5}{9}$, ratio of prices = $P_x/P_y = \frac{30}{45} = \frac{6}{9}$, so the ratios are not equal. (Economic theory shows this indicates you could reallocate your spending on the two goods to increase your total utility.)

1.11 (a) $3x/4$

      (b) $15M \times P \times P/(P^2 \times 15M) = 1$

      (c) $x(5x - 3)/x(1 + 3x) = (5x - 3)/(1 + 3x)$

      (d) $4x \times 2/4x(3x - 4) = 2/(3x - 4)$

      (e) $[7(2x - 1) + 5x(x + 2)]/(x + 2)(2x - 1) = (14x - 7 + 5x^2 + 10x)/(x + 2)(2x - 1)$

          $= (5x^2 + 24x - 7)/(x + 2)(2x - 1)$

1.12 (a) x^{15} (b) $x^{7/12}$

      (c) x^5 (d) x^5

      (e) $x^{1/2} = \sqrt{x}$ (f) x^8

      (g) $16x^4$ (h) 1

      (i) $12x^{1/4}y^{-1/2}(x^{3/4}y^{-1/2}) = 12xy^{-1} = 12x/y$

      (j) $x^{a-1} \cdot x^{-a} = x^{-1} = 1/x$

1.13 $1^t/(1 + r)^t = 1/(1 + r)^t = (1 + r)^{-t}$

1.14 (a) 25   (b) 32

      (c) $1/81 = 0.012$ (d) 1

      (e) 8 or −8   (f) 3

      (g) 6561   (h) 1

      (i) 163   (j) 49

      (k) $(81 + 40)^{1/2} = 11$ or −11

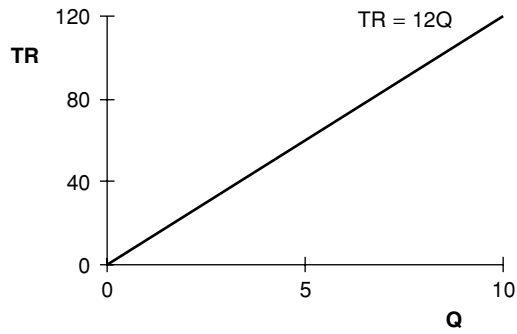1.15 Price constant, TR a line through the origin. At $Q = 10$, TR = 120.

Figure 1.28

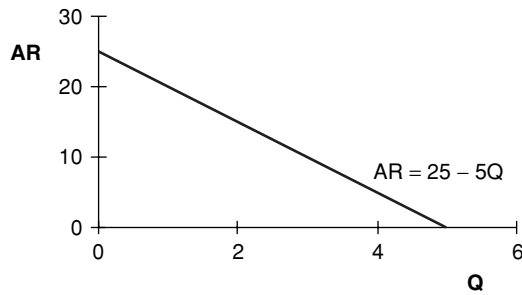1.16 The term in Q is negative, so as Q increases AR decreases.

Q	0	5	6
AR	25	0	−5

Figure 1.29

1.17

Q	0	1	2	3	4	5	6	7	8	9	10	11	12
AC	9	8.5	9	10.5	13	16.5	21	26.5	33	40.5	49	58.5	69
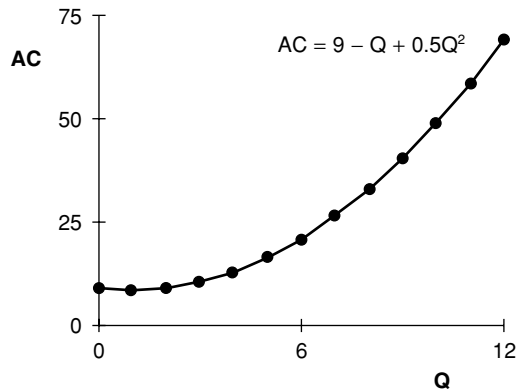
Figure 1.30

1.18 $y = 180 + 0.05x$

x	0	1000	5000	10,000
y	180	230	430	680
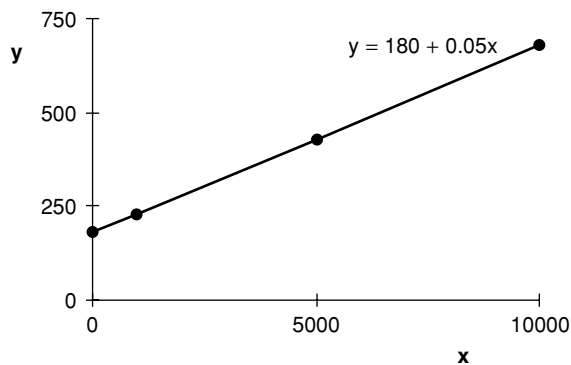
Figure 1.31

