Truncated Sequential Change-point Detection based on Renewal Counting Processes

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ABSTRACT. The typical approach in change-point theory is to perform the statistical analysis based on a sample of fixed size. Alternatively, one observes some random phenomenon sequentially and takes action as soon as one observes some statistically significant deviation from the “normal” behaviour. Based on the, perhaps, more realistic situation that the process can only be partially observed, we consider the counting process related to the original process observed at equidistant time points, after which action is taken or not depending on the number of observations between those time points. In order for the procedure to stop also when everything is in order, we introduce a fixed time horizon \( n \) at which we stop declaring “no change” if the observed data did not suggest any action until then. We propose some stopping rules and consider their asymptotics under the null hypothesis as well as under alternatives. The main basis for the proofs are strong invariance principles for renewal processes and extreme value asymptotics for Gaussian processes.

Key words: change-point, extreme value asymptotics, first passage time, increments, renewal counting process, sequential tests, stopping time, strong approximation, Wiener process

1. Introduction

A typical situation in a series of observations is that if everything is in order, then the observations follow some kind of common pattern, whereas if something goes astray at some time point, then, from there on, the observations follow a different pattern. The simplest example is when the observations are an i.i.d. sequence of random variables if everything is in order (\( H_0 \) is true) with an alternative that amounts to, say, that the mean changes at some time point (known or unknown) to some other value (prespecified or not). One obviously wishes to find out as soon as possible if something goes wrong in order to take appropriate action, and, at the same time, minimize the probability of taking action if nothing is wrong.

In this setting one talks about the case with at most one change point (AMOC). A variation on the theme is when there is a temporary change, after which the system “returns” to the original pattern, or several such alternative periods of behaviour. In this paper we only consider the first setting, which is the natural one anyhow, in a sequential approach.

One possibility is to consider a sample of fixed size and perform the statistical analysis. Alternatively, one observes some random phenomenon sequentially and takes action as soon as one observes a statistically significant deviation from the ideal situation. In many situations it may even be more realistic (or less expensive, or both) to assume that the process is, or can be, only partially observed, for example once a day, once a month, and so on. Our approach, therefore, is to consider the counting process, that is, the first passage time process, related to the original process. More precisely, we suggest an analysis based on the number of occurrences during each such time interval: that is between the actual observation points, after which action is taken or not depending on what is observed during those time intervals (if the data are i.i.d.
exponential, then the counting process is, of course, a Poisson process). Also, in order for the procedure to stop if everything is in order, we introduce a fixed time horizon \( n \) at which we stop declaring “no change” if the observed data did not suggest any action until then.

Throughout, our models are non-parametric. To the best of our knowledge this approach based on renewal counting processes has not been investigated so far. Gombay (1995) suggested a non-parametric truncated sequential change-point detection using sequential ranks, which, in turn, improves on an earlier procedure of Bhattacharya & Frierson (1981) when relatively large changes occur early in the sequence of observations. For other non-parametric and asymptotic change-point procedures confer also Csörgő & Horváth (1988, 1997) and Steinebach (1994).

The plan of the paper is as follows. In section 2 we present the problems to be investigated more formally, after which we provide some prerequisites concerning invariance principles for renewal counting processes in section 3. Asymptotics for the problems under consideration under the null hypothesis with known and unknown parameters are derived in sections 4 and 5, respectively. One-sided alternatives are discussed in section 6, and two-sided ones in sections 7 and 8. Section 9 contains some concluding remarks and comments. We close with an appendix devoted to some stopping times related to the Wiener process. For a more detailed discussion we refer to Gut & Steinebach (2000).

