VALUING THE OPTION TO PURCHASE AN ASSET AT A PROPORTIONAL DISCOUNT

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Abstract

I analyze the value of a nonstandard call option that allows the holder to purchase an underlying asset at a discount proportional to the asset’s market price. Several applications for this type of option exist, including its use in employee compensation contracts. I derive the value of this option for a dividend-paying asset and for an option whose exercise price reflects a time-varying discount factor. The derived value incorporates the optimal time at which the option should be exercised. One application of this option relates to a residential real estate program in China.

JEL Classifications: G12, G13, J33

I. Introduction

Option contracts have existed since at least the eighteenth century in Europe and the United States. They became increasingly popular after the establishment of the Chicago Board Options Exchange in 1973 and the publication of the seminal article on option pricing by Black and Scholes (1973). In this article, I analyze the value of a nonstandard, American-style call option whose exercise price is a proportion of the price of its underlying asset. I also consider possible uses and advantages of this type of option, as well as an application found in the Chinese real estate market.

The option under consideration is different from a conventional call option in that its exercise price is not a fixed price but rather a proportion of the price of its underlying asset. For example, an option might give its owner the right to buy a share of common stock at a price equal to 80% of the stock’s market price at any time before the option’s expiration date. Because its exercise price is a portion of the underlying asset’s price, the option is always in the money as long as the value of the underlying asset is greater than zero.

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A potential use for such an option may be by employers who wish to provide performance-sensitive compensation to their employees. Companies are increasingly issuing stock or conventional stock options, or both, to employees in compensation contracts. An alternative, which is a hybrid of stocks and conventional stock options, would be to compensate employees with options to buy company stock at a proportional discount. Such an option is always in the money, yet its value is sensitive to the firm’s performance. For a given number of underlying shares, these proportional-exercise price options would be less expensive to the company than directly issuing stock, and, unlike standard, fixed-exercise price options, they would not lose their beneficial incentive features should the company’s stock decline because of exogenous swings in the stock market.

Compensation in the form of proportional-exercise price options can be customized to meet a company’s needs. The option’s proportional-exercise price can be based on an employee’s seniority, a feature that can enhance loyalty to the company and thereby reduce employee turnover. An exercise price of, say, 80% of the current stock price could be set for a new hire, with the exercise price being reduced by 1 percentage point for each additional year of employment. Thus, a company could give an employee the right to buy a certain number of shares of stock at a price equal to 80% of the stock’s market value if purchased during the employee’s first year, 79% if purchased during the second year, 78% if purchased during the third year, and so on until the employee retires or leaves the company. Although there would be no further reduction in the option’s proportional exercise price following retirement, the option’s maturity could be designed to follow the employee’s retirement date, at which time his/her tax liability on the option’s capital gain may be lower because of a lower post-retirement tax bracket.

As part of a compensation scheme, these proportional-exercise price options need not be written on the employer’s stock. They may be written on another asset such as real estate. For example, governments or nonprofit organizations, such as universities, may establish a housing policy that allows their employees to buy homes at a discount to their market prices. The program might apply only to areas where the employer wishes to promote home ownership. I consider such a program offered by the government of China.

II. The Model

Let $P$ be the current, date 0 price of an underlying asset. An option written on this asset is assumed to have an exercise price at any future date $t \geq 0$ equal to $XP_t$, where $P_t$ is the price of the underlying asset at date $t$ and $0 < X < 1$.

First, consider an option on a non-dividend-paying asset, such as a non-dividend-paying stock. Suppose the holder of the option decides to exercise the option at some arbitrary future date, $T$. One can value the cash flows from this
option using a simple application of risk-neutral pricing developed by Cox and Ross (1976). Let \( r \) be the continuously compounded risk-free interest rate for borrowing or lending between dates 0 and \( T \). Then the present value of this option, \( V \), is simply

\[
V = e^{-rT} E^* [P_T - XP_T] = (1 - X)e^{-rT} E^* [P_T]
\]

\[
= (1 - X)e^{-rT} Pe^{rT} = (1 - X)P, \tag{1}
\]

where \( E^* \) is the risk-neutral expected value of the payoff at date \( T \).\(^1\) It is well known that this expectation can be computed by assuming that the underlying asset has an expected rate of return equal to the risk-free rate, \( r \), rather than its true expected rate of return.

