A COOPERATIVE GAME THEORY OF NONCONTIGUOUS ALLIES

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Abstract
This paper develops a cooperative game-theoretic representation of alliances with noncontiguous members that is based on cost savings from reducing overlapping responsibilities and sequestering borders. For various scenarios, three solutions (the Shapley value, nucleolus, and core’s centroid) are found and compared. Even though their underlying ethical norm varies, the solutions are often identical for cases involving contiguous allies and for rectangular arrays of noncontiguous allies. When transaction costs and/or alternative spatial configurations are investigated, they may then differ. In all cases the cooperative approach leads to a distribution of alliance costs that need not necessarily coincide with the traditional emphasis on gross domestic product size as a proxy for deterrence value (the exploitation hypothesis). Instead, burdens can now be defined based upon a country’s spatial and strategic location within the alliance.

1. Introduction
Since the birth of the nation-state, countries have allied as a means of economizing on defense spending while presenting a united front against would-be aggressors. Alliances represent a way of achieving defense economies without the need to integrate politically. In a recent study, Sandler (1999) presented a cooperative game theory of alliances based on the cost

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savings from the sequestration of borders.¹ This alliance theory concerns only a single solution concept for the distribution of the benefits, namely, the core, with the center of mass of the core (the centroid) used to give a point estimate of the bargaining outcome when the core is set-valued. As formulated, this sequestration theory cannot explain why noncontiguous allies with no common borders would ever join forces, because in such cases the core is either empty or assigns noncontiguous members what they would get in autarky (Sandler 1999, p. 738). Many current and historical alliances involve noncontiguous allies: for example, NATO, ANZUS (Australia, New Zealand, and the United States), the present U.S.-Japanese strategic alliance, the Triple Entente (Britain, France, and Russia) prior to World War I, the Allies in World War II, the Franco-Czech alliance (1924–1938), the Franco-Polish alliance (1921–1938), and the Anglo-Japanese alliance (1902–1922).² If a cooperative game theory of alliances is to prove even more useful, it must be extended to allow for noncontiguous allies, separated by bodies of water (e.g., the Atlantic Ocean or the English Channel), neutral territories, or an enemy. Questions concerning the enlargement of NATO involve noncontiguous allies such as two of the Baltic States (i.e., Latvia and Estonia). Even the linking of two or more alliance pacts from diverse regions of the world is an example of noncontiguous allies.

The literature on alliances is based on two alternative paradigms—the pure public good model and the joint product model.³ Olson and Zeckhauser (1966) originally formulated the pure public good model, in which allies share deterrence or the threat of a retaliatory response of devastating proportions to forestall an attack. If this threat is credible and automatic, then the benefits to the allies are nonrival and nonexcludable, so that deterrence is purely public. The joint product model represents a generalization of this approach by allowing a defense activity to yield multiple outputs that can vary in their degree of publicness.⁴ For example, an ally’s arsenal may yield purely public deterrence, impurely public damage-limiting protection, and ally-specific private benefits (e.g., disaster relief, control of domestic terrorism). Damage-limiting protection is needed when deterrence fails and an attack ensues in which borders must be guarded from enemy infiltration. Such protective forces are subject to a spatial rivalry in the form of force thinning as a given army and arsenal

²On these alliances, see Thies (1987) and Conybeare and Sandler (1990).
³These two models and their implications were discussed in detail in Sandler and Hartley (2001). The particular issue of NATO expansion is addressed from a public goods perspective in Michta (1999).
⁴The joint product model was originated by van Ypersele de Strihou (1967), Sandler and Cauley (1975), and Sandler (1977) and was later refined by Murdoch and Sandler (1982, 1984).
are spread over a longer exposed border. Coalescing forces at select places along an alliance’s perimeter leads to vulnerabilities elsewhere, and it is these vulnerabilities that lead to a rivalry in consumption.

Although these noncooperative game models of alliances are well suited to investigating burden sharing, defense demands, and allocative efficiency, they are ill suited for studying alternative distributions of net benefits among allies, the implications of allies’ size and location, alliance enlargement of spatially diverse allies, and the stability of alternative allied coalitions. In particular, these earlier theories require a knowledge of allies’ preferences that is difficult to ascertain in practice. Our alternative spatial approach requires easy-to-obtain cost information and lends itself to addressing precisely the questions ignored by these earlier paradigms.

The cooperative game model involves a mutual defense game in which we are able to capture both the benefits and costs of alliance formation. The majority of theoretical and empirical treatments of alliance formation quantify costs, stemming from Olson and Zeckhauser’s (1966) seminal finding that defense spending burdens are shared unevenly, with larger allies (in terms of gross domestic product [GDP]) shouldering the burden (i.e., the exploitation hypothesis). “Deterrence” is the primary benefit of such an alliance, implicitly implying that those with the larger GDP have more to lose if the alliance is ineffective. In this paper we measure the benefits of alliance in a tangible way—each country must defend its perimeter, and if a border is made interior through an alliance then it no longer needs protecting. A benefit then accrues as each ally’s (shared) exposed border decreases. Moreover, we focus our analysis on cases where nonadjacent countries must independently protect land or water areas in between them in the absence of an alliance. Thus, as an alliance expands, benefits from allying represent cost savings both from reducing overlapping security activities and creating interior borders between allies. Spatial considerations in terms of allies’ position and shape become an essential factor in this process.

Several interesting results follow from the analysis. If noncontiguous allies have overlapping responsibilities in these disjoint areas, then the core is nonempty. Another is that given the ethical/normative considerations that underlie our single-valued solution concepts (centroid, nucleolus, value), there is no a priori intuition that they should coincide, but in many mutual defense games they do. Yet, the introduction of transaction costs and/or alternative spatial configurations can cause these solution values to differ. Finally, the distribution of defense burden—as determined by the spatial location of allies—can be very different than that predicted by the exploitation hypothesis, in which the primary determinant is relative GDP.

