A Generalized Framework for Quantity Discount Pricing Schedules

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ABSTRACT

The use of price to influence a buyer’s purchasing behavior and thus improve supply chain coordination has received considerable attention. The vendor and buyer are independent economic entities, each maximizing its own profit. We consider the case of a buyer with fixed annual demand, independent of cost. The vendor’s objective is to set a price schedule that encourages the buyer to raise its order quantity, increasing the vendor’s profits. We present a unified treatment of the problem, categorize different variations, and provide a common solution procedure for all cases.

Subject Areas: Materials Management, Mathematical Programming/Optimization, and Pricing.

INTRODUCTION

The treatment of quantity discounts in the inventory and purchasing literature has tended to be from the buyer’s perspective. A few authors (Crowther, 1967; Dolan, 1978; Drezner & Wesolowsky, 1989; Kim & Hwang, 1988; Lal & Staelin, 1984), however, have examined the question of how the vendor should best set the discount schedule. Most of that work deals with the case of a single vendor selling to a single buyer, as does this paper. Our intent is to categorize the major variations on the problem and to develop a consistent procedure for finding the vendor’s optimal discount schedule.

Consider the case of a vendor that has negotiated a long-term contract with a customer for regular purchases of a single item, which the vendor either manufactures or resells. The buyer selects its economic order quantity (EOQ) as the standard purchase quantity, but the vendor would like to move the buyer to an order size more efficient from the vendor’s perspective. As noted by Rubin and Carter (1990), among others, the ideal is for the two parties to negotiate the vendor’s profit margin, determine an order quantity that minimizes the sum of the vendor’s and buyer’s costs (other than the vendor’s margin), and then negotiate a method of

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splitting the overall savings between the parties. In practice, however, cooperation at this level may not be possible, due to legal prohibitions, reluctance of either party to disclose cost data, or other managerial concerns. The vendor therefore seeks other means to induce the buyer to shift to a different order size.

Assuming that the vendor prefers larger orders than does the buyer, a reasonable choice for the vendor is to offer a quantity discount. Most authors consider one (but not both) of the two common discount policies: incremental discounts, in which a lower unit price is applied to all units in excess of a qualifying amount (the breakpoint); or all-units discounts, in which the lower price is applied to the entire order once the order exceeds the breakpoint. Monahan (1984) and Lee and Rosenblatt (1986) take a slightly different approach, offering a discount only for a specific quantity rather than for all quantities above some cutoff. Lal and Staelin (1984) use a continuous decaying function of quantity for the unit price, which can be approximated at implementation by an incremental discount with multiple breakpoints.

When a single buyer is involved, only a single discount range is required, and so the vendor needs to determine just two parameters, the breakpoint and the discount price. How they are determined depends on a number of modeling assumptions. In the following sections, we categorize the different variations on the problem (and note which variants are considered by which authors), examine the problem from first the buyer’s and then the vendor’s perspective, present a solution procedure, and illustrate the solution procedure through a numerical example.

PROBLEM FRAMEWORK

Assumptions

Common assumptions

Certain assumptions are common to the bulk of the work cited above. A single vendor sells a single product to a single buyer. The buyer’s demand is deterministic, is constant over the indefinite future, and occurs at a steady rate. A critical assumption is that discounts do not affect the buyer’s annual demand. Thus the vendor’s sole purpose in offering a discount is to increase the periodic order quantity, not to increase overall sales. Weng (1995) and Abad (1994) deviate from this assumption by tying the buyer’s demand to the discount.

Replenishment lead times are assumed to be deterministic and hence can be neglected. Neither stockouts nor backorders are allowed. The buyer is assumed to use an EOQ. In addition to the purchase price, the buyer incurs a fixed cost for each order placed, independent of the order size, and incurs material holding costs that increase with order size. Lal and Staelin (1984) start with these assumptions but then generalize by allowing order placement cost to vary with order size.

The vendor additionally is assumed to have adequate capacity to meet the buyer’s annual demand, and to have negotiated a unit price adequate to cover the vendor’s variable cost. The vendor incurs a fixed charge for each replenishment order, which may combine acquisition costs (Crowther, 1967), processing costs (Dolan, 1978), and setup costs. The vendor is also assumed to be able to estimate the buyer’s cost parameters. This knowledge is necessary for the vendor to
anticipate the buyer’s reaction to any possible discount schedule. Contrary to these assumptions, Corbett and de Groote (2000) and Burnetas, Gilbert, and Smith (2002) assume that the buyer’s demand is stochastic and that the vendor has incomplete knowledge of the buyer’s cost parameters.