Note that for this very simple case, the option’s value \( V = (1 - X)P \) does not depend on when exercise occurs; that is, the option’s value does not depend on date \( T \). What this tells us is that it does not matter when one decides to exercise the option.

Next, consider the same type of option but on a dividend-yielding asset. The underlying asset is assumed to continuously pay a dividend that yields a proportion, \( q \), of its value. For example, if the option is written on a stock index where the portfolio of stocks underlying the index pay a 3% dividend, then \( q = 0.03 \). Another example would be where the underlying asset is real estate. Residential real estate pays a dividend in the form of housing services. The value of the dividend equals the implicit rental rate of housing minus any required maintenance and other expense rate (Smith 1980). For example, if the annual rental rate equals 5% of the value of the house, then \( q = 0.05 \).

Again, assume the option is exercised at some arbitrary future date, \( T \); then the value of the option is

\[
V = e^{-rT} E^* [P_T - XP_T] = (1 - X)e^{-rT} E^* [P_T]
\]

\[
= (1 - X)e^{-rT} Pe^{(r-q)T} = (1 - X)Pe^{-(r-q)T}. \tag{2}
\]

Here, \( E^* [P_T] = Pe^{(r-q)T} \) is the risk-neutral expected value of the underlying asset price at date \( T \). This means the price of the asset is expected to appreciate at the rate \((r-q)\) under the risk-neutral measure. This occurs because, in equilibrium, the owner of the asset must obtain a total expected rate of return equal to \( r \), which is the case because the dividend, \( q \), plus price appreciation, \( r-q \), equals \( q + r - q = r \).

In the case of a dividend-yielding asset, the option’s value is a function of the exercise date \( T \). For \( q > 0 \), the value of the asset is maximized when \( T = 0 \),

\(^1\)In other words, the expectation is taken with respect to the equivalent martingale measure, not the true probability measure, of the asset’s date \( T \) probability distribution.
implying \( V = (1 - X)P \). Hence, the optimal exercise strategy for this option on a dividend-yielding asset is to exercise immediately.

Now, consider one additional case where the proportional exercise price \( X \) changes over time according to some deterministic function of time, \( X = x(t) \). For example, an employee may have the option to buy stock or a house at a discount and the discount changes as a function of the employee’s seniority or level of promotion. In this case, the value of the option is similar to the previous case:

\[
V = e^{-rT}E^*[P_T - X(T)P_T] = [1 - x(T)]e^{-rT}E^*[P_T] \\
= [1 - x(T)]e^{-rT}Pe^{(r-q)T} = [1 - x(T)]Pe^{-qT}.
\] (3)

Depending on the specification of \( x(T) \), it may be optimal to exercise the option immediately or at some future date. For example, suppose that \( x(t) = Ke^{-gt} \), so that the exercise price is assumed to decline at rate \( g \) from its initial value of \( K \). Then

\[
V = [1 - Ke^{-gT}]Pe^{-qT} = P[e^{-qT} - Ke^{-(q+g)T}].
\] (4)

Taking the derivative with respect to \( T \) gives

\[
\frac{\partial V}{\partial T} = P[-qe^{-qT} + (q + g)Ke^{-(q+g)T}] = 0,
\] (5)

or

\[
(q + g)Ke^{-(q+g)T} = qe^{-qT},
\] (6)

which implies that the optimal date to exercise this option is

\[
T^* = \frac{1}{g} \ln \left[ \frac{K}{q + g} \right].
\] (7)