The remainder of the paper contains six sections. Preliminaries that include key concepts and the contiguous allies case are presented in
Section 2. Section 3 contains the benchmark analysis of one noncontiguous ally, while Section 4 addresses the case of two noncontiguous allies. In Section 5, transaction costs are introduced, followed by alternative spatial configurations in Section 6. Section 7 contains concluding remarks and briefly puts forward some nonmilitary applications of our analysis.

2. Preliminaries and Contiguous Allies Case

Any agreement in the bargaining game of mutual defense must be such that members of the pact receive a benefit that is as at least as good as they can obtain in autarky. Allies with strategically advantageous positions will add more to the overall gain from a proposed alliance and, as such, possess a bargaining advantage with an expectation of a greater share of the alliance gain. A coalition is any subset \( S \) drawn from a population \( N = \{1, 2, \ldots, n\} \) of countries and may be a singleton set \( \{i\} \) of just the \( i \)th nation or the grand coalition, \( N \), of all \( n \) nations. An alliance is a coalition of two up to \( n \) allies.

The characteristic or coalition function is denoted by \( v(\{S\}) \) for any subset \( S \) of \( N \) and indicates the sum of net payoffs to the members of \( S \). Henceforth, \( v(\{S\}) \) is depicted by \( v(S) \) for notational simplicity, so that \( v(i,j) \) is the characteristic value for an alliance consisting of allies \( i \) and \( j \). For any coalition \( S \), then \( v(S) \) represents the disagreement payoff to be distributed among \( S \), since \( S \) will not accept less than this amount in any alternative alliance. The net payoffs going to the set of \( s \) allies in coalition \( S \) are the vector \( \hat{u} = (\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_s) \), where \( \hat{u}_i \) is ally \( i \)’s net gain from membership. These net gains or utilities are assumed to be linearly transferable, thus allowing for side payments among participants. In reality, allies do engage in side payments to one another. Nations heavily dependent on Middle East oil gave large cash transfers to the United States during the Gulf War of 1991. In other instances, the transfers assumed more subtle forms such as political concessions to those allies underwriting the defense (Morrow 1991). NATO side payments have taken the form of technology transfers (e.g., the United States sharing its nuclear weapon technology with the United Kingdom and France), special discounts given for weapon sales to allies, and allies’ disproportionate support of alliance infrastructure expense. Given the assumption of transferable utility, any positive affine transformation of utility, \( u_i = a\hat{u}_i + b \) where \( a > 0 \) for all \( i \), has no influence on the strategic features of the associated bargaining game (Moulin 1995; Luce and Raiffa 1957, pp. 186–187). Such transformations merely change the origin and measurement units and can serve to translate cost-sharing problems with negative payoffs into an equivalent representation with positive payoffs for cost savings.

A key concept is that of an imputation.
DEFINITION: An imputation is a payoff vector, associated with a coalition agreement, that satisfies individual rationality, 

\[ v(i) \leq \hat{u}_i \]  

for all \( i \) in \( N \), and Pareto optimality, 

\[ v(N) = \sum_{i=1}^{n} \hat{u}_i. \]  

Individual rationality is necessary because any country has the option of leaving an alliance and will do so if its payoff, \( \hat{u}_i \), from the alliance is below what it can earn on its own, \( v(i) \). Article 13 of the NATO alliance allows any ally to leave after providing a year’s notice. Pareto optimality ensures that the grand coalition receives the same aggregate payoff as any agreement of the whole.

DEFINITION: Imputations are in the core if the following inequalities are satisfied:

\[ v(S) \leq \sum_{i \in S} \hat{u}_i \quad \text{for } S \subseteq N. \]  

The core implies that there are no blocking coalitions that can do better by striking out on their own. Because the singleton sets and the grand coalition are subsets of \( N \), equation (3) is consistent with the satisfaction of (1) and (2). In comparison, the core additionally implies coalition rationality for any nonsingleton coalition \( S \neq N \).

Another solution concept is the Shapley value. Define \( \varphi(v) = (\varphi_1(v), \varphi_2(v), \ldots, \varphi_n(v)) \) to be an allocation of payoffs to the \( n \) members of population \( N \).

DEFINITION: The Shapley value for player \( \{i\} \), \( \varphi_i(v) \), is the imputation that satisfies

\[ \varphi_i(v) = \sum_{S \in \wp(N)} q(s) \cdot [v(S) - v(S - \{i\})], \]  

where \( \wp(N) \) is the power set of \( N \), \( s = |S| \) (the number of members/elements in \( S \)) and

\[ q(s) = (s - 1)!(n - s)/n!. \]

The Shapley value rewards participants according to the average marginal contribution of each participant over its member coalitions. The

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\[ ^5 \text{On Shapley value, see Eichberger (1993, pp. 287–298), Luce and Raiffa (1957, pp. 247–250), and Shapley (1953).} \]
expression in brackets in (4) represents the marginal impact that the \(i\)th participant (entrant) has on coalition \(S\)’s aggregate net benefits. Since \(S\) may assume any subset in the power set, we must account for \(i\)’s impact on any such subset and thus must sum over all subsets containing \(i\). If \(i\) is not in \(S\), then \(S - \{i\} = S\) and the bracketed expression is 0. The weighting factor \(q(s)\) denotes the probability that a subset \(S\) of \(s\) elements is drawn from a population of \(n\) elements. The Shapley value is intended to give an economic perspective on the marginal contribution of each ally. Unlike the core, the value is unique. Moreover, the value need not be contained in the core.