We assume (as do the authors cited in Table 1) that the order quantity and discount price are divisible rather than discrete, and that the buyer can be induced to switch from one order size to another if the buyer at least breaks even. These assumptions are for mathematical convenience, and can be dealt with through minor adjustments to the solution with little impact on total cost.

Finally, we assume that, absent any discounts, the negotiated contract is profitable for the vendor, meaning that the vendor’s total annual cost is strictly less than the annual revenue at the contracted price. This allows us to avoid certain pathological situations, such as an optimal discounted price set below the vendor’s variable cost to mitigate unavoidable losses on setups and inventory.

**Points of departure**

One obvious difference among the cited works is whether they consider all-units or incremental discounts. Another difference is their treatment of holding costs on both sides. On the buyer’s side, the issue is whether holding costs per unit per year are fixed or proportional to the average unit price paid; the former is independent of discounts, while the latter is clearly influenced by discounts. On the vendor’s side, three distinct possibilities emerge. One version has the vendor incurring no holding costs; only order setup and direct production costs apply. A second version has the vendor incurring holding costs, proportional to the size of the order, during the portion of each order cycle in which the item is being produced. This version is reasonable if the item is manufactured, rather than acquired; if the time to fill an order is nontrivial; and if the vendor incurs work-in-process costs during production. The question of fixed versus proportional holding costs does not arise here, since the vendor’s costs are independent of any discounts. A third variant has the vendor receiving a credit proportional to the order size, representing either a reduction in capital cost stemming from receipt of payment for each order, or, for vendors who make to stock, a reduction in holding costs by shifting inventory to the buyer.

Finally, most authors assume that the vendor produces or procures the item on a lot-for-lot basis. Some authors (Abad, 1994; Weng, 1995), though, assume instead that the vendor produces or purchases an integer multiple of the buyer’s order quantity; the integer multiple to use is part of the vendor’s pricing decision.

Table 1 shows the combination of assumptions used in each of several pertinent papers. All those tabulated assume lot-for-lot production. Though all save Monahan (1984) and Lee and Rosenblatt (1986) delve into the problem of setting a single discount schedule for use with multiple buyers, the cited works all cover the case of interest here, that of a single buyer. We will see subsequently that while the type of discount has a significant effect on the solution process, the types of buyer and seller holding costs, while affecting the actual solution, do not materially affect the solution process.
Table 1: Assumptions used by various authors.

<table>
<thead>
<tr>
<th>Authors</th>
<th>Discount</th>
<th>Buyer Holding</th>
<th>Vendor Holding</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drezner &amp; Wesolowsky</td>
<td>All-Units</td>
<td>Proportional</td>
<td>Credit</td>
</tr>
<tr>
<td>Kim &amp; Hwang</td>
<td>Incremental</td>
<td>Proportional</td>
<td>None</td>
</tr>
<tr>
<td>Lal &amp; Staelin</td>
<td>Continuous</td>
<td>Fixed</td>
<td>Credit</td>
</tr>
<tr>
<td>Monahan</td>
<td>Single Quantity</td>
<td>Proportional</td>
<td>None</td>
</tr>
<tr>
<td>Lee &amp; Rosenblatt</td>
<td>Single Quantity</td>
<td>Proportional</td>
<td>Proportional</td>
</tr>
</tbody>
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Notation

**Buyer’s parameters**

- $D$: buyer’s annual demand (units)
- $S_b$: buyer’s order setup cost
- $H_b$: buyer’s holding cost (absolute) per unit per year
- $H'_b$: buyer’s holding cost (fraction of price) per unit per year
- $Q_e$: buyer’s economic order quantity without discounts
- $B$: buyer’s annual cost using the undiscounted EOQ

**Vendor’s parameters**

- $P$: unit price negotiated with buyer (before discounts)
- $S_v$: vendor’s setup cost per order
- $H_v$: vendor’s holding cost (absolute) per unit per year
- $H'_v$: vendor’s rate of return on capital (fraction)
- $C$: vendor’s direct cost per unit to produce/acquire the item
- $R$: vendor’s annual production/acquisition rate operating at full capacity

**Variables**

- $q$: buyer’s order quantity
- $p$: vendor’s discounted price
- $\bar{q}$: breakpoint above which orders qualify for discount

**Functions**

- $ap(q; p, \bar{q})$: average price per unit for an order of $q$ units given discount policy $(p, \bar{q})$
- $ceoq(p, \bar{q})$: buyer’s conditional economic order quantity with discount policy $(p, \bar{q})$
- $bc(q; p, \bar{q})$: buyer’s annual costs at order size $q$ given discount policy $(p, \bar{q})$
- $vc(q; p, \bar{q})$: vendor’s annual costs at order size $q$ given discount policy $(p, \bar{q})$

**Buyer’s problem.** The buyer seeks to minimize total annual costs, given by

$$bc(q; p, \bar{q}) = \frac{DS_b}{q} + D \times ap(q; p, \bar{q}) + \frac{(H_b + H'_b \times ap(q; p, \bar{q}))q}{2}.$$  \hspace{1cm} (1)

Traditionally, authors have assumed that the buyer’s inventory holding costs are either absolute ($H'_b = 0$) or proportional to the value of the material ($H'_b = 0$),
Table 2: Buyer cost coefficients (when discounts apply).

