Efficiencies of rounded optimal approximate designs for small samples

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Optimal exact designs are notoriously hard to study and only a few of them are known for polynomial models. Using recently obtained optimal exact designs (IMHOF, 1997), we show that the efficiency of the frequently used rounded optimal approximate designs can be sensitive if the sample size is small. For some criteria, the efficiency of the rounded optimal approximate design can vary by as much as 25% when the sample size is changed by one unit. The paper also discusses lower efficiency bounds and shows that they are sometimes the best possible bounds for the rounded optimal approximate designs.

Key Words and Phrases: lower efficiency bounds, equivalence theorems, exact designs, $G$-, $A$-, $I$- and $MV$-optimal designs, quadratic models, rounding-off procedure.

1 Introduction

The importance of choosing efficient designs in a study cannot be over-emphasized because poor designs, in the extreme case, cannot provide answers to the research questions. The rising cost of experimentation further requires that design issues be carefully considered at the onset of the study. Optimal design strategies are increasingly employed to provide guidance for practitioners. They can be used, for example, as benchmarks for comparing designs of interest.

A common design technique is to treat designs as approximate designs. This means that designs are viewed as probability measures. For example, in the context of a dose-response study, this means a design consists of a set of dose levels and the assignment of a proportion of the fixed number of subjects to each dose level.
detailed description of this technique can be found in design monographs, for example, Silvey (1980), Pázman (1986) and Fedorov (1972). Some recent papers using this approach to design experiments include Atkinson, Chaloner, Herzberg and Juritz (1993), Atkinson and Haines (1996), Atkinson and Bogacka (1997), Spears, Brown and Atkinson (1997), Huang and Wong (1998a, b), Zhu, Ahn and Wong (1998) and Dette and Wong (1999). The design problems considered in these papers range from finding optimal approximate designs to estimate the rate and order of chemical reaction in esterification, estimating several effective dosage levels simultaneously in quantal dose-response studies and optimum return from fertilizer dressing to optimal sampling times in pharmacokinetic experiments.

Kiefer (1959) was the first to advocate optimal approximate designs when the experiments have large sample sizes. He found optimal approximate designs much easier to study than exact designs. Both types of optimal designs require the optimum choice of the design points to be determined. The main difference between the two approaches is that exact designs require the number of observations to be taken at each of the design points to be specified and approximate designs require only the proportion of observations to be taken at the design points to be specified. In general, exact design problems are much harder to solve because they involve number-theoretic questions which are frequently mathematically intractable. In contrast, there is a general methodology for constructing optimal approximate designs using tools from convex analysis. Consequently, optimal approximate designs are easier to understand and characterize. In particular, there are computer algorithms for generating many types of optimal approximate designs (Pázman, 1986, Chapter 5) and there are simple ways of verifying whether designs are optimal. These are perhaps the main reasons for the increased use of optimal approximate designs in recent years.

When the sample size is large, the two approaches yield similar optimal designs and there is not much loss in efficiency in using the rounded optimal approximate designs. When the sample size is small, which is often the case when low degree polynomial models are used, then large sample results may be invalid and new methods are required. For instance, Storer (1993) recently proposed a method of constructing small-sample confidence sets for estimating the maximum tolerated dose (MTD) in a Phase I clinical trial.

The purpose of this paper is to show that a similar rule applies here; optimal approximate designs should be used with caution when the sample size is small. For some criteria, we demonstrate that changing the sample size by one unit can affect the efficiency of the rounded optimal approximate design by about 25%. In addition, we evaluate and improve, where possible, efficiency bounds described in Pukelsheim and Rieder (1992) and Pukelsheim (1993).

We focus on polynomial models in one variable of the form

\[ E[y(x)] = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k, \quad \text{var}[y(x)] = \sigma^2, \quad x \in X. \]  

(1)

Here \( y(x) \) is the univariate response at the \( x \)-level of the independent variable and...
the model parameters are $a_0, a_1, \ldots, a_k$. All observations are assumed to be independent and normally distributed.