A third solution concept due to Schmeidler (1969) is the nucleolus, which also provides a unique solution. Like the Shapley value, the nucleolus always exists, but unlike the value the nucleolus is always contained in the core (if the core exists). To motivate the use of the nucleolus, we begin with the realization that for any possible agreement, \(\hat{u}\), the excess of a coalition, \(e(S|\hat{u})\), is given as

\[
e(S|\hat{u}) = v(S) - \sum_{i \in S} \hat{u}_i.
\]

The excess represents the loss (or gain, if positive) to coalition \(S\) if its members depart from an agreement \(u\) in order to form their own coalition. For agent \(i\), the excess is the difference between its stand-alone payoff \(v(i)\) and its payoff \(\hat{u}_i\) from an agreement. For any two allocations \(u\) and \(\hat{u}\), if \(e(S|u) < e(S|\hat{u})\) then \(u\) gives more to \(S\) than does \(\hat{u}\). Hence, it is in each coalition’s interest to minimize \(e(S|\hat{u})\). By contrast, if \(e(S|u^*)\) defines \(S\)’s maximum excess, then \(S\) would do better under any allocation other than \(u^*\).

**Definition:** The nucleolus, \(N(v)\), is the set of imputations that minimize the maximum excess across all coalitions.

The nucleolus provides a Rawlsian perspective on alliance building for mutual defense. A priori each coalition knows that if the nucleolus is to be used to allocate costs and benefits, then the maximum objection to any outcome will be minimized. This is akin to Rawls’s (1971) veil of ignorance. Moreover, Maschler, Peleg, and Shapley (1979, p. 337) identify an operational procedure that may lead an arbitrator to select the nucleolus. The idea is that an arbitrator may wish to regard the excess of a coalition at point \(\hat{u}\) as a measure of dissatisfaction of that coalition with \(\hat{u}\). He might therefore aim to select an imputation such that the maximum excess is minimal. Having minimized the maximum excess he may then attempt to minimize the second highest excess. He will then continue with this procedure until he is left with the lexicographic center (relative to the excess functions), namely, the nucleolus.
CONTIGUOUS ALLIES

For the purpose of comparison with noncontiguous ally scenarios, we shall briefly review Sandler’s (1999) treatment of the core for contiguous allies, and then we will extend this analysis to the various single-valued solution concepts. Panel a of Figure 1 displays three contiguous countries that are contemplating a mutual defense pact. Each country is a unit square with 4 inches of border to protect from a conventional attack that can come...
from anywhere. Suppose that each inch costs 1 to protect, so that \( v(i) = -4 \) is the costs of defending nation \( i \) on its own. If countries 1 and 2 were to form an alliance, then the border between them is sequestered, saving 2 in costs since an attack from either direction must be defended against (prior to the alliance). These benefits are reflected in the characteristic function value \( v(1, 2) = -6 \). To be in the core, the imputation \( \bar{u} \) for the three-country alliance must satisfy

\[
\begin{align*}
v(i) &\leq -4 \leq \bar{u}_i, \\
v(1, 2) &\leq -6 \leq \bar{u}_1 + \bar{u}_2, \\
v(2, 3) &\leq -6 \leq \bar{u}_2 + \bar{u}_3, \\
v(1, 3) &\leq -8 \leq \bar{u}_1 + \bar{u}_3, \\
v(1, 2, 3) &\leq -8 \leq \bar{u}_1 + \bar{u}_2 + \bar{u}_3.
\end{align*}
\]

(6)

To simplify the analysis, we find an affine transformation to apply to the \( u_i \) payoffs so that each transformed \( u_i \) is nonnegative, and the maximum gain to the grand coalition is 4, equal to the maximum cost savings from sequestering two interior borders. The required transformation is

\[
u_i = \hat{u}_i + 4, \quad i = 1, 2, 3,
\]

so that the core must satisfy

\[
\begin{align*}
v(i) &\leq 0 \leq u_i, \\
v(1, 2) &\leq 2 \leq u_1 + u_2, \\
v(2, 3) &\leq 2 \leq u_2 + u_3, \\
v(1, 3) &\leq 0 \leq u_1 + u_3, \\
v(1, 2, 3) &\leq 4 = u_1 + u_2 + u_3.
\end{align*}
\]

(7)

These equations can be solved graphically using an equilateral triangle or simplex whose vertices correspond to a lone ally receiving the entire gain of 4. Such a triangle is given in Figure 2—but we caution the reader that the analysis within this triangle is not for the game in panel a (it instead corresponds to the upcoming benchmark game in panel c). The solution of panel a is given by Sandler (1999, pp. 733–734), who shows that the core is defined by a rhombus within the triangle with vertices at \((0, 2, 2), (2, 0, 2), (2, 2, 0), \) and \((0, 4, 0)\). The centroid of the core is its center of gravity or mass from which the sum of the displacements of all points is zero. In the case of a rhombus, the centroid corresponds to

\(^6\)Sandler (1999) analyzed alternative attack scenarios, natural defenses, heterogeneous sized allies, and other aspects. These extensions can be easily applied to the spatial environments examined in this paper. Also, sometimes allies do attack each other (for a discussion, see James 1988). We are implicitly ruling out this possibility.
the intersection of its diagonals, which in this case is (1, 2, 1). Ally 2 gets twice as much as the end allies. The middle ally is at a bargaining advantage since its position saves protection costs in two directions, unlike the end allies, which economize protection expense in a single direction.

The centroid has at least two disadvantages, and these motivate our study of other solutions. First, the centroid is obviously not defined when the core is empty, which is not an uncommon occurrence. Second, infinitesimal changes to the characteristic function can result in finite changes to the centroid. Thus, the centroid may be discontinuous with respect to the characteristic function, as is demonstrated in footnote 8 of Section 3.

Next, we apply the formula in equation (4) to derive the Shapley value, denoted by \( \varphi_1, \varphi_2, \varphi_3 \). We have

\[
\varphi_1 = (2/6)[v(1) - v(\emptyset)] + (1/6)[v(1, 2) - v(2)] + (1/6)[v(1, 3) - v(3)] + (2/6)[v(1, 2, 3) - v(2, 3)],
\]

indicating the expected marginal benefit derived from ally 1’s presence in the various pacts. Employing the convention that \( v(\emptyset) = 0 \) and substituting from (7) yields \( \varphi_1 = 1 \).\(^7\) By symmetry, \( \varphi_3 = 1 \). Since the three payoffs must add to 4, we can conclude that \( \varphi_2 = 2 \) so that the Shapley value coincides with the centroid. An ally’s marginal contribution is equivalent to its mean contribution for this configuration of contiguous allies.