2. Statement of the problem

The AMOC problem was introduced in Page (1954, 1955), in the context of quality control/control charts in order to test for a possible shift in the distribution at some intermediate time point. More formally, \( \{ X_i; i = 1, 2, \ldots, n \} \) with distribution functions \( \{ F_i \} \) are given and one wishes to test the null hypothesis

\[ H_0: F_i \equiv F, \quad i = 1, 2, \ldots, n, \tag{1} \]

against

\[ H_1: F_i \equiv F, \quad 1 \leq i \leq i^* < n, \quad \text{and} \quad F_i(\cdot) = F(\cdot - \Delta), \quad i^* < i \leq n, \tag{2} \]

where either \( \Delta \neq 0, \Delta < 0 \) or \( \Delta > 0 \). A particular case, and the one we shall focus on, is when the hypotheses concern no change in the mean against a change (at some \( i^* < n \)). A variation of the above setting is the sequential one in which the observations are collected one after another, and action is taken as soon as some critical region has been entered by some relevant test statistic.

As outlined in the introduction, we shall consider the corresponding counting process and assume that the only information at our disposal is the increments of this process with the aim to find criteria for how to handle a counting process analogue of Page’s model. Based on these observations we wish to determine how to proceed in order to detect, sequentially, whether or not there is a “change” in the process, with \( \Delta \) now being a (possible) change in the mean. Our aim is to derive null and non-null asymptotics for our test statistics via strong invariance principles, and to construct rejection zones of sequential tests with (asymptotic) level \( \alpha \) and power 1. We also study the corresponding stopping times.

Along with our sequence \( \{ X_i; i = 1, 2, \ldots \} \) we shall therefore introduce a counting process \( \{ N(t); t \geq 0 \} \), which we suppose is observed at equidistant time points \( k = 0, 1, \ldots, n \). The analysis will be based on test statistics which are constructed from these counting variables, or their corresponding increments taken over subintervals of size \( h_n \), where \( \{ h_n; n = 1, 2, \ldots \} \) is an integer sequence satisfying some growth conditions specified below. At this point we only remark that the window size should be chosen with some care. Namely, if it is “small”, then
usual local randomness might play a too important role and cause false alarms, whereas, on
the other hand, a “large” window size might have the effect that action is unnecessarily
delayed. An initial sample size of (at least) one window size is always tacitly assumed.

3. Invariance principles for renewal counting processes

Assume that we observe a renewal counting process up to time \( n \), such that it is first based on
an i.i.d. sequence \( X_1, X_2, \ldots \), and later, after (say) time \( k_n^* \), on another i.i.d. sequence
\( X_1^*, X_2^*, \ldots \), independent of the first one. Set \( EX_1 = \mu > 0 \), \( 0 < \var{X_1} = \sigma^2 < \infty \),
\( EX_1^* = \mu^* = \mu - \Delta > 0 \), and \( 0 < \var{X_1^*} = \sigma^{*2} < \infty \). Let

\[
N(t) = \begin{cases} 
N_0(t), & \text{for } 0 \leq t \leq k_n^*, \\
N_0(k_n^*) + N_1(t - k_n^*), & \text{for } k_n^* < t \leq n,
\end{cases}
\]

where

\[
N_0(t) = \min \left\{ k \geq 1 : \sum_{i=1}^{k} X_i > t \right\}, \quad t \geq 0,
\]

\[
N_1(t) = \min \left\{ k \geq 1 : \sum_{i=1}^{k} X_i^* > t \right\}, \quad t \geq 0.
\]

So, there is a “change” in the renewal counting process \( \{N(t) : 0 \leq t \leq n\} \) at time-point \( k_n^* \),
where, without loss of generality (w.l.o.g.), \( \{k_n^*\} \) will be assumed to be some integer sequence.
Note that, according to the positive mean of the sequences \( \{X_i\} \) and \( \{X_i^*\} \), the counting
variables \( N_0(t) \) and \( N_1(t) \) are a.s. finite for every \( t \geq 0 \). Moreover, \( N(t) = N_0(t) \) for all \( t \in [0, n] \)
under the null hypothesis that there is no change in the mean.

Remark 1. The term “renewal theory” traditionally assumes non-negative summands,
whereas the more general case we consider runs under the heading renewal theory for random
walks (Gut, 1988).