The worth of a design $\hat{\xi}$ is measured by its normalized Fisher information matrix

$$M(\hat{\xi}) = \int f(x) f^T(x) d\hat{\xi}(x),$$

where $f^T(x) = (1, x, \ldots, x^k)$. This matrix is proportional to the expected second derivative matrix of the log-likelihood and has the interpretation that its inverse is proportional to the variance-covariance matrix of the parameter estimates. Good designs for estimating model parameters require that $M(\hat{\xi})$ is large in some sense. Some popular design criteria are:

the $D$-optimality criterion:

$$\Phi_D[M(\hat{\xi})] = (\det M(\hat{\xi}))^{-1/(k+1)},$$

the $A$-optimality criterion:

$$\Phi_A[M(\hat{\xi})] = \sum_i \text{var}_{\hat{\xi}}(\hat{\alpha}_i),$$

the $I$-optimality criterion (with respect to the uniform measure):

$$\Phi_I[M(\hat{\xi})] = \int_\chi f^T(x) M^{-1}(\hat{\xi}) f(x) dx,$$

the $G$-optimality criterion:

$$\Phi_G[M(\hat{\xi})] = \max_{x \in \chi} f^T(x) M^{-1}(\hat{\xi}) f(x).$$

The $D$- and $A$-optimality criteria are used for estimating model parameters and the $I$- and $G$-optimality criteria are used for estimating the response surface, partly or entirely. These criteria are discussed further in the above references and in 

The efficiency of a design $\hat{\xi}$ is the ratio of its criterion value and the best possible value, that is

$$\text{eff}(\hat{\xi}) = \frac{\Phi(\hat{\xi}^*)}{\Phi(\hat{\xi})},$$

where $\hat{\xi}^*$ is the $\Phi$-optimal design. Clearly, $0 < \text{eff}(\hat{\xi}) \leq 1$. If, for example, one uses a design with efficiency 0.5, one has to perform twice as many experiments as if one uses the optimal design.

Section 2 is a brief review of approximate design techniques and the development of exact designs from a chronological perspective. Rounding-off methods are reviewed in Section 3 and various types of optimal designs for the quadratic model are presented. The efficiencies of the rounded optimal approximate designs are

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presented in Section 4 in two parts; the first part reports the efficiencies relative to the optimal exact designs, and the second part reports the efficiencies relative to the optimal approximate designs. The latter comparison has theoretical interest because only a very limited number of exact optimal designs are known to date. Additionally, we provide optimal lower efficiency bounds for comparing rounded optimal approximate designs with optimal approximate designs. The rounding procedures used here are closely related to the design problem of augmenting a single design point to a given exact design. One such problem is considered in Section 5. Section 6 contains a discussion. Justifications for the optimal lower efficiency bounds are provided in the appendix.

2 Optimal approximate and exact designs

A major advantage with approximate designs is that the above optimality criteria are all convex and so we can verify the optimality of approximate designs using directional derivatives. These considerations form the basis of equivalence theorems which are extremely useful in practice. For example, if there is only one independent variable in the regression model, equivalence theorems permit us to check optimality of a given design by plotting its directional derivative and examining its behavior. Algorithms for generating optimal designs and lower efficiency bounds for any design are also obtained from the equivalence theorems.

In practice, approximate designs have to be rounded off before they can be implemented. For instance, if the sample size is $n = 9$ and we have the straight line regression model, the $A$-optimal approximate design requires equal proportion of observations at both ends of the design space. Do we assign 4 and 5 observations at both ends arbitrarily? In this case, it is immaterial which end of the design space receives 4 observations. However, in general it is far from obvious how to transform an optimal approximate design into a good exact design, cf. PUKELSHEIM and RIEDER (1992).

The construction of optimal exact designs differs from that of optimal approximate designs. A major distinction is that there is no general methodology for solving exact design problems. The relatively recent date of the work in this area is a testament of the difficulty of finding optimal exact designs.