<table>
<thead>
<tr>
<th></th>
<th>All-Units</th>
<th>Incremental</th>
</tr>
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<tbody>
<tr>
<td>$\alpha$</td>
<td>$DS_b$</td>
<td>$D(S_b + (P - p)\bar{q})$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$Dp$</td>
<td>$Dp + \frac{1}{2}H_b'(P - p)\bar{q}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\frac{1}{2}(H_b + H'_b p)$</td>
<td>$\frac{1}{2}(H_b + H'_b p)$</td>
</tr>
</tbody>
</table>

but there is no mathematical difficulty in accommodating both types of cost. This allows, for example, the inclusion of capital costs and shelf loss (proportional) as well as warehousing expenses (absolute).

In the absence of discounts, $ap(q; p, \bar{q}) = P$ and so the cost formula has the hyperbolic form

$$\frac{\alpha}{q} + \beta + \gamma q,$$

with $\alpha = DS_b$, $\beta = DP$, and $\gamma = (H_b + H'_b P)/2$. Throughout the paper, we will repeatedly encounter functions of the form (2) with domain $q > 0$. When $\alpha > 0$, as will always be the case here, the function is strictly convex and thus unimodal, with global minimum at

$$\sqrt{\alpha/\gamma},$$

when $\gamma > 0$. (When $\gamma = 0$ the function is asymptotically minimized as $q \to \infty$, and when $\gamma < 0$ the function is unbounded below.) This leads to

$$Q_e = \sqrt{\frac{2DS_b}{H_b + H'_b P}},$$

which reduces to the well-known Wilson EOQ formula when one of $H_b$ or $H'_b$ is set to zero.

The introduction of discounts changes the average unit price. In the all-units case,

$$ap(q; p, \bar{q}) = \begin{cases} P, & 0 < q < \bar{q} \\ p, & q \geq \bar{q} \end{cases};$$

in the incremental case,

$$ap(q; p, \bar{q}) = \begin{cases} P, & 0 < q < \bar{q} \\ p + \frac{(P - p)\bar{q}}{q}, & q \geq \bar{q} \end{cases}. \tag{4}$$

In both cases, $bc(q; p, \bar{q})$ retains the form (2) over each quantity range, with coefficients (for the discount range $q \geq \bar{q}$) as indicated in Table 2.

Figures 1(a) and (b) show the buyer’s annual total cost as a function of the order size, assuming an all-units discount. Cost functions using both the originally negotiated price (top curve) and the discounted price (bottom curve) are displayed. The original EOQ and conditional EOQ are indicated. The conditional EOQ is the economic order quantity based on the discounted price; it minimizes the actual
buyer cost only if it is feasible (i.e., large enough to qualify for the discount). The thick curve in the figures is the actual annual cost to the buyer; the jump discontinuity occurs at the breakpoint.

With all-units discounts, the total cost curves are nested. We assume that the vendor wants the buyer to increase its order size and thus will set the breakpoint above the original EOQ. It follows that there are only three order quantities the buyer need consider: the original EOQ \((Q_e)\); the conditional EOQ \(( \text{eoq}(p, \bar{q}) )\); and the breakpoint \((\bar{q})\).

**Remark 1** If the conditional EOQ is feasible, it is the optimal choice for the buyer. Otherwise, the buyer will choose the less costly of the original EOQ and the breakpoint.

**Remark 2** When the buyer’s holding cost is absolute, the conditional EOQ is the same as the original EOQ (since neither \(\alpha\) nor \(\gamma\) depends on the discount price \(p\) when \(H'_b = 0\)) and thus presumably is infeasible. (This is visible in Figure 1(a).)

Figures 2(a) and (b) show the undiscounted and conditional total costs assuming an incremental discount policy. The conditional cost curve extrapolates (backward) the formula for the average unit price of qualifying orders; that is, the dotted portion uses \(p + (P - p)\bar{q} / q\) as the unit price for orders below the qualifying amount. Unlike the all-units case, the cost curves are not nested, but rather intersect. As is visible in Figure 2(b), the conditional EOQ, even if feasible, is no longer automatically the buyer’s best choice.