According to Csörgö et al. (1987) (see also Csörgő & Horváth (1993) and Steinebach (1988)),
the Komlós et al. (1975) strong approximation for partial sums can be converted into the
following strong invariance principle for \( \{N(t) : t \geq 0\} \).

Proposition 1

Assume \( E|X_1|^r < \infty \) and \( E|X_i^*|^r < \infty \) for some \( r > 2 \). Then

\[
\sup_{0 \leq t \leq n} |N(t) - V(t)| \overset{a.s.}{=} o(n^{1/r}),
\]

where

\[
V(t) = \begin{cases} 
t \theta + \eta W_0(t), & \text{for } 0 \leq t \leq k_n^*, \\
V(k_n^*) + (t - k_n^*) \theta^* + \eta^* W_1(t - k_n^*), & \text{for } k_n^* < t \leq n,
\end{cases}
\]

with \( \theta = 1/\mu, \eta^2 = \sigma^2/\mu^3, \theta^* = 1/\mu^*, \eta^{*2} = \sigma^{*2}/\mu^{*3} \), and two independent (standard) Wiener
processes \( \{W_0(t) : t \geq 0\} \) and \( \{W_1(t) : t \geq 0\} \).

Remark 2. Our analysis will solely be based on the strong approximation in (6) and (7).
Actually, a weak invariance principle would have been sufficient. Such weak embeddings are
available for much wider classes of stochastic processes, including partial sums under

4. Asymptotic distributions under $H_0 : \theta, \eta$ known

We wish to construct an (asymptotic) sequential test, either for the (one-sided) hypotheses

$$H_0 : k^*_n = n \quad \text{vs} \quad H^+_1 : 1 \leq k^*_n < n, \ A > 0;$$

or for the (two-sided) hypotheses

$$H_0 : k^*_n = n \quad \text{vs} \quad H_1 : 1 \leq k^*_n < n, \ A \neq 0,$$

based upon observations of the process $\{N(t) : t \geq 0\}$ at the equidistant time points $k = 0, 1, \ldots, n$.

Towards this end, set

$$Y_k = Y_{k,n} = \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}}, \quad k = h_n, \ldots, n,$$

$$Z_k = Z_{k,n} = \frac{N(k) - k \theta}{\eta \sqrt{k}}, \quad k = 1, \ldots, n.$$

Based upon $\{Y_k\}$ and $\{Z_k\}$, respectively, we define four stopping rules as follows:

**Stopping rule $1^+$**: stop at the first $k$, $h_n \leq k \leq n$, such that $Y_k > c^*_n$;

**Stopping rule 1**: stop at the first $k$, $h_n \leq k \leq n$, such that $|Y_k| > c_n$;

**Stopping rule $2^+$**: stop at the first $k$, $k_n \leq k \leq n$, such that $Z_k > d^*_n$;

**Stopping rule 2**: stop at the first $k$, $k_n \leq k \leq n$, such that $|Z_k| > d_n$,

where $\{k_n, n = 1, 2, \ldots\}$ is another integer sequence given below.

In the following two subsections we show how the critical constants $c^*_n, c_n$ and $d^*_n, d_n$, respectively, can be specified via the following null asymptotics:

4.1. Stopping rules $1^+$ and 1

**Theorem 1**

Let $\{h_n : n = 1, 2, \ldots\}$ be a sequence of integers satisfying $1 \leq h_n \leq n$, $h_n \uparrow \infty$, $h_n/n \to 0$, and $n^{2/r} \log(n/h_n)/h_n \to 0$, as $n \to \infty$. Consider

$$M^{(1)}_n = \max \{Y_k : h_n \leq k \leq n\} \quad \text{and} \quad M^{(1)}_n = \max \{|Y_k| : h_n \leq k \leq n\}.$$