One of the most important contributions to the theory of exact designs for polynomial models is arguably the paper of SALAEVSKII (1966). He showed that the $D$-optimal exact designs for (1) are obtained by placing the observations at the support points of the known $D$-optimal approximate design (cf. PUKELSHEIM, 1993, Sect. 9.5) with frequencies as equal as possible. This result, however, was only shown to be true if the number of observations is sufficiently large, with no indication of how large is large enough. Salaevskii conjectured that the designs he gave are in fact always optimal, and it is easily seen that this is true for straight line regression. The case of quadratic regression was investigated by WYNN (1972), but there was a slight fallacy in his argument. A complete solution for the quadratic model was given by
GAFFKE and KRAFFT (1982). The general Salaevskii problem was subsequently reconsidered by GAFFKE (1987) who was able to give a bound for the critical value for the number of observations. In particular, Gaffke disproved the Salaevskii conjecture for polynomial regression of degree four. That the conjecture does hold for cubic regression was shown by CONSTANTINE, LIM and STUDDEN (1987).

KRAFFT and SCHAEFER (1992) and IMHOF (2000a) examined a linear-quadratic multivariate model and found several exact \( D \)-optimal designs. It turned out that for this model no Salaevskii type result can be established. This means that there are infinitely many sample sizes for which the \( D \)-optimal exact design cannot be obtained from the approximate designs.

In the approximate theory, the \( D \) and \( G \)-optimal designs coincide but this is not true for the exact theory. In fact, the exact \( G \)-optimal design problem appears to be nearly intractable at present. The \( G \)-optimal exact designs for straight line regression were found by JUNG (1971) and a conjecture for the quadratic case is given in GAFFKE and KRAFFT (1982) and in CONSTANTINE et al. (1987). The conjecture was later proved by IMHOF (1997). For regression models of degree greater than 2 no \( G \)-optimal exact designs are known, except, of course, for those cases where the \( G \)-optimal approximate designs can be directly realized as exact designs.

The \( A \)-optimal exact \( n \)-observation designs for quadratic regression on \([-1, 1]\) for \( n = 4p + q, q = -1, 0, 1, \) were found by CHANG and YEH (1998). They also gave a conjecture for the case \( n = 4p + 2, \) which was proved (for \( p > 3 \)) by IMHOF (1998). Elfving minimax designs and \( I \)-optimal designs for quadratic regression are described in KRAFFT and SCHAEFER (1995) and IMHOF (2000b), respectively.

The \( A \), \( G \)- and \( I \)-optimal exact designs for the quadratic model on the interval \([-1, 1]\) are displayed in Table 1. The value of \( z \) in the table is the root of a polynomial, which varies from criterion to criterion. For \( A \)-optimality, the polynomial is \( t^4 - 2nt^3 + 6t^2 + 4(2/n - n)t + 1 \) \( (n = 4p + 2, p > 3) \); for \( G \)-optimality, the polynomial is \( (n - 6)t^4 + (q - 9p)t^2 + 2pq \) \( (n = 3p + q, q = 1, 2 \) and \( n = 4 \) or \( n > 13) \) and for \( I \)-optimality, the polynomial is \( 5t^4 - 7nt^3 + 12t^2 - (5n - 16/n)t - 1 \) \( (n = 4p + 2, p > 4) \). The exceptions are cases when \( n = 5, 7, 8, 10, 11 \) or 13 for \( G \)-optimality, \( n = 6, 10 \) or 14 for \( A \)-optimality, and \( n = 5, 6, 10, 14 \) and 18 for \( I \)-optimality. The exact design problems for these situations remain unsolved.

Apart from the \( D \)-optimal designs, optimal exact designs for cubic or higher degree polynomial models remain elusive and are available only for very few isolated cases. The degree of difficulty of finding optimal exact designs can be seen by considering the exemplary problem of finding the \( I \)-optimal exact \( (k + 1) \)-observation designs for model (1) with \( X = [-1, 1] \). This problem is equivalent to the problem of finding interpolation nodes \( x_0, \ldots, x_k \in X \) which minimize

\[
\int_{-1}^{1} \hat{P}_0(x) + \cdots + \hat{P}_k(x) \, dx,
\]

where \( l_0, \ldots, l_k \) are the Lagrange interpolation polynomials associated with

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This second problem, however, is a long-standing and as yet unsolved problem in the theory of algebraic interpolation, see e.g. Erdös (1995).