**Remark 3** Since the composite total cost function (the thick curve in Figure 2) is continuous when discounts are incremental, the breakpoint will never be the buyer’s best choice. (This observation is straightforward: The cost at the breakpoint is the same with or without the discount, and this is more than the original EOQ’s cost.)

The buyer will choose the less costly of the original EOQ and, if feasible, the conditional EOQ.
Figure 2: Buyer annual cost with incremental discounts.

Vendor’s problem. We begin by assuming that the vendor operates lot-for-lot, and defer the question of lot aggregation to later. The vendor’s task is to steer the buyer, through a carefully crafted discount schedule, to an order quantity \( q \) that minimizes the vendor’s annual cost, given by

\[
\text{vc}(q; p, \bar{q}) = DS_v \frac{q}{q} + D(C + P - ap(q; p, \bar{q})) + \frac{1}{2} H_v D \frac{q}{R} q
\]

\[
- \frac{1}{2} H'_v((ap(q; p, \bar{q}) - C)q - S_v).
\]  

The vendor’s setup cost (first term) is straightforward, but the other terms bear explanation. In the second term, we charge the lost revenue on the (captive) demand \( D \) as a cost, along with the variable cost of producing or procuring the product. The third term expresses the vendor’s holding cost, based on the assumption that the vendor builds the order at a constant rate \( R \) over a time period \( q/R \), incurring holding costs until the order is completed and shipped, and does so \( D/q \) times per year. The final term expresses the vendor’s capital credit. It assumes that the vendor collects a payment of \( ap(q; p, \bar{q})q \) each time an order arrives, nets out the order production/acquisition cost \( Cq \) and setup cost \( S_v \), and draws down the balance linearly over time to cover other expenses (so that, on average, half the margin is invested). Drezner and Wesolowsky (1989), citing Crowther (1967) and Dolan (1987), omit the setup cost adjustment in the capital credit. Lal and Staelin (1984) write the last term as \(-H_s q/2\) and define \( H_s \) as the “seller’s cost of capital per year” without explaining how it is calculated.

The cost function \( \text{vc}(q; p, \bar{q}) \) again takes the hyperbolic form (2). Table 3 lists the coefficients for both types of discounts. Previous work has assumed that one or both of \( H_v \) and \( H'_v \) are zero, but we can accommodate in one model both holding costs for work-in-process inventory and a capital credit for accelerated payments.

We noted earlier that in the case of all-units discounts, the buyer will choose the conditional EOQ if it is feasible and will otherwise choose the less expensive of
the original EOQ and the breakpoint. It turns out that it is never in the best interest of the vendor to have the conditional EOQ be feasible in the all-units case. (This result is stated, without formal proof, as Property 1 in Drezner and Wesolowsky, 1989.)

**Proposition 1**

The vendor should never design an all-units discount policy that makes the buyer’s conditional EOQ feasible.

**Proof** In the all-units discount case, the conditional EOQ is given by

\[
ceoq(p, \bar{q}) = \sqrt{\frac{2DS_b}{H_b + H_b'p}},
\]

where one (but not both) of \( H_b \) and \( H_b' \) may be zero, and is independent of the breakpoint \( \bar{q} \). We dispense immediately with the case \( H_b' = 0 \) (buyer’s holding costs are absolute) by noting that the conditional EOQ matches the original EOQ; since the vendor’s intent was to stimulate a larger order size, clearly the vendor needs to set the breakpoint \( \bar{q} \) above the EOQ. Henceforth we assume that the buyer incurs some proportional holding costs (\( H_b' > 0 \)), and as a consequence that the conditional EOQ is a decreasing function of \( p \).

Consider any discount policy \((p, \bar{q})\) and suppose that the conditional EOQ is feasible and that the policy minimizes the vendor’s total cost. Because the buyer’s total cost functions with and without discount are strictly convex and nested, the buyer’s total cost at the conditional EOQ is strictly less than it is at the original EOQ:

\[
bc(ceoq(p, \bar{q}); p, \bar{q}) < B = bc(Q_e; p, \bar{q}).
\]

Since the buyer’s cost function is continuous in both \( p \) and \( \bar{q} \), the vendor can make small adjustments in either one without raising the buyer’s cost enough to shift the buyer back to the original EOQ.