By assuming (6)–(7) for some $r > 2$, we have, under $H_0$ (i.e. for $k^*_n = n$), with

$$a_n^{(1)} = \sqrt{2 \log(n/h_n)},$$

$$b_n^{(1)} = 2 \log(n/h_n) + \frac{1}{2} \log \log(n/h_n) - \frac{1}{2} \log \pi,$$

the following asymptotic distributions:

$$a_n^{(1)} M^{(1)}_n - b_n^{(1)} \xrightarrow{d} E^+, \quad (12)$$

$$a_n^{(1)} M^{(1)}_n - b_n^{(1)} \xrightarrow{d} E, \quad (13)$$
as \( n \to \infty \), where

\[
P(E^+ \leq x) = \exp(-e^{-x}), \quad \text{and} \quad P(E \leq x) = \exp(-2e^{-x}), \quad -\infty < x < \infty.
\]

**Remark 3.** In view of theorem 1, the critical constants \( c_n^+ \) and \( c_n^- \) related to stopping rules \( 1^+ \) and \( 1^- \) may be chosen as

\[
c_n^+ = \frac{(E_{1-x}^+ + b_n^{(1)})}{a_n^{(1)}}, \quad \text{and} \quad c_n^- = \frac{(E_{1-x}^- + b_n^{(1)})}{a_n^{(1)}},
\]

where \( E_{1-x}^+ \) and \( E_{1-x}^- \) denote the \((1 - x)\)-quantiles of \( E^+ \) and \( E \), respectively.

A possible choice of the sequence \( \{h_n\} \) is, for example, \( h_n = n^2 \) with \( 2/r < x < 1 \).

**Proof.** We follow (a discretized version of) the arguments used in Steinebach & Eastwood (1995, proof of th. 2.5):

Via the strong approximation (6)–(7), with \( k_n^* = n \), we have

\[
d_n^{(1)}M_n^{(1)} - b_n^{(1)} = \max_{h_n \leq k \leq n} \left\{ \frac{W_0(k) - W_0(k - h_n)}{\sqrt{h_n}} \right\} - h_n^{(1)} + o\left( \frac{n^{1/r} \log(n/h_n)}{h_n} \right).
\]

(14)

On the other hand, in view of th. 1.2.1 of Csörgő & Révész (1981),

\[
\sup_{0 \leq t \leq n} |W_0(t) - W_0([t])| \leq C(\sqrt{\log n}),
\]

(15)

where \([t]\) denotes the integer part of \( t \).

Finally,

\[
\sup_{h_n \leq t \leq n} \left\{ \frac{W_0(t) - W_0(t - h_n)}{\sqrt{h_n}} \right\} \equiv \sup_{1 \leq s \leq n/h_n} \{ \tilde{W}_0(s) - \tilde{W}_0(s - 1) \},
\]

(16)

where \( \{\tilde{W}_0(s) : s \geq 0\} \) denotes a (standard) Wiener process.

The proof of (12) is completed by combining (14)–(16) with th. 12.3.5 of Leadbetter et al. (1983).

Assertion (13) is a consequence of (12), the symmetry of \( \{\tilde{W}_0(s) - \tilde{W}_0(s - 1) : s \geq 1\} \), and the asymptotic independence of the maxima and minima. Confer also Deheuvels & Révész (1986, lem. 8), for a detailed discussion of the extreme value asymptotics for \( \sup_{0 \leq s \leq t \leq n} (\tilde{W}_0(s) - \tilde{W}_0(s - 1)) \) as well as for \( \sup_{0 \leq s \leq t \leq n} |\tilde{W}_0(s) - \tilde{W}_0(s - 1)| \).

**Remark 4.** Note that \( \{\tilde{W}_0(s) - \tilde{W}_0(s - 1) : s \geq 1\} \) is a stationary Gaussian process with covariance function \( \text{cov}(\tilde{W}_0(s) - \tilde{W}_0(s - 1), \tilde{W}_0(t) - \tilde{W}_0(t - 1)) = 1 - |t - s| \), if \( |t - s| \leq 1 \), \( = 0 \), if \( |t - s| > 1 \).