1.19

Q	0	1	2	3	4	5	6	7	8	9	10	11	12	13
TC	300	331	348	357	364	375	396	433	492	579	700	861	1068	1327
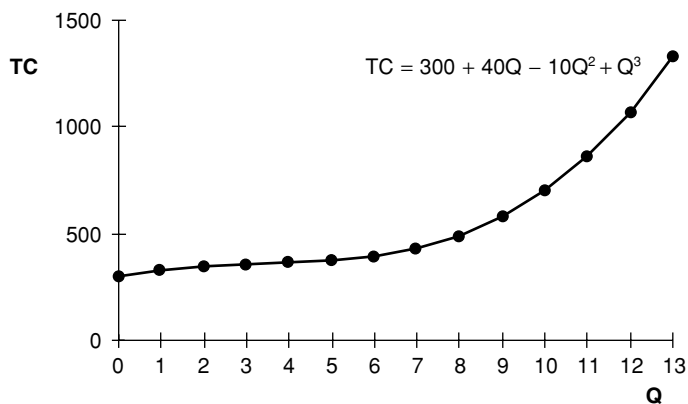
Figure 1.32

$AC = 300/Q + 40 - 10Q + Q^2$

$FC = 300$

$VC = 40Q - 10Q^2 + Q^3$

$AVC = 40 - 10Q + Q^2$

1.20 $\quad$ TR $= P \cdot Q = 100Q - 0.5Q^2$

$\quad$ Profit $=$ TR $-$ TC $= 100Q - 0.5Q^2 - (300 + 40Q - 10Q^2 + Q^3)$

$\qquad = -300 + 60Q + 9.5Q^2 - Q^3$
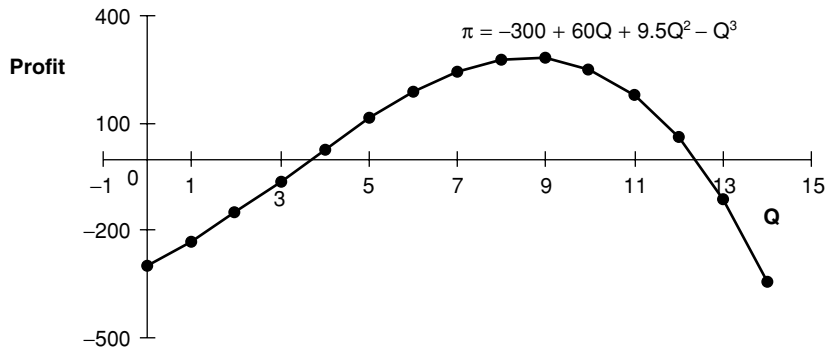
Figure 1.33

1.21 $\quad$ Given $K = 1$ we have

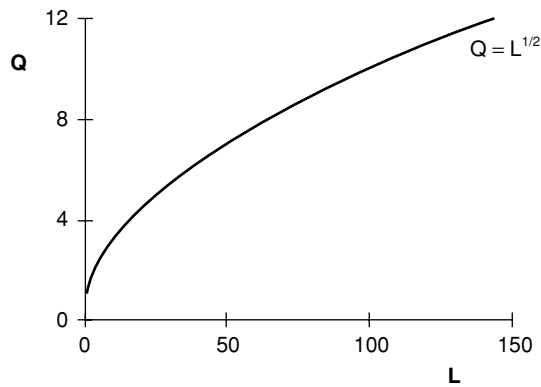$$\begin{array}{llllll} L & 1 & 4 & 16 & 64 & 144 \\ Q & 1 & 2 & 4 & 8 & 12 \end{array}$$

Figure 1.34

If $K = 32$ and $L = 4$, $Q = 4$. This point and the point $K = 1$, $L = 16$ form the basis for the isoquant shown in figure 1.35.

Figure 1.35   Isoquant
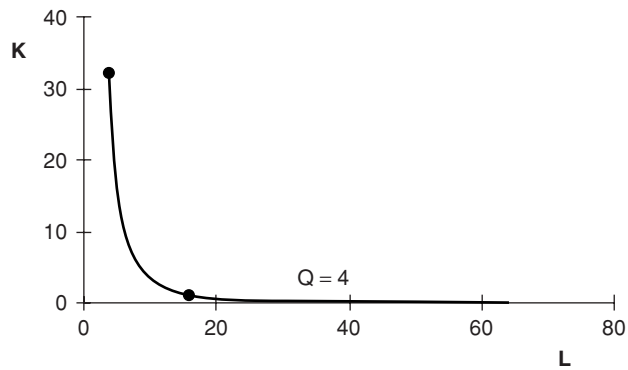
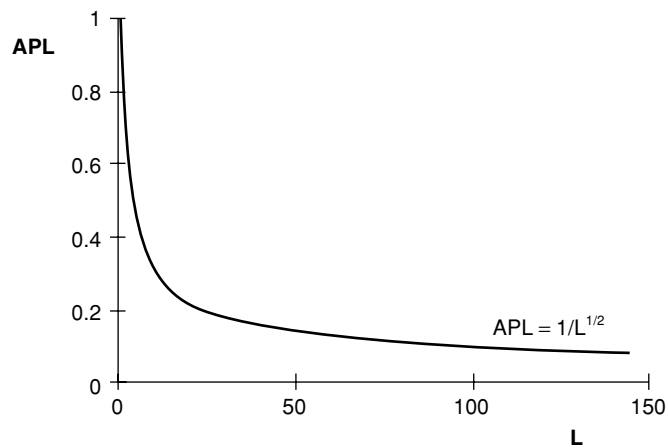when $K = 1$, APL is as shown in figure 1.36.

Figure 1.36   Average product of labour