Now suppose that the vendor’s total cost is a decreasing function of order size at the conditional EOQ, that is,

\[
\frac{d}{dq}vc(q; p, \bar{q})|_{q=ceoq(p,\bar{q})} < 0.
\]

Then if the vendor leaves the discount price at \( p \) and sets the breakpoint \( \bar{q} \) slightly greater than \( ceoq(p, \bar{q}) \), the conditional EOQ is infeasible, and the breakpoint remains more attractive to the buyer than is the original EOQ while being less
costly to the vendor. Hence for the original policy to be optimal for the vendor, the partial derivative in (7) must be nonnegative. Assuming that to be true, consider the total derivative with respect to $p$ of the vendor’s cost function, restricted to the conditional EOQ:

$$
\frac{d}{dp} vc(ceoq(p, \bar{q}); p, \bar{q}) = \frac{d}{dq} vc(q; p, \bar{q})|_{q=ceoq(p,\bar{q})} \times \frac{d}{dp} ceoq(p, \bar{q})
+ \frac{d}{dp} vc(q; p, \bar{q})|_{q=ceoq(p,\bar{q})}.
$$

(8)

The first of the three derivatives we have just assumed to be nonnegative, the second we observed previously to be negative, and the third is strictly negative: from (5), setting $ap(q; p, \bar{q}) = p$, $d/dp vc(q; p, \bar{q}) = -D - \frac{1}{2} H'_{v} < 0$. Thus the total derivative in (8) is negative. This implies that a small increase in the discounted price $p$ will decrease the conditional EOQ slightly, and will reduce the vendor’s total cost while keeping the conditional EOQ more attractive to the buyer than was the original EOQ. This contradicts the assumption that the original policy was optimal. It therefore follows that an all-units discount policy that leaves the conditional EOQ feasible can never be optimal for the vendor.

Before proceeding, we note that both $bc(q; p, \bar{q})$ as given in (1) and $vc(q; p, \bar{q})$ as given in (5) depend on the discount policy $(p, \bar{q})$ only through the average price $ap(q; p, \bar{q})$. Let $bc(q, \bar{p})$ and $vc(q, \bar{p})$ denote $bc(q; p, \bar{q})$ and $vc(q; p, \bar{q})$ respectively, with $\bar{p}$ substituting for $ap(q; p, \bar{q})$. We will pose the vendor’s problem in two phases. The first phase is to determine the vendor’s ideal buyer order size $q$ and the ideal discounted average price $\bar{p}$ to charge at that order size, without regard to whether the discount is posed as all-units or incremental. The second phase is to determine a discount policy $(p, \bar{q})$ of the appropriate type that will drive the buyer to order quantity $q$ with average price $\bar{p}$.

An important property of this approach is that the outcome does not depend on the type of discount offered. More precisely, the breakpoint $\bar{q}$ and discounted unit price $p$ will differ between all-units and incremental discounts, but the effective average price paid by the buyer, the buyer’s ultimate order quantity, and the vendor’s ultimate annual cost will be the same either way. Weng (1995) reaches a similar conclusion in a somewhat different scenario, although he initially treats the all-units and incremental cases separately.

**Vendor’s optimal quantity and average price.** The vendor’s ideal order size and average price form the optimal solution to the following mathematical programming problem:

$$
\begin{align*}
\text{minimize} & \quad vc(q, \bar{p}) \\
\text{s.t.} & \quad bc(q, \bar{p}) \leq B \\
& \quad \bar{p} \geq C \\
& \quad q \geq Q_{e}.
\end{align*}
$$

(9)

The first constraint requires the target order quantity, at the discounted average price, to be attractive to the buyer, to the extent that the buyer must at least break even switching to that quantity. This prevents the vendor from raising the price.
As noted above, the vendor will in practice have to “sweeten the pot” a bit to induce the buyer to shift. The second constraint requires that the vendor at least recover variable costs. The final constraint requires an order size no less than the undiscounted EOQ. Problem (9) is well posed in the sense that it has an optimum.

If the optimal solution sets \( \tilde{p} = P \), then the vendor cannot improve its costs by offering any discount. Recall the earlier assumption that the vendor’s initial contract (equivalent to \( q = Q_e, \tilde{p} = P \) here) is profitable. Barring a pathological case in which \( H' \) is so large that capital credits by themselves are worth more than the contract’s profits, \( \tilde{p} = C \) is unprofitable, since there are no margins to compensate for setups and inventory. Henceforth we will assume that any solution of (9) has \( C < \tilde{p} < P \). A consequence of this is that \( q > Q_e \); if the vendor improves its profits by offering a discount of any size, it must do so by inducing the buyer to increase, not decrease, its order quantity.