Note that \( T^* > 0 \) if \( K(q + g)/q > 1 \), which occurs if \( g \) and the original \( K \) is sufficiently large. If, instead, \( K(q + g)/q < 1 \), it is optimal to exercise the option immediately. Figure I shows the optimal exercise date \( T^* \) with respect to different values of \( g \), assuming \( q = 0.03 \) and \( K = 0.9 \). It would be optimal for the option holder to exercise immediately if the value of \( g \) is close to zero. \( T^* \) reaches its maximum value when \( g = 0.02 \) and declines as the value of \( g \) increases. The track of \( T^* \) further implies that this option is attractive to both the employer and the employee. Companies generally allow new hires to exercise their stock options after one to six years of service (Huddart and Lang 1996), with this option, employees would delay the exercise voluntarily. In the early stage of employment, employees usually do not need to exercise the option and the optimal exercise date is far away. Employees usually gain the most seniority and promotions as they near retirement, when the optimal exercise date is only a few years away, and they then need to exercise the option.
There may be other factors, such as personal taxes, which might affect the optimal exercise date. But the preceding analysis should clarify the most important issues regarding the valuation and optimal exercise of this option.

III. An Application

An application of a proportional-exercise price option is found in China, where a housing savings program in cities allows state or semistate employees to buy their housing at a portion of the market price. Here, market price refers to the following: (a) the bid price where an employer contracts with a contractor to build the apartment building, (b) the estimated market price of comparable apartments in the free market where price is determined by supply and demand as in a market economy, and (c) the price at which the employer or employee buys the apartment from the free market.

To assist employees in buying their housing and to accelerate housing privatization, China started a nationwide housing savings program in 1995 (Gu and Colwell 1997; Gu and Yang 2000). Under this program, employees of government agencies, state enterprises, universities, hospitals, and some semistate companies are eligible for the housing subsidy. According to the program, 5% or more of an employee’s monthly salary is automatically deducted, and the employer also contributes an equal amount (some employers offer to match up to 10%). That is, if an employee contributes 50 yuan, the employer would also contribute 50 yuan,
providing a one-for-one match. These employee and employer contributions are deposited in state banks and the principal and interest in this savings account belong to the employee. The employee can only use these funds for purchasing his/her housing unit, and the bank can only use the funds for housing investments within the city.

Only employees who join this housing savings program can purchase public (government-owned) housing at a reduction to the market price. The price reduction generally ranges from 60% to 90% (the buyer paying 10% to 40%) of the market price, depending on the employee’s occupational rank, seniority, performance, and other factors. As an example, for each year an employee works for a state employer, the proportional exercise price may be reduced by 2 percentage points. Being promoted to the rank of section director may worth an additional 15 to 20 percentage points.

The reduced price applies only to housing for eligible categories. For example, in Beijing as of 2000, a senior staff member is eligible for a 75-square-meter unit; a section director, 120-square-meter unit; and a department director, 150-square-meter unit. Until recently, only the rationed housing in which the participant resided or some other government-owned housing was available for purchase under the program. Since 1998, participants may buy new or old housing from the employer or from the market. But few buy from the market if the employer has the housing because employers offer lower prices, better service, and usually better locations. An employee may buy a larger or better quality housing unit, but he/she will have to pay full market price for the additional size or quality.

Different cities and employers may use different parameters for calculating the price reduction, but the total reduction rates are usually similar among employees with the same rank and seniority in different cities. Market prices of housing can vary significantly among cities. At the time of housing purchase, the employee may also qualify for a subsidized mortgage in the form of a long-term loan from state banks at an interest rate substantially below the market rate.

After purchasing a house, the employee can remain in the housing savings program. The salary deduction, matched by the employer, continues and the savings can be used toward a future housing purchase. The contributions continue to be kept in a state bank in the city where the employee works, and deduction points earned thereafter (additional years of seniority, higher rank) apply to the next housing purchase. In case of relocation, the employee can sell his/her home in the free market and buy one in the new location at the market price; other things being equal, there would be no discount subsidy. If the employee gains additional years of seniority or a higher rank, the additional discount subsidy would apply. When an employee retires, all of the balance and interest in the account belong to the employee. In case of an employee’s death, all of the balance and interest generated in the account are passed to his/her heirs. Similarly, the purchased housing unit and the outstanding mortgage is part of the employee’s bequest. The government does
not tax balances or interests in this savings program in order to stimulate housing privatization. In this article I try to value the option to a participant until its exercise; the option following one’s first home purchase can be valued in a similar way. I believe the housing savings program is transitional.