It was shown only recently by Erdös et al. (1995) that the zeros of \((1 - x^2)P_k(x)\), where \(P_k\) denotes the Legendre polynomial of degree \(k\), provide a very good but not optimal choice. Translating their result in the language of experimental design, one may say that the \(G\)-optimal exact \((k + 1)\)-observation designs, which are based on the zeros of \((1 - x^2)P_k(x)\), are very good but not \(I\)-optimal. Currently, only the \(I\)-optimal exact design for the cubic model with \(n = 4\) is known (Imhof, 1997).

### 3 Optimal designs and rounding methods

The rounded optimal approximate designs are found using rounding methods discussed in Kiefer (1971), Fedorov (1972) and Pukelsheim and Rieder (1992). All of the rounding methods proposed there are based on adjusting the weights at the design points and leaving the design points fixed. An interesting rounding procedure which leads to exact designs with possibly additional design points is described in Bandemer and Näther (1980, p. 269).

Kiefer’s method (1971) is perhaps the simplest. If the optimal approximate design has weights \(w_1, w_2, \ldots, w_l\) at its support points such that \(\sum_{i=1}^{l} w_i = 1\), then the rounded optimal approximate design is obtained by choosing integers \(n_1, n_2, \ldots, n_l\) such that \(n = n_1 + n_2 + \cdots + n_l\) and \(\max_{j \leq l} |n_j/n - w_j|\) is minimized.

Pukelsheim and Rieder (1992) considered multiplier methods and showed that the well-known John Quincy Adams method produced the smallest loss in efficiency. The Adams rounding-off method consists of two steps. In the first step, a multiplier \(v\) is chosen. The rounding of the apportionment \(vw_i\) is \(R(vw_i) \in \|vw_i\|\), where

\[
\|z\| = \begin{cases} 
\{m + 1\} & \text{for } z \in (m, m + 1) \\
\{m, m + 1\} & \text{for } z = m
\end{cases}
\]

for all integers \(m\). The design now has size equal to \(n\) or \(n \pm s\) for some positive integer \(s\). In order to ensure that \(s\) is small, Pukelsheim and Rieder (1992) used the multiplier \(v = n - k/2\) instead of the multiplier proposed by Fedorov (1972), \(v = n - k\). The second step guarantees the rounded approximate design has size \(n\) by removing or adding additional points. For a transition from \(n\) to \(n + 1\), a frequency \(n_i\) such that \(n_j/w_j\) attains \(\min_{i \leq k} n_i/w_i\) ought to be augmented to \(n_j + 1\). Similarly for a transition from \(n\) to \(n - 1\), a frequency \(n_i\) such that \(n_j/w_j\) attains \(\max_{i \leq k} (n_i - 1)/w_i\) ought to be reduced to \(n_j - 1\). An advantage of the Adams method is that it is sample size monotone. This means that when a new point is added to \(n\), the rounded design for \(n + 1\) completes the rounded design for \(n\) without taking any observation out. This method applies also for weights if they do not sum to one, which can happen in computer computations.

In this paper, we use the Adams method for rounding off. The method does not always guarantee a unique design and when this happens, we pick the design as symmetric as possible. Some of these rounded optimal approximate designs are the
same as those obtained using the Kiefer method, but not always. We checked all the rounded optimal approximate designs for the quadratic model considered in this paper and found minimal difference in efficiencies between Kiefer’s method and Adams’ method; the maximum difference in efficiencies between the two rounded optimal approximate designs obtained from either method is 5%. Table 1 lists some rounded optimal approximate designs from the optimal approximate designs for the quadratic model when \( X = [-1, 1] \).