**Buyer does not gain.** Before proceeding to the second phase, we must take note of an obvious but important result. In any optimal solution of (9), the first constraint will be binding. That is, within the limitations of the mathematical program (which ignores the need to provide an incentive to the buyer to switch), the buyer will break even but not gain from an optimal discount policy.

**Proposition 2** Any optimal solution \((\tilde{p}, q)\) of (9) that improves the vendor’s results over the undiscounted case satisfies the first constraint of the respective problem as an equality.

**Proof** Suppose \((\tilde{p}, q)\) is an optimal solution of (9) such that \( bc(q, \tilde{p}) < B \). For the vendor to see improvement, we must have \( C < \tilde{p} < P \), and so for sufficiently small \( \varepsilon > 0 \) we have \( C < p + \varepsilon < P \) and, because \( bc \) is continuous with respect to \( \tilde{p} \), \( bc(q, \tilde{p} + \varepsilon) < B \). From Table 3, clearly \( vc(q, \tilde{p}) \) is a decreasing function of \( p \) for qualifying order sizes \( q > \bar{q} \), and so \( vc(q, \tilde{p} + \varepsilon) < vc(q, \tilde{p}) \), contradicting the assumption of optimality.

**Vendor’s optimal discount schedule: All-units.** Let \((q, \tilde{p})\) be an optimal solution to (9), and assume that the vendor’s preference is to use an all-units discount. Since the average and marginal prices are the same with an all-units discount, we must choose \( p = \tilde{p} \). In view of Proposition 1, the vendor’s target quantity must be the breakpoint, and so \( \bar{q} = q \). In other words, the optimal solution to (9) is the optimal discount policy. What remains is to verify that the buyer will in fact order \( q \) under this policy; that is, we must check that \( c eoq(p, \bar{q}) = c eoq(\tilde{p}, q) \leq \bar{q} = q \). This is a direct consequence of Proposition 2. Since the cost curves are nested in the all-units case, \( bc(Q_e, \tilde{p}) < bc(Q_e, p) = B \). Because \( bc(\cdot, \tilde{p}) \) is strictly convex as a function of the first argument, \( bc(q, \tilde{p}) = B > bc(Q_e, \tilde{p}) \) and \( q > Q_e \) together imply that \( bc(\cdot, \tilde{p}) \) is rising at \( q \) and hence that \( q > c eoq(\tilde{p}, q) \).

**Vendor’s optimal discount schedule: Incremental.** The vendor’s problem with incremental discounts differs from the all-units version in that the vendor steers the buyer not to the breakpoint (which is never attractive to the buyer) but rather to the conditional EOQ. Thus the second phase is to find a breakpoint \( \bar{q} \) and unit
price $p$ such that $\text{ceoq}(p, \bar{q}) = q$ and $ap(q; p, \bar{q}) = \bar{p}$. The second condition allows us to express $\bar{q}$ in terms of $\bar{p}$ and $q$. From (4),

$$\bar{p} = ap(q; p, \bar{q}) = p + \frac{(P - p)\bar{q}}{q}$$

implies that

$$\bar{q} = \frac{(\bar{p} - p)q}{P - p}.$$  \hspace{1cm} (10)

From (3) and Table 2, the conditional EOQ in the incremental case is

$$\text{ceoq}(p, \bar{q}) = \sqrt{\frac{2D(S_b + (P - p)\bar{q})}{H_b + H_b p}},$$

so the first condition (after substituting the right-hand side of (10) for $\bar{q}$) becomes

$$\sqrt{\frac{2D(S_b + (\bar{p} - p)q)}{H_b + H_b p}} = q.$$  

The solution to that equation is

$$p = \frac{2D(S_b + \bar{p}q) - H_b q^2}{q(2D + H_b q)}.$$ \hspace{1cm} (11)

**Lot aggregation.** To this point, we have assumed that the vendor produces or acquires the item in the same size lots that the buyer orders. We now turn to the case where the vendor attempts to reduce setup costs by producing or purchasing $N$ lots at a time, for some integer $N > 1$. Two changes need to be made to the vendor’s total cost function (5). First, the vendor’s setup costs need to be divided into two parts: those, such as paperwork processing, incurred each time a buyer replenishment occurs (which we represent by $S'_v$); and those, such as production setups, that occur once per $N$ buyer orders (which we represent by $S''_v$). Second, the vendor’s inventory holding costs now must reflect not only work-in-process (WIP) inventory during a production cycle (assuming the item is manufactured) but also the carrying of inventory between buyer replenishments. The WIP inventory term in (5) remains unchanged. We assume that the vendor produces or purchases $Nq$ units just in time to ship $q$ of them, so that the maximum inventory carried is $(N - 1)q$. The on-hand inventory level is then a step function, dropping $q$ units every $q/D$ years, until it reaches zero and remains there for $q/D$ years. The average on-hand inventory is $\frac{N-1}{2}q$, charged at the same holding rate $H_v$ as the WIP inventory is. Note that the capital credit term is unaffected, since payments still arrive with each replenishment order. The vendor’s adjusted annual cost function is