Let \( S_T \) be the date \( T \) value of the savings in an employee’s housing savings account begun at date 0. This includes both the employee’s deposit and the employer’s match, as well as the returns on these contributions until date \( T \). This can be expressed as

\[
S_T = \sum_{i=0}^{T} C_i e^{r(T-i)} = \sum_{i=0}^{T} C_0 e^{wi} e^{r(T-i)} = C_0 e^{rT} \sum_{i=0}^{T} e^{(w-r)i}
\]

where \( C_i = C_0 e^{wi} \), with \( C_0 \) being the contribution made at date 0 and \( w \) being the per-period wage growth rate from date 0 to date \( T \). The value of the subsidy on the low interest mortgage loan can be expressed as the product of a proportion \( \gamma \) and the difference between the price paid for the house and the savings account value.

\[
Sub_T = \gamma (x(T)P - S_T).
\]

For example, if the amount borrowed, \( x(T)P_T - S_T \), is a mortgage with a maturity of \( \tau \) years made at the subsidized interest rate of \( r_s \), the monthly payment on this subsidized mortgage, \( PM \), would equal

\[
PM = (x(T)P_T - S_T) \frac{\frac{1}{12}r_s}{1 - (1 + \frac{1}{12}r_s)^{12\tau}}.
\]

At the competitive mortgage interest rate of \( r \), the present value of this stream of mortgage payments equals

\[
PV = PMT \frac{1 - (1 + \frac{1}{12}r)^{12\tau}}{\frac{1}{12}r} = (x(T)P_T - S_T) \frac{r_s}{r} \frac{1 - (1 + \frac{1}{12}r_s)^{12\tau}}{1 - (1 + \frac{1}{12}r_s)^{12\tau}}.
\]

Thus, in this example,

\[
\gamma = 1 - \frac{r_s}{r} \frac{1 - (1 + \frac{1}{12}r)^{12\tau}}{1 - (1 + \frac{1}{12}r_s)^{12\tau}}.
\]

Assuming \( x(t) = Ke^{-gt} \) and using risk-neutral valuation, the present value of the
The subsidy is

$$Sub_0 = \gamma \left( Ke^{-qT} e^{-rT} E^* [P_T] - C_0 \left[ \frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right] \right)$$

$$= \gamma \left( Ke^{-(q+g)T} P - C_0 \left[ \frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right] \right). \tag{10}$$

From equation (8) it is clear that the longer an employee waits to purchase a house, the greater is the value of the subsidy received from his or her employer in the form of matching savings contributions. However, from equation (10), one can see that waiting lowers the employee’s mortgage loan subsidy. Assuming and incorporating both of these factors in equation (4) gives

$$V = \left[ 1 - (1 - \gamma) Ke^{-qT} \right] e^{-qT} P - \gamma C_0 \left[ \frac{1 - e^{(w-r)(T+1)}}{1 - e^{w-r}} \right]. \tag{4'}$$

Taking the derivative with respect to $T$ gives

$$\frac{\partial V}{\partial T} = [(q + g)(1 - \gamma) Ke^{-qT} - q] e^{-qT} P + \frac{\gamma C_0 (w - r) e^{(w-r)(T+1)}}{1 - e^{w-r}} = 0, \tag{5'}$$

or

$$[q - (q + g)(1 - \gamma) Ke^{-qT}] e^{-qT} P = \frac{\gamma C_0 (w - r) e^{(w-r)(T+1)}}{1 - e^{w-r}}, \tag{6'}$$

which implies that the optimal date to exercise one’s option can be found by solving for

$$T^* = f(q, g, k, g, w, r) \tag{7'}$$

in (6'). In general, this optimal date must be solved numerically.