\(^7\)This follows from \( \varphi_1 = (2/6)(0 - 0) + (1/6)(2 - 0) + (1/6)(0 - 0) + (2/6)(4 - 2) = 1 \).
The nucleolus is also \((1, 2, 1)\). The geometric process for finding the nucleolus is exactly as given in the Maschler et al. (1979) arbitration scheme outlined in the previous section. The formulas corresponding to this process are contained in Section 3—where we address our main issue of noncontiguous allies.

In Sandler (1999), the three-ally case is extended to allow for alternative border lengths, natural defenses, more allies, and other considerations. In particular, we turn our attention to the case where two allies are noncontiguous, as in panel \(b\) of Figure 1. If \(\{2\}\) and \(\{3\}\) do not guard the north and south perimeters of the area between them, then the core is not fully dimensional. It only includes imputations where the two adjacent allies divide the cost savings of 2, that is, \((u_1, u_2, u_3) = (1, 1, 0)\) (Sandler 1999, p. 738). This implies that countries such as Estonia and Latvia have nothing to offer NATO in a mutual defense pact motivated by cost savings/interior borders.

3. One Noncontiguous Ally: Benchmark Case

The Triple Alliance of Germany, Austria-Hungary, and Italy during 1882–1925 is an instance of the contiguous case. In contrast, the foe of the Triple Alliance—the Triple Entente (Great Britain, France, and Russia) was not contiguous. The analysis of panel \(b\) casts doubt on the spatial benefits of the latter alliance. Can cooperative gains arise from cost savings when allies are noncontiguous? Many important alliances include noncontiguous members, and so this is a relevant question. We establish that cost savings may still occur and provide a justification for allying even without outside political or strategic benefits added to our framework.

For the benchmark case, refer to the three-ally grouping displayed in panel \(c\) of Figure 1 where allies 2 and 3 are noncontiguous. In contrast to panel \(b\), in panel \(c\) the shaded region in between must have its borders protected, and this gives overlapping responsibilities to countries 2 and 3 in the absence of an alliance. Without an alliance, country 1 incurs a defense cost of 4, while both countries 2 and 3 incur individual defense costs of 6. In autarky, the perimeter defense of country 2 includes the thickened perimeter, and country 3 has a similar responsibility. An alliance involving countries 1 and 2 or countries 2 and 3 defends 8 inches of perimeter at a cost of 8, while an alliance involving countries 1 and 3 guards 10 inches of perimeter at a cost of 10. The three-nation alliance would also incur a defense cost of 10 because the interior border shared by \(\{1,2\}\) is sequestered, and \(\{2\}\) and \(\{3\}\) achieve a cost savings by limiting overlapping responsibility as allies can rely on one another and assign responsibilities to perimeters for in-between regions. An example is the English Channel between Britain and France. The maximum cost savings achieved by the three-country alliance is again 4.
The positive affine transformation that reflects a cost savings of 4 for $v(N)$ requires that, when applied to the $\hat{u}_i$’s, it must also give a nonnegative disagreement payoff for $v(i)$, $i = 1, 2, 3$. Normalizing the autarky outcomes for $\{2\}$ and $\{3\}$ to be $v(2) = v(3) = 0$ yields the transformation $u_i = 0.5(\hat{u}_i + 6)$. Given this transformation, the core is defined by the following seven constraints:

$$
\begin{align*}
  v(1) &= 1 \leq u_1, \\
  v(2) &= 0 \leq u_2, \\
  v(3) &= 0 \leq u_3, \\
  v(1, 2) &= 2 \leq u_1 + u_2, \\
  v(2, 3) &= 2 \leq u_2 + u_3, \\
  v(1, 3) &= 1 \leq u_1 + u_3, \\
  v(1, 2, 3) &= 4 = u_1 + u_2 + u_3.
\end{align*}
$$

The associated core is illustrated in Figure 2, which displays an equilateral triangle whose vertices correspond to one ally receiving the entire gain of 4. Each point in or on the triangle is a triple $(u_1, u_2, u_3)$ representing an imputation. The three sides correspond the gain going to just two allies: $u_1 + u_2 = 4$ is the triangle’s base, while $u_2 + u_3 = 4$ and $u_1 + u_3 = 4$ are the triangle’s left-hand and right-hand sides, respectively. The perpendicular distance of a point in the triangle from any side indicates the payoff of the unlisted ally on that side of the triangle; for example, the payoff to nation 3 is the height from the base.

In Figure 2, the rhombus defining the core is formed by the lines: $u_1 = 1; u_1 + u_2 = 2; u_2 + u_3 = 2; \text{ and } u_3 = 0$. The shaded areas indicate imputations not satisfying the various inequalities in (8). Notice that some constraints in (8) do not bite—for example, $u_1 + u_3 = 1$ is made redundant by $u_1 = 1$. The core’s vertices are $(1, 1, 2), (2, 0, 2), (2, 2, 0), \text{ and } (1, 3, 0)$, whose average coordinates result in a centroid at $(1.5, 1.5, 1.8)$. This case differs from that of contiguous allies in that country 1’s rationality constraint, $u_1 = 1$, now bites, thereby shrinking the core. End ally 1 draws equal to its adjacent middle ally in contrast to the earlier case, while ally 3 remains at a bargaining disadvantage given its inability to sequester an interior border. Yet ally 3 is not so disadvantaged that it receives nothing, as was the case in panel $b$.