$$vc(q; p, \bar{q}) = \frac{DS'_v}{q} + \frac{DS''_v}{Nq} + D(C + P - ap(q; p, \bar{q})) + \frac{1}{2}H_v\frac{D}{R}q$$

$$+ \frac{1}{2}H_v(N - 1)q - \frac{1}{2}H'_v((ap(q; p, \bar{q}) - C)q - S_v).$$ \hspace{1cm} (12)
For fixed $N$, (12) has the same basic form as (5). More precisely, if we replace $S_v$ and $H_v$ in (5) with $S'_v + \frac{S_v}{N}$ and $H_v \left[ 1 + \frac{(N-1)R}{D} \right]$ respectively, we have (12).

**SOLUTION PROCEDURE**

Again, we start with a lot-for-lot assumption and subsequently generalize. The second phase of the vendor problem, as described above, is trivial for the all-units case and a simple computation for the incremental case. The work lies in solving (9). Proposition 2 allows us to reduce the problem to one dimension. We use the first constraint of (9) to express $q$ as a function of $\tilde{p}$. The resulting quadratic equation has two solutions. When $\tilde{p} = P$ (no discount), both solutions equal $Q_e$. As $\tilde{p}$ decreases, the solutions diverge on either side of $Q_e$. Clearly, we are interested in the larger solution. We substitute that expression for $q$ in $vc(q, \tilde{p})$, reducing it to a function of just $\tilde{p}$.

From the vendor cost function, expressed in terms of just $\tilde{p}$ with numeric values replacing the symbolic parameters, we can either derive and solve the first-order optimality condition using a symbolic algebra program such as Mathematica or Maple, or we can numerically minimize using a variety of software tools, including symbolic algebra packages, programs such as Matlab, or spreadsheet optimizers (such as the Solver in Excel). Since discount schedules would be computed infrequently in practice, and would require a degree of human judgment to determine the incentives needed to move the buyer to the vendor’s target order size, automated solution of the problem is not necessarily a goal; thus, it would also be reasonable to solve the single-dimension restriction graphically.

Our approach differs from those of other authors cited above, all of whom tie the calculation of the optimal order quantity to the type of discount being offered. Crowther (1967) employs trial and error to find an acceptable, but not necessarily optimal, discount schedule. Dolan (1978) assumes that the breakpoint for an incremental discount will be set equal to the buyer’s undiscounted EOQ (which in general will not be optimal), reducing the problem to one dimension (discounted price); if the result is unfavorable to the vendor, Dolan uses ad hoc methods to seek a better schedule. Drezner and Wesolowsky (1989) take a similar approach to ours for the specific scenario covered in their paper, obtaining a quartic equation involving $\bar{q}$, which can be solved in closed form. Unfortunately, when applied to the case of incremental discounts (retaining their other assumptions), the equation for $\bar{q}$ involves a sixth-degree polynomial and so cannot be solved in closed form. The solution procedure specified by Kim and Hwang (1988) for the case of multiple buyers (and a single discount) essentially reduces to that described above when there is only one buyer. As noted above, Lal and Staelin (1984) employ a radically different approach, approximating the discount schedule with a decaying exponential function for price in terms of quantity, while Monahan (1984) and Lee and Rosenblatt (1986) offer a discount only for a particular multiple of the undiscounted EOQ.

Solution of the lot-for-lot case requires relatively little computational effort. This allows us to solve the lot aggregation variant by enumeration. For $N = 2, 3, \ldots$ we adjust the vendor cost coefficients as explained earlier and repeat the solution.
Table 4: Parameters for examples.

<table>
<thead>
<tr>
<th></th>
<th>Buyer</th>
<th>Vendor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annual demand ($D$)</td>
<td>1,540</td>
<td></td>
</tr>
<tr>
<td>Setup ($S_b$)</td>
<td>3</td>
<td>48</td>
</tr>
<tr>
<td>Holding ($H_b$)</td>
<td>0.05</td>
<td>0.03</td>
</tr>
<tr>
<td>Holding ($H_v$)</td>
<td>0.068</td>
<td>0.04</td>
</tr>
<tr>
<td>Undiscounted EOQ ($Q_e$)</td>
<td>280</td>
<td></td>
</tr>
<tr>
<td>Undiscounted annual cost ($B$)</td>
<td>1,573</td>
<td>6,158</td>
</tr>
</tbody>
</table>

procedure. This continues until we reach a lot multiple at which the vendor’s production/acquisition quantity becomes unrealistically high. When finished, we simply adopt the lot multiple whose optimal vendor cost is minimal.