**Remark 5.** Recently, Haiman (1999) was able to derive a representation of the distribution of \( \sup_{0 \leq s \leq t \leq n} (\tilde{W}_0(s) - \tilde{W}_0(s - 1)) \) (see also Haiman (2000, th. 1.2)).

### 4.2. Stopping rules \( 2^+ \) and \( 2^- \)

**Theorem 2**

Let \( \{k_n : n = 1, 2, \ldots\} \) be a sequence of integers satisfying \( 1 \leq k_n \leq n \), \( k_n \uparrow \infty \), \( k_n/n \to 0 \), and \( n^{2/r} \log \log(n/k_n)/k_n \to 0 \), as \( n \to \infty \). Consider

\[
M_n^{(2)} = \max\{Z_k : k \leq n\} \quad \text{and} \quad M_n^{(2)} = \max\{|Z_k| : k \leq n\}.
\]

Assuming (6)–(7) for some \( r > 2 \), we have, under \( H_0 \) (i.e. for \( k_n^* = n \)), with
For example, similar to (14) and (15) above, we obtain

\[ a_n^{(2)} = \sqrt{2 \log \log(n/k_n)}, \]
\[ b_n^{(2)} = 2 \log \log(n/k_n) + \frac{1}{2} \log \log \log(n/k_n) - \frac{1}{2} \log(4\pi), \]

the following asymptotic distributions:

\[ a_n^{(2)} M_n^{(2)} - b_n^{(2)} \xrightarrow{d} E^+, \]
\[ a_n^{(2)} M_n^{(2)} - b_n^{(2)} \xrightarrow{d} E, \]

as \( n \to \infty \), with \( E^+ \) and \( E \) as in theorem 1.

**Remark 6.** In analogy with remark 3 the present theorem allows for the following (asymptotic) choice of critical constants \( d_n^+ \) and \( d_n \) related to stopping rules 2+ and 2:

\[ d_n^+ = (E_{1-\alpha}^+ + b_n^{(2)})/a_n^{(2)}, \quad \text{and} \quad d_n = (E_{1-\alpha} + b_n^{(2)})/a_n^{(2)}, \]

where, again, \( E_{1-\alpha}^+ \) and \( E_{1-\alpha} \) denote the \((1-\alpha)\)-quantiles of \( E^+ \) and \( E \), respectively.

The sequence \( \{k_n\} \) could e.g. be chosen as \( k_n = n^\alpha \) with \( 2/r < \alpha < 1 \).

**Proof.** Here we follow (a discretized version of) the proof of th. 2.5 in Steinebach (1988).

For example, similar to (14) and (15) above, we obtain

\[ a_n^{(2)} M_n^{(2)} - b_n^{(2)} \xrightarrow{a.s.} \sup_{k_n \leq t \leq n} \left\{ \frac{W_0(t)}{\sqrt{t}} \right\} - b_n^{(2)} + o \left( n^{1/r} \sqrt{\log \log(n/k_n)/k_n} \right). \]

A time transformation \( t = e^t \) leads to study the (stationary) Ornstein–Uhlenbeck process \( \{U_0(s) : log k_n \leq s \leq log n\} \), where \( U_0(s) = W_0(e^s)/\sqrt{e^s} \). Its extreme value limiting distribution is given by the well-known Darling & Erdös (1956) asymptotic resulting in (17) (see e.g. Csörgő & Révész, 1981, th. 1.9.1; Leadbetter et al., 1983, th. 12.3.5).

Assertion (18) follows, as above, from symmetry and the asymptotic independence of the maxima and minima.