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*The potential discontinuous behavior of the centroid alluded to earlier can be illustrated for this case by changing the characteristic function in (8). In particular, replace $v(1) = 1$ with $v(1) = 1 - \epsilon$ for $\epsilon > 0$. This displaces the $u_1 = 1$ line to the left in Figure 2 by an infinitesimal distance and results in five vertices of the core being averaged instead of four. As a consequence, a finite change in the centroid results from an infinitesimal displacement.*
In the periods before and during World War II, the United States and Britain had to defend the North Atlantic to keep shipping lanes open. The benchmark case would predict that the United States, in the position of ally 3, is at a distinct bargaining disadvantage, leading it to assume greater relative burdens. The same is true of the U.S. position in NATO, where many of the European allies are contiguous. Empirical observations on NATO indicate that the United States assumed a disproportionate share of the burdens during the cold war (Sandler and Hartley 1999, Chap. 2), which is consistent with the theoretical prediction here. Unlike the Olson–Zeckhauser (1966) model, which attributes this disproportionality to free riding and relative income considerations, the cooperative game model attributes it to a bargaining disadvantage. Another example consists of the Franco-Belgian (1920–1936) and Franco-Czech (1924–1938) mutual defense pacts against Germany. As war approached, Czechoslovakia narrowed its defense gap with France despite the former’s inferior economic position (Thies 1987, pp. 314–317). In the cooperative game based on position and perimeter size, GDP is no longer a predictor of burden sharing, unlike in earlier economic models of alliances.

To find the Shapley value, we apply equation (4) using the \( v \) values given by (8). The Shapley value computes to be \( \varphi_1 = 1.5, \varphi_2 = 1.5, \) and \( \varphi_3 = 1 \), which coincides with the centroid, so that the mean and marginal contributions of the allies to the pact match. Ally 3’s marginal influence puts it at a disadvantage. From an economic perspective, each country’s contribution to the alliance should be inversely related to its Shapley value. Consequently, this theory predicts that ally 3 should shoulder more of the burden of the alliance.

Next, we turn to the nucleolus. Moulin (1988, pp. 136–137) provides an exhaustive characterization of the nucleolus for three-person superadditive games\(^9\) when the characteristic functions are alternatively normalized so that \( \hat{v}(i) = 0 \) for all \( i \). Without loss of generality, up to a permutation of the agents \( \{i, j, k\} \) one can also assume \( 0 \leq \hat{v}(j, k) \leq \hat{v}(i, k) \leq \hat{v}(i, j) \leq v(N) \). For all games analyzed in Sections 2–5 it is the case that Moulin’s Class 3 condition holds, which identifies the explicit formula for the nucleolus as \( \gamma_i = 0.5 \cdot [\hat{v}(i, j) + \hat{v}(i, k)]; \gamma_j = 0.5 \cdot [\hat{v}(N) - \hat{v}(i, k)]; \) and \( \gamma_k = 0.5 \cdot [\hat{v}(N) - \hat{v}(i, j)] \).

A judicious examination of these formulas additionally reveals that for this class the nucleolus is the centroid of the core. For example, with respect to \( \gamma_i \), the core’s coalition rationality constraints for dyads involving \( \{i\} \) are defined by \( \hat{v}(i, j) \) and \( \hat{v}(i, k) \). The formula for \( \gamma_i \) then gives the center of mass of these constraints. In the same way, the numerical values corresponding to the level curves/constraints for the dyads involving \( \{j\} \) and \( \{k\} \) are given in the bracketed terms for \( \gamma_j \) and \( \gamma_k \), respectively. Their

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\(^9\)A game is superadditive if, for any partition \( \{S_1, \ldots, S_k\} \) of \( N \), it is the case that \( \sum_{i=1}^{k} v(S_i) \leq v(N) \). Superadditivity is a necessary condition for core nonemptiness.
nucleoli formulas then take the average over their coalition rationality constraints, that is, the centroid of the core.

This observation is helpful for several reasons. First, for the games under consideration it is always the case that each endpoint that defines the core geometrically will be such that the sum of its entries equals \( v(N) \). As such, the centroid can be found by adding up the coordinates of each endpoint componentwise and dividing by the number of endpoints. Consequently, we have a simplified procedure for calculating the nucleolus/centroid for the games under study. Interested readers who wish to verify Maschler et al.’s (1979) geometric procedure for deriving the nucleolus can consult the authors for the associated diagrams. Second, a novel result arises from our analysis. There is no reason to anticipate that the centroid, Shapley value, and nucleolus should coincide because they are based on different negotiation structures and criteria of fairness. Yet for our benchmark case they do.

4. Multiple Disjoint (Noncontiguous) Allies

The next three-country scenario is depicted in panel \( d \) of Figure 1, where countries 1 and 2, as well as 2 and 3, are separated by a region of mutual defense responsibilities. Examples include the Triple Entente, ANZUS, United States–Japan–NATO Europe alliance, and networks of alliance pacts. In this new scenario, countries 1 and 3 incur a defense cost of 6, while country 2 incurs a defense cost of 8. An alliance consisting of either allies 1 and 2, or 2 and 3, must cover 10 in protection expense. A pact involving the two end countries incurs a defense cost of 12, which reflects the defense responsibilities of protecting 12 sides. To translate these negative payoffs into strategically equivalent nonzero imputations, we apply the transformation, \( u_i = (\hat{u}_i + 8)/3. \) The core must then satisfy

\[
\begin{align*}
    v(1) &= 2/3 \leq u_1, \\
    v(2) &= 0 \leq u_2, \\
    v(3) &= 2/3 \leq u_3, \\
    v(1, 2) &= 2 \leq u_1 + u_2, \\
    v(2, 3) &= 2 \leq u_2 + u_3, \\
    v(1, 3) &= 4/3 \leq u_1 + u_3, \\
    v(1, 2, 3) &= 4 = u_1 + u_2 + u_3.
\end{align*}
\]  