**NUMERICAL EXAMPLES**

We illustrate our method by computing all-units and incremental discount schedules for a single scenario. Scenario parameters are given in Table 4. To demonstrate the flexibility of the models, we include both fixed and proportional holding costs for the buyer and both holding costs and capital credits for the vendor. Both combinations are plausible in practice. A buyer’s holding cost drivers would reasonably include space and labor, both largely independent of the value of the product, and capital investment and insurance, both tied to the price. A vendor could well incur both holding costs on work-in-process inventory and capital savings from acceleration of payments.

We start by assuming that the vendor operates lot-for-lot. In the absence of a discount, the buyer’s annual cost is 1,573 and the vendor’s annual cost is 1,095.2. Given contractual revenue of 1,540, the vendor’s annual profit margin is 444.8.

**Phase 1** The Phase 1 problem (9) becomes

$$\begin{align*}
\text{minimize} & \quad \frac{73,920}{q} + (2,372.56 - 1,540 \tilde{p}) + (0.0145512 - 0.02 \tilde{p})q \\
\text{s.t.} & \quad \frac{4,620}{q} + 1,540 \tilde{p} + (0.025 + 0.034 \tilde{p})q \leq 1,573 \\
& \quad \tilde{p} \leq 1 \\
& \quad \tilde{p} \geq 0.54 \\
& \quad q > 0.
\end{align*}$$

Solving the first constraint as an equality yields

$$q = \frac{23,132.6 - 22,647.1 \tilde{p} + 22,647.1 \sqrt{1.04315 - 2.04315 \tilde{p} + \tilde{p}^2}}{0.735294 + \tilde{p}}. \quad (13)$$

(There is a second solution, which we rule out because it produces an order quantity smaller than $Q_e$.) Substitution into the objective function gives the univariate objective function
Figure 3: Vendor cost as function of $\tilde{p}$.

\begin{align*}
10^7(5.59674 + 5.47548\eta - (6.2608 + 0.779474\eta)\tilde{p} \\
- (0.679744 + 1.43609\eta)\tilde{p}^2 + 1.43609\tilde{p}^3) \\
(0.735294 + \tilde{p})(23,132.6 - 22,647.1\tilde{p} + 22,647.1\eta)),
\end{align*}

where

$$\eta = \sqrt{1.04315 - 2.04315\tilde{p} + \tilde{p}^2}.$$ 

Figure 3 depicts the univariate objective function. Both algebraic solution of the first-order conditions and numerical optimization of (14) yield a target average price of $\tilde{p} = 0.973$; substitution into (13) gives a target order quantity of $q = 1,222$. Those values make the vendor’s annual cost 928.8 (a 15% reduction) and the vendor’s profit 611.2 (a 37% increase).

**Phase 2: All-units discounts** For an optimal all-units schedule, the vendor sets the discounted price at 0.973, a 2.7% discount, and the qualifying quantity (breakpoint) at 1,222. From (6), we can calculate the conditional EOQ to be $ceoq(p; \tilde{q}) = 282$, which, as expected, is infeasible.

**Phase 2: Incremental discounts** Given $\tilde{p} = 0.973$ and $q = 1,222$, equation (11) sets the discounted unit price at $p = 0.930$, a 7% discount. From that, (10) sets the qualifying quantity (breakpoint) at 746.

**Lot Aggregation.** Continuing the example, suppose that the vendor’s setup cost $S_v = 48$ breaks down into a cost of 12 ($S'_v$) to process a buyer order and 36 ($S''_v$) to set up a production lot. If we repeat the Phase 1 calculations using the aggregation substitutions with $N$ ranging from 2 to 10, we obtain the vendor costs depicted in Figure 4. We see that the vendor can reduce its cost from 928.8 to 920.9 by producing or purchasing two times the buyer’s order quantity at a time. The revised Phase 1
solution calls for a buyer order size of 891 (down from 1,222) and an average discounted price per unit of 0.984 (up from 0.973). Using an all-units schedule, the vendor would set the breakpoint at 891 and the discounted unit price at 0.984. Using an incremental schedule, the vendor would set the breakpoint at 583 and the unit price for units in excess of 583 at 0.954. Note that the Phase 2 calculations are performed identically with or without aggregation. [Received: July 4, 2002. Accepted: December 23, 2002.]

REFERENCES


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