Sensitivity analyses of the optimal exercise date with respect to the parameters are summarized in Table 1. A “$>$” indicates that an increase in the value of the

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Note: $T$ is the exercise date, $q$ is the dividend yield as a proportion of the asset’s value, $\gamma = Sub_T / (S(T)P - S_T)$, $Sub$ is the subsidy on the low interest mortgage loan, $S$ is the savings account value, $w$ is the wage-growth rate, $C_0$ is the amount of contribution, $K$ is the initial exercise price, $P$ is the price of the underlying asset, $r$ is the risk-free interest rate, and $g$ is the decline rate of exercise price.
parameter will cause an increase in the value of $T^*$ or will make the optimal exercise date farther away. On the other hand, a “$<$” indicates that an increase in the value of the parameter will result in a decrease in the value of $T^*$ or will make the optimal exercise date nearer. The effect of the decline rate $g$ of the exercise price on the optimal exercise date $T^*$ is undetermined. The optimal exercise date with respect to different $g$ values from 0.05 to 0.5 is depicted in Figure II. It is optimal to exercise the option immediately when $g < 0.05$. $T^*$ reaches its maximum value when $g = 0.2$ and declines as the value of $g$ increases. These results show the program provides participants with the incentive to purchase apartments sooner rather than later.

In the preceding analysis, I assume the proportional discount is reduced at a continuous rate $g$. However, if the decline were to occur suddenly, as might happen if the employee received a promotion, the problem could be solved but a (binomial) numerical solution technique would be required.  

To illustrate the value of this option and the optimal exercise date for the simple case in which $g$ is assumed to be a continuous rate, consider the following example. From late 1998 to early 2000, the average of the interest rates was 5.742% for program participants and 6.94% for nonparticipants on a twenty-five-year mortgage, $P = 170,000$ yuan (for a standard three-bedroom, 100-square-meter apartment). $C_0 = 900$ yuan per year; $w = 0.075$ per year; and $r = 0.06$ per year.

For illustrations of such a technique, see Cox, Ross, and Rubinstein (1979) and Boyle (1988). 

These are rates reported by the People’s Bank of China.
Assuming \( q = 0.06 \) per year, \( K = 0.67 \), and the annual reduction in the proportional price since the current date is \( g = 0.14 \) (this, for example, will allow \( K \) to decline to 0.33 five years from the current date). Given these parameter values, \( \gamma = 0.182 \). From equation (5') it can be calculated that the optimal time for exercising the option is 4.15 years.

From (4') I calculate the current value of the housing purchase option with the parameters described previously and come up with a figure of 98,887 yuan. Hence, this subsidy is equal to 58% of the current market price of the house. If the employee exercises the option immediately (at time 0), he or she could buy an apartment at 67% of the market price of public housing. The value of this option is \((1 − 0.67) \times 170,000 = 56,100\) yuan, which is 42,787 yuan smaller than 98,887 yuan.

**IV. Conclusion**

A call option having an exercise price that is a proportion of the underlying asset’s price can be an attractive part of an employee compensation package. If the employer is a private firm, this option could be written on the equity of the firm’s shareholders, making such compensation sensitive to performance. Alternatively, the employer could provide proportional-exercise price options on real estate. Such a program could be used to promote private homeownership in a particular geographic area.

In this article I show how such options can be valued using a risk-neutral pricing approach. My model’s results imply that if the exercise price of the option is a fixed proportion of the underlying asset, it is optimal to exercise the option immediately if the asset pays dividends not received by the option owner. However, if the proportional exercise price declines at a sufficiently high rate, the optimal exercise of the option occurs at a specified future date.

The model is used to value the option to purchase housing under a program offered by governments in Chinese cities. Under this program, state employees can purchase a residence at a price that is a proportion of its market value. The purchase proportion declines with the employee’s seniority and rank. A simple numerical calculation shows that this real estate option can be of significant value and should provide a strong incentive for housing privatization.

**References**


Gu, Y. A. and T. Yang, 2000, Rent or buy and when to buy, Paper presented at the ASSA-AREUEA 2000 annual meetings, Boston.