\( ^{10} \) We thank Alvin Swimmer, professor of mathematics at Arizona State University, for pointing out this property of the centroid to us.
A diagrammatic procedure identical to that of the benchmark case gives the core as a rhombus \(^{11}\) with vertices at \((2/3, 4/3, 2), (2, 0, 2), (2, 4/3, 2/3),\) and \((2/3, 8/3, 2/3)\) and sides defined by \(u_1 = 2/3, u_1 + u_2 = 2, u_2 + u_3 = 2,\) and \(u_3 = 2/3.\) The nucleolus (and the core’s centroid) is at \((4/3, 4/3, 4/3)\), so that the three allies now make identical average contributions to mutual defense. The middle ally has lost its bargaining advantage compared with the contiguous and benchmark cases, since it now has two adjacent regions to defend when on its own. This even-division outcome is identical to a three-country contiguous scenario where the middle ally is 50% longer than the end allies (Sandler 1999, pp. 736–737); thus, the adjacent regions on either side of ally 2 are analogous to a lengthened width of ally 2 itself. The bargaining advantage associated with an ally’s placement is consequently sensitive to what lies between the allies. This model implies that when Japan is considered along with the United States and NATO-Europe as an alliance network, the United States relinquishes its equal bargaining advantage as a result of its long perimeter compared with Japan or its European allies. Thus, the United States is again viewed as needing to do disproportionately more. This case is especially germane to the post–cold war era where peacekeeping and policing nonproliferation in adjacent nonallied regions represent current defense challenges.

The Shapley value is again equivalent to the nucleolus. From a marginal viewpoint, the middle ally’s position helps the end allies as much as the latter allies assist the middle ally, leading to a value where \(\varphi_1 = \varphi_2 = \varphi_3 = 4/3.\)

5. Transaction Costs and Other Considerations

Thus far, the three solution concepts have not deviated from one another within a given situation of contiguous or noncontiguous allies. They may differ, however, when transaction costs have a differential impact on the allies. Consider, for example, the benchmark case of panel \(c\) in Figure 1, where transaction costs are now assumed to be 1 for any coalition containing noncontiguous allies and 0 otherwise. These transaction costs arise from the increased costs required to coordinate efforts to patrol the region in between without duplicating efforts. They also stem from the extra costs needed to transport forces between allies in times of crisis or for maneuvers. The maximum gain from the three countries forming a mutual defense pact is now just 3 since transaction costs of 1 must be deducted from cost savings of 4. The characteristic function is as follows: \(\nu(1) = -4, \nu(2) = \nu(3) = -6, \nu(1, 2) = -8, \nu(2, 3) = -9, \nu(1, 3) = -11 = \nu(1, 2, 3).\) In this scenario, the costs of defending allies 2 and 3 involve guarding the eight 1-inch sides plus 1 in transaction costs, while the costs of protecting

\(^{11}\)The computations are henceforth suppressed and available from either author upon request.
allies 1 and 3 or the three-ally coalition include protecting the ten 1-inch sides plus 1 in transaction costs.

Because of transaction costs, the underlying equilateral triangle has vertices at $(0, 0, 3)$, $(3, 0, 0)$, and $(0, 3, 0)$. The required transformation for the imputations’ coordinates becomes $u_i = 3(\hat{u}_i + 6)/7$. Application of this transformation results in equations that define the core as a rhombus with vertices at $(12/14, 0, 30/14)$, $(24/14, 0, 18/14)$, $(24/14, 18/14, 0)$, and $(12/14, 30/14, 0)$ and nucleolus/centroid at $(18/14, 15/14, 9/14)$. In comparison to the benchmark case, ally 1’s share has increased relative to allies 2 and 3, with ally 3 being the greatest loser. Hence, in the case of NATO, there is further justification for the United States’ bargaining disadvantage leading it to assume a disproportionately large burden that is independent of the exploitation hypothesis or free-riding considerations raised by Olson and Zeckhauser (1966). Quite simply, the U.S. position with an ocean between it and its European friends limits its average contribution whether it be during the two world wars or the cold war, and so it had to do more than others.

The Shapley value is $(17/14, 17/14, 8/14)$, which differs from the nucleolus. This highlights the theoretical significance that even a simple introduction of transaction costs can make to mutual defense pacts. From a marginal influence perspective, allies 1 and 2 add identical contributions, but this is not true of ally 3, whose position is the cause of transaction costs when it joins a pact with either ally 1 or 2, or both. As a consequence, ally 3’s value suffers, leading its share of the gain from the grand coalition to be smaller than for any scenario considered thus far.\(^\text{12}\)

This case represents the first instance in which the value, nucleolus, and centroid do not coincide. The reasoning is as follows. Under Moulin’s (1988, p. 137) Class 3 formula for the nucleolus, if $\hat{v}(i,j) + \hat{v}(j,k) + \hat{v}(i,k) = \hat{v}(N)$—where $\hat{v}$ denotes the transformation such that $\hat{v}(i) = 0 \forall i = 1, 2, 3$—then it is straightforward to show that the nucleolus coincides with the Shapley value. The economic interpretation of this equality is that the dyads can cumulatively achieve the cost savings produced by the grand coalition. Once transaction costs are introduced, however, this produces an externality that cannot be fully internalized bilaterally. In particular, $\hat{v}(i,j) + \hat{v}(j,k) + \hat{v}(i,k) < \hat{v}(N)$, Joint cost savings occur under the grand coalition that cannot be replicated across all dyads.