As a further example, consider the \( A \)-optimal approximate design for the cubic model. This design is symmetrically supported at \(-1, -x^0, x^0 \) and \( x^0 = \sqrt{(\sqrt{7} - 2)/3} \). Table 2 shows the rounded \( A \)-optimal approximate designs for different sample size \( n \) when the rounding-off method is applied. A similar procedure can be carried out for the numerical \( A \)-optimal designs for higher degree polynomials listed in PUKELSHEIM and TORSNEY (1991).

Similarly, the rounded optimal approximate \( G \)-optimal designs for the cubic model can be obtained from the \( G \)-optimal approximate design, which puts equal weight at \(-1, -1/\sqrt{5}, 1/\sqrt{5} \) and 1 (FEDOROV, 1972). The \( I \)-optimal design for the cubic model is symmetrically supported at \(-1, -y_0, y_0 \) and 1 where \( y_0 \) is the largest zero of \( 686y^8 + 882y^6 - 105y^4 - 64y^2 + 9 \). The weight at 1 is \( 1/(2 + 2/c) \), where \( c = y_0\sqrt{(9 - 28y_0^2 + 35y_0^4)/(14y_0^2 + 2)} \).

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4 Main results

In this section, we provide efficiency calculations for the quadratic model with one independent variable. When optimal exact designs are available, we provide corresponding results for polynomial models of degree three as well. The quadratic model is the simplest curvilinear model and is widely used in practice. For example, Cox (1972) estimated the relative potency in a bioassay assuming a quadratic log-dose response relationship, and Efron and Tibshirani (1991) used a quadratic model to study the relationship between cholesterol reduction scores and percentage of compliance among a cohort of 164 men in the Stanford arm of the cholestyramine study. Generalizations of this model are frequently used for modelling quadratic response surfaces. A recent application is Peace (1990, p. 285) where he used a quadratic model with several independent variables in a Phase II clinical trials to estimate dose and frequency of dosing of a new drug believed to have antianginal efficacy prior to a Phase III antianginal trials.

We first evaluate the efficiencies of the rounded optimal approximate designs relative to the optimal exact designs. When the latter designs are not available, we compare the efficiencies of rounded optimal approximate designs with the optimal approximate designs. The latter comparison is potentially useful because polynomial models of degree higher than two are used in practice and – except for the $D$-optimal designs – only very few optimal exact designs are known for polynomial models of degree three or higher. Consequently, approximations are needed.

Figures 1–3 show the efficiencies of the rounded optimal approximate designs for the quadratic model relative to the optimal exact designs for the $A$-, $I$- and $G$-optimality criterion respectively. In each graph, the efficiency is plotted against the sample size $n$. The efficiencies of the rounded optimal approximate designs relative to the optimal exact designs are denoted by dots in the plots. Depending on the sample size $n$, the plots show that the efficiencies of the rounded approximate designs can change significantly. For instance, in Figures 1–2, when $n$ changes from 6 to 7, the efficiency changes from about 94% to 100% for both the $A$- and $I$-optimality

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Table 2. Rounded $A$-optimal approximate designs for cubic regression model.

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criterion. More dramatic changes occur for the $G$-optimality criterion where the efficiency increases at least 25% from $n = 5$ to 6 (Figure 3).

For those cases where the exact optimal designs are still unknown (see Section 2), we assume that the exact optimal designs have the same form as in Table 1. This means that for these cases, the corresponding dots in the plots do not represent the true efficiencies of the rounded optimal approximate designs but they represent upper bounds for their true efficiencies. This implies that the changes in the efficiencies of

Fig. 1. $A$-efficiencies of rounded optimal approximate designs relative to $A$-optimal exact designs for the quadratic model, when $n = 4p + q, q = 0, 1, 2, 3$.

Fig. 2. $I$-efficiencies of rounded optimal approximate designs relative to $I$-optimal exact designs for the quadratic model, when $n = 4p + q, q = 0, 1, 2, 3$.
the rounded optimal approximate designs as $n$ changes are at least as great as those shown in the plots.

Figures 4–9 show the efficiencies of the rounded optimal approximate designs relative to the optimal approximate designs for the quadratic and cubic models.

PUKELSHEIM and RIEDER (1992) showed that $1 - k/n$ is a lower efficiency bound
for any rounded design of the efficient apportionment $E(\xi, n)$, where $k$ is the number of parameters in the model. The definition of the efficient apportionment is

$$E(\xi, n) = \left\{ (n_1, \ldots, n_k) : n = \sum_{i \leq k} n_i, n_i \in \|\nu w_i\| \text{ for some } \nu \geq 0 \right\}.$$
They noted that the constant $k$ may be improved for specific criteria and for differentiable criteria, a bound of the kind $O(n^{-2})$ could be used.

Our plots suggest that improvements can be made for the lower efficiency bounds given by PUKELSHEIM and RIEDER (1992) when the model is quadratic, and in some cases, for cubic models as well. The continuous lines in Figures 4–9 are the graphs...
of $1 - r/n$ for various $r$ values. The case $r = k$ corresponds to Pukelsheim and Rieder's bound. Improved bounds are shown in the plots; for example, from Figures 4–6, the choice with $r = 2/3$ (or $r = 0.56477$) provides a better $A$-efficiency bound for the quadratic (cubic) model respectively. Likewise, Figure 7 shows $r = 0.58751$ gives a better efficiency bound of the $I$-efficiency for the cubic model than $r = 4$ does.

There are some nice relationships between the efficiency of rounded optimal approximate designs relative to the optimal approximate design. If the model is quadratic, the $A$- and $I$-efficiency of the rounded optimal approximate design relative to the optimal approximate design is $1 - q^2/n^2$, $n = 4p + q$, $q = 0, 1, 2, -1$. For the $G$-criterion, we have a more general result; we argue in the appendix that when we have a homoscedastic polynomial model of degree $k$, the $G$-efficiency of the rounded design relative to the optimal approximate design is $1 - q/n$, where $n = (k + 1)p + q$. $p$ is a positive integer and $q$ is an integer between 0 and $k$. This shows that the efficiency bound given in Lemma 12.8 b of PUKELSHEIM (1993) cannot be improved in that situation.

5 An augmentation problem

The rounding procedure considered above has two steps. In the first step, observations are taken at each design point. If the total number of observations is greater than the sample size, one or more points are removed from the design. Conversely, if the sum is less than the sample size, one or more points are added to the design. This procedure gives rise to an interesting problem. Once an optimal
exact design has been chosen the chance of observing a new point shows up. How does one choose this point in order to get some desirable property? Let us consider, for example, the augmentation problem for the $G$-optimal exact design for polynomial regression when the interest now is to do a lack-of-fit test to ascertain if the degree of the polynomial has to be increased.

Suppose that one has already observed the random variable $y(x)$ at the design points $-1 = x_0 < x_1 < \cdots < x_n = 1$ of the $G$-optimal exact design for the polynomial model of degree $n$. Suppose further that one is willing to make one additional observation so as to check whether the highest coefficient in a hypothetical model of degree $n + 1$ is in fact (nearly) zero. If it is, this would clearly substantiate the original assumption that the model is of degree $n$. The power of the $F$-test for the hypothesis $H_0: \alpha_{n+1} = 0$ is maximized if $\det M_{n+1}(x)/\det M_n(x)$ is a maximum. Here $x$ denotes the additional design point and

$$M_k(x) = f_k(x)f_k^T(x) + \sum_{i=0}^{n} f_k(x_i)f_k^T(x_i), \quad f_k(x) = (1, \ldots, x^k)^T.$$  

The numerical solution of this optimization problem for $n = 2, \ldots, 10$ is given in Table 3. The values in the table suggest that for larger values of $n$ a reasonable choice of $x$ is given by $x = 0$ or $x = x_{n/2+1}/2$ according as $n$ is odd or even.