Next, we apply the same transaction costs arrangement to the case of two noncontiguous allies of panel $d$ in Figure 1, where transaction costs are 1 for a pact containing two allies separated by an intervening region. These transaction costs are 2 if there are two intervening regions between allies within a two-ally or three-ally pact. With reference to the previous

\(^{12}\) Its share of the Shapley value equals $(8/14)/(42/14)$, or about 0.19 compared with 0.214 at the centroid of this case. For the benchmark and contiguous allies cases, ally 3’s share was 0.25. In the multiple noncontiguous allies example, it rose to 0.333.
paragraph this means that the grand coalition internalizes transaction costs only as efficiently as the dyads do cumulatively. Thus, the characteristic function is as follows: \( v(1) = -6 = v(3); v(2) = -8; v(1, 2) = -11 = v(2, 3); \) and \( v(1, 3) = -14 = v(1, 2, 3), \) which includes relevant protection and transaction costs. The transformation used to find the core graphically is \( u_i = (\hat{u}_i + 8)/5. \) In the equilateral triangle with vertices at \((0, 0, 2), (2, 0, 0), \) and \((0, 2, 0), \) a rhombus again defines the core,\(^{13}\) whose centroid is at \((0.7, 0.6, 0.7).\) With transaction costs, middle ally 2 loses some bargaining strength to the end allies as compared with the earlier no transaction costs scenario. This follows because ally 2 on average adds less to the pact given the regions on either side of it, which augments transaction costs to a larger extent than the other allies. All three solution concepts coincide, so that this example indicates that transaction costs alone need not cause the solutions to differ. The key is the effectiveness of the grand coalition for internalizing the costs, versus the way that they accumulate across all possible dyads.

**Other Considerations**

The methodology used to include transaction costs lends itself to other kinds of costs or benefits. In the case of benefits, we shall illustrate with the benchmark model an example where the geographically separated country offers a strategic gain to the alliance, not possessed by the two contiguous allies. Suppose that ally 3 offers strategic advantages—for example, a staging area from which to launch attacks against a potential adversary or a listening post (as with Turkey for NATO). Further assume that this country’s strategic advantage is worth 1 to any alliance for which it is a member. The characteristic function is as follows: \( v(1) = -4; v(2) = -6; v(3) = -5; v(1, 2) = -8; v(2, 3) = -7; v(1, 3) = -9; \) and \( v(1, 2, 3) = -9. \) The required transformation is \( u_i = 5(\hat{u}_i + 6)/9, \) which when applied yields a triangular core with vertices at \((10/9, 10/9, 25/9), (20/9, 0, 25/9), \) and \((20/9, 10/9, 15/9)\) and a nucleolus/centroid at \((100/54, 40/54, 130/54).\) This is the first case in which ally 3 gains a bargaining advantage over its counterparts. Thus, strategic factors can be added to spatial ones to present a more complete picture. In the case of NATO, some contiguous allies along the front lines (e.g., Germany) possessed a strategic advantage during the cold war over the United States, implying yet again why the United States might pay a larger share without being “exploited.”

When marginal contributions are taken into account, the Shapley value equals \((110/54, 50/54, 110/54).\)\(^{14}\) Allies 1 and 3 possess the same bargaining advantage from a marginal perspective with the middle ally gaining some ground over its nucleolus allocation. Now, instead of recurrent

\(^{13}\)The core’s vertices are at \((2/5, 3/5, 1), (1, 0, 1), (1, 3/5, 2/5), \) and \((2/5, 6/5, 2/5).\)

\(^{14}\)This follows from the transformed characteristic function, which equals \( v(1) = 10/9, v(2) = 0, v(3) = 5/9, v(1, 2) = 20/9, v(2, 3) = 25/9, v(1, 3) = 35/9, \) and \( v(1, 2, 3) = 5, \) and equation (4).
transaction costs across dyads, we have a situation where some dyads are as capable as the grand coalition in capturing the transaction benefit. Coalition \{1,3\} and \{2,3\} capture the “listening post” benefit as easily as the grand coalition does.\footnote{In terms of Moulin’s transformation, we now have \(\hat{v}(i,j) + \hat{v}(j,k) + \hat{v}(i,k) > \hat{v}(N)\).} Coalition \{1,2\} cannot achieve this strategic benefit, and this partially offsets the geographic cost of alliance \{1,3\}. Thus, a strategic advantage can neutralize a positional disadvantage. Since the middle ally is no longer as desirable a position, this model highlights that noncontiguous alliances may result in drastically different solutions as compared with contiguous alliances.

6. Two Alternative Spatial Configurations

To this point, the allied configurations have been in a linear array, so that the alliance has been rectangular. The pattern of the allies can also make a difference that is now demonstrated with two alternative T-shaped configurations (panels e and f of Figure 1). This exercise indicates that the alliance configuration, like transaction costs, can influence whether or not solution concepts give different outcomes. The allies’ positions are immaterial in the traditional public good models of alliances, so that the results here further differentiate the insights gained from a cooperative game approach. A T-shape applies to the dual alliances of Franco-Czech (1924–1938) and Franco-Polish (1921–1938), since Poland is contiguous and north of Czechoslovakia. It also represents the combined alliances of Franco-Belgian (1920–1936) and Franco-Czech, as well as other alliances (e.g., United States–Canada–NATO Europe).

In this first case, the T is “centered” (panel e) so that countries 1 and 2 share a common border and each also has a half inch exposed to the intervening region between it and country 3. Country 1 must guard 3.5 inches of its own perimeter and 3.5 inches of the intervening region’s perimeter (including the half inch bordering country 2) at a cost of 7. A similar situation characterizes country 2. Country 3 must in isolation defend 6 inches at a cost of 6. With some care, the characteristic function can be shown to be as follows: \(v(1) = v(2) = -7; v(3) = -6; v(1, 2) = -8; v(1, 3) = v(2, 3) = -9\); and \(v(1, 2, 3) = -10\). Based on the transformation \(u_i = 4(\hat{u}_i + 7)/11\), the core is a trapezoid with vertices at \((0, 24/11, 20/11)\), \((24/11, 0, 20/11)\), \((24/11, 16/11, 4/11)\), and \((16/11, 24/11, 4/11)\). The nucleolus/centroid is at \((16/11, 16/11, 12/11)\) with countries 1 and 2 displaying the same bargaining advantage over country 3.\footnote{This and the following game satisfy Moulin’s (1988, p. 137) Class 4 condition for calculating the nucleolus. Here again it is the case that if \(\hat{v}(i,j) + \hat{v}(j,k) + \hat{v}(i,k) = \hat{v}(N)\), then the nucleolus and Shapley value are equivalent.} The core is no longer a rhombus because the restriction on the coalition involving allies
1 and 3 now bites. In the earlier scenarios, these allies were so disjoint that \( v(1, 3) \) did not yield a binding restriction on the core. Now that it does, ally 3’s share is slightly improved over the benchmark case with or without transaction costs.