### Table 3. Optimal augmentation points for the $G$-optimal exact design for different polynomial degrees.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x$</th>
<th>$x_{n/2+1}/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\pm0.594586$</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\pm0.341291$</td>
<td>0.327327</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\pm0.238909$</td>
<td>0.234424</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$\pm0.183532$</td>
<td>0.181559</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$\pm0.148915$</td>
<td>0.147879</td>
</tr>
</tbody>
</table>

6 Discussion

We are not aware of any previous work in the literature reporting on the sensitivities of the efficiencies of the rounded optimal approximate designs relative to the optimal exact designs. This work was motivated by much work in the biometry literature on the use of approximate designs and recently-found optimal exact designs. The comparisons in this paper are necessarily limited because very few optimal exact designs are available to date. Our results here suggest that the performance of the rounded optimal approximate design can be very sensitive when
the sample size is small and their efficiencies can vary as much as 25% per unit change in the sample size.

Similar calculations can be performed for other criteria if the exact optimal design is known. For instance, suppose we are interested in \( MV \)-optimal designs found by minimizing

\[
\Phi_{MV} [M(\hat{\xi})] = \max_i \text{var}(\hat{\alpha}_i).
\]

This criterion minimizes the maximum variances across all estimated parameters. Some optimal approximate designs are given in LÓPEZ-FIDALGO, TORSNEY and ARDANUY (1998). The optimal exact design for the quadratic model was recently found by KRAFFT and SCHAEFER (1995) and shown to have a similar form given in Table 1, where \( z \) now is the only root of the polynomial \((2p + 1)^2 t^3 - 3(2p + 1)t^2 + (20p^2 + 20p + 3)t - 2p - 1 \) and \( n = 4p + 2 \) on \((0, 1)\). The optimal approximate design is the same as the \( A \)- and \( I \)-optimal design in the quadratic case. It turns out that the \( MV \)-efficiency of the rounded optimal approximate design relative to the optimal approximate design is the same as that for \( A \)- and \( I \)-optimality when we have a quadratic model (Figures 10 and 11).

7 Acknowledgments

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Fig. 10. \( MV \)-efficiencies of rounded optimal approximate designs relative to \( MV \)-optimal exact designs for the quadratic model, \( n = 4p + q \).
visited the Department of Biostatistics at UCLA. These two authors would like to thank the department for the hospitality.

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Appendix

**Theorem.** Consider the kth degree regression model (1) with \( X = [a, b] \) and the rounded G-optimal approximate design with sample size \( n \). Write \( n = (k + 1)p + q \), where \( p \) is any positive integer and \( q \in \{0, \ldots, k\} \). The G-efficiency of the rounded design relative to the approximate design is \( 1 - q/n \).

**Proof.** Let \( \xi^* \) be the G-optimal approximate design for regression model (1) and let \( \xi_n \) be an exact \( n \)-observation design obtained from \( \xi^* \) by the Adams rounding method. Denote the \( k + 1 \) support points of \( \xi^* \) by \( x_0, \ldots, x_k \). It is well known that \( \xi^* \) puts probability \( 1/(k + 1) \) at each of these points and it is easily seen that \( \xi_n(x_i) \) is either \( p/n \) or \((p + 1)/n\). There is clearly at least one \( x_s \) such that \( \xi(x_s) = p/n \). Now let \( l_0, \ldots, l_k \) denote the fundamental Lagrange interpolation polynomials associated with \( x_0, \ldots, x_k \). Then

\[
\sup_{x \in X} f^T(x)M^{-1}(\xi_n)f(x) = \sup_x \sum_{i=0}^k \frac{l_i^2(x)}{\xi_n(x_i)} \leq \sup_x \sum_{i=0}^k \frac{nl_i^2(x)}{p} = \frac{n}{p}.
\]

On the other hand,

\[
\sup_{x} f^T(x)M^{-1}(\xi_n)f(x) \geq f^T(x_s)M^{-1}(\xi_n)f(x_s) = \sum_{i=0}^k \frac{l_i^2(x_s)}{\xi_n(x_i)} = \frac{n}{p},
\]

and it follows that the efficiency of \( \xi_n \) is

\[
\frac{\sup_x f^T(x)M^{-1}(\xi^*)f(x)}{\sup_x f^T(x)M^{-1}(\xi_n)f(x)} = \frac{k + 1}{n/p} = 1 - \frac{q}{n}.
\]

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