The Shapley value equals \( 14.67 \) because each of the three allies makes the same marginal contribution to the pact, despite locational differences. From a marginal perspective, the sequestering of the border between allies 1 and 2 is offset by the need to guard the intervening region. As was the case for transaction costs, the pattern of spatial distribution can cause the Shapley value to differ from the nucleolus/centroid.

Off-Centered T-Shaped Alliance

In panel f of Figure 1, the off-centered T-shaped alliance is displayed, where ally 1 has only a quarter inch adjacent to the intervening region, whereas ally 3 has three-quarters inch next to this region. Insofar as ally 1 must protect its own territory and this intervening region, this ally defends 3.75 inches of its perimeter and that of the adjacent area for a total cost of 7.5. Only its common quarter-inch border with the shaded area provides any defense relief. Ally 2 is now in a relatively improved position because it only has to protect 6.5 inches and economizes on a full three-quarters of an inch of common borders with the adjacent area. Similar calculations yield the following characteristic function: \( v(1) = -7.5, v(2) = -6.5, v(3) = -6, v(1, 2) = -8, v(2, 3) = -8.5, v(1, 3) = -9.5, \) and \( v(1, 2, 3) = -10. \) The appropriate transformation is \( u_i = 8(\hat{u}_i + 7.5)/25, \) which makes for four binding constraints defining a trapezoidal core with vertices of \((0, 56/25, 44/25), (48/25, 8/25, 44/25), (48/25, 40/25, 12/25), \) and \((32/25, 56/25, 12/25).\)

The nucleolus/centroid is \((96/75, 120/75, 84/75), \) which differs from the Shapley value of \((88/75, 112/75, 100/75), \) where ally 3 gains at the expense of allies 1 and 2. As ally 1 has less border in common with the intervening region, its bargaining position worsens until the limit at which it has nothing in common. Longer common boundaries to such areas are advantageous, which is a surprising finding that is explained once the strategic implications are examined. An interesting discontinuity is present because once ally 1 has nothing in common with the shaded area, the resulting solutions for the L-shaped configuration are identical to the benchmark, where allies 1 and 2 draw equal and ally 3 is greatly disadvantaged.

7. Conclusion and Extensions to Nonmilitary Applications

In this paper we examine the spatial incentives for alliance formation when prospective allies are not contiguous. We find that when individual

\[\text{17} \text{These four constraints are as follows: } u_1 + u_2 = 44/25; u_1 + u_2 = 56/25; u_2 + u_3 = 52/25; \text{ and } u_3 = 12/25.\]
countries have overlapping responsibilities for the defense of areas lying in between, then there is a great deal of scope for cooperation. The benefits for such alliances rest in the cost savings that are accrued through mutual defense of shared perimeters. Moreover, in contrast to the standard explanation of defense burdens in alliances, which is based on the relative GDP of allies, burdens can now be defined in terms of each country’s strategic spatial location within the alliance. For example, one would expect that the contribution that a country makes to an alliance is inversely related to its marginal contribution to the defense of the alliance—as measured by the Shapley value of the underlying cooperative game. We show that the Shapley value is sensitive to the spatial relationship among members, a characteristic that is not addressed in the GDP/exploitation thesis of burden sharing.

In addition, competing cooperative solution concepts such as the core, its centroid, the Shapley value, and the nucleolus are meant to represent the outcomes generated from alternative bargaining criteria. We show that if allies are distributed along a straight line then conditions exist such that the centroid, nucleolus, and Shapley value coincide. This coincidence is broken, however, if the spatial layout is not linear, for example, if the geographical dispersion is a “T.” Moreover, the introduction of transaction costs in the bargaining process can also cause these solution concepts to diverge. As a consequence, we provide a justification for the existence of noncontiguous alliances such as ANZUS and the Triple Entente, the expansion of NATO, and the distribution of defense burdens within these alliances.

Our theory of alliance building also puts a new twist on the cooperative approach to cost sharing. For example, spatial relations are of prime importance in environmental treaties. Belgium and the Netherlands have to “defend” their seaward perimeters from water borne pollutants in the North Sea Southern Bight. Ensuring an abatement alliance with France would increase their effectiveness because France’s spatial location is upcurrent (Tulkens 1979). Indeed, pollution often has a unidirectional, rather than bidirectional, flow, and the transaction cost approach introduced here can accommodate this observation. Moreover, not all pollution abatement efforts involve contiguous countries. A poignant example is the ongoing negotiations over acid rain between China and Japan. Finally, the environmental analog to the exploitation hypothesis for military alliances is the global assessment scale used by the United Nations to differentiate between nations’ abatement targets. The scale is typically a function of technological and economic differences; however, it is clear from our analysis that spatial location has a role to play as well.

From a competitive perspective, open skies treaties for airline agreements carry spatial considerations. In this case both the surface area of a country and its network of routes must be addressed. The open skies concept implies that the entire area of an alliance member can be serviced.
Once access to this area is assured, the location of domestic hubs is a paramount consideration for the purposes of rerouting previously direct routes in order to consolidate those with lower volumes. The automobile industry is another area where noncontiguous alliances require explanation. The Daimler-Chrysler merger of 1998 was almost unique in the history of transatlantic automotive alliances, but it set off a wave of such mergers that has yet to subside. Here, development costs that would otherwise be unilaterally sunk can instead be partially sequestered in an alliance.

References