5. Asymptotic distributions under \( H_0; \theta, \eta \) unknown

In this section the unknown parameters \( \theta \) and \( \eta \) have to be replaced by suitable estimates. This has to be done sequentially, that is, instead of \( Y_{k,n} \) and \( Z_{k,n} \) one should think of using

\[ \hat{Y}_k = \hat{Y}_{k,n} = \frac{N(k) - N(k - h_n) - h_n \hat{\theta}_k}{\eta_k \sqrt{h_n}}, \]
\[ \hat{Z}_k = \hat{Z}_{k,n} = \frac{N(k) - k \hat{\theta}_k}{\eta_k \sqrt{k}}, \]

for \( k = 2h_n, \ldots, n \), where

\[ \hat{\theta}_k = \hat{\theta}_{k,n} = \frac{N(k - h_n)}{k - h_n}, \]
\[ \eta_k^2 = \frac{1}{k/h_n} \sum_{i=1}^{[k/h_n]} \left( \frac{N(ih_n)}{h_n} - \frac{N((i-1)h_n)}{h_n} - h_n \hat{\theta}_{k/h_n} \right)^2, \]

with \( \hat{\theta}_k = (N(k) - N(0))/k \). It turns out that, in view of (21),

\[ N(k) - N(k - h_n) - h_n \hat{\theta}_k = N(k) - k \hat{\theta}_k, \]

so that, up to normalization, the statistics \( \hat{Y}_k \) and \( \hat{Z}_k \) are equivalent.
We shall see below that corresponding stopping rules based, for example, on the $\hat{Y}_{k,n}$’s can be similarly applied, whereas their counterparts based on the $\hat{Z}_{k,n}$’s cannot work any longer.

We consider the following stopping rules:

**Stopping rule $\hat{I}^+$:** stop at the first $k$, $\hat{h}_n \leq k \leq n$, such that $Y_k > \hat{c}_n^+$;

**Stopping rule $\hat{I}$:** stop at the first $k$, $\hat{h}_n \leq k \leq n$, such that $|Y_k| > \hat{c}_n$,

where $\{\hat{h}_n; n = 1, 2, \ldots\}$ is an integer sequence to be specified below.

**Theorem 3**

Let $\{h_n; n = 1, 2, \ldots\}$ be as in theorem 1, and $\{\hat{h}_n; n = 1, 2, \ldots\}$ be another integer sequence such that $h_n/(h_n \log(n/h_n) \log \log n) \to \infty$, but $\log(n/h_n) = O(\log(n/h_n))$. Consider

$$\hat{M}^{(1)}_n = \max \{\hat{Y}_k; \hat{h}_n \leq k \leq n\} \quad \text{and} \quad \hat{M}^{(1)}_n = \max \{|\hat{Y}_k|; \hat{h}_n \leq k \leq n\}.$$

Assume that, under $H_0$, as $n \to \infty$,

$$\max_{h_n \leq k \leq n} |\hat{H}_k^2 - \eta^2| = o_p(1/\log(n/h_n)). \quad (23)$$

Then, under the assumptions of theorem 1, the null asymptotics (12) and (13) retain for $\hat{M}^{(1)}$ and $\hat{M}^{(1)}$, respectively, i.e. we have

$$a_n^{(1)} M^{(1)} - b_n^{(1)} d \to E^+,$$

$$a_n^{(1)} M^{(1)} - b_n^{(1)} d \to E,$$

as $n \to \infty$, where $a_n^{(1)}, b_n^{(1)}, E^+$, and $E$ are as in theorem 1.

**Remark 7.** Theorem 3 suggests a choice of $\hat{c}_n^+$ and $\hat{c}_n$ in stopping rules $\hat{I}^+$ and $\hat{I}$ as follows:

$$\hat{c}_n^+ = (E_{1-\alpha} + b_n^{(1)})/a_n^{(1)} \quad \text{and} \quad \hat{c}_n = (E_{1-\alpha} + b_n^{(1)})/a_n^{(1)},$$

that is, they coincide with the constants suggested in remark 3.

**Remark 8.** A choice of $h_n = [n^r]$ and $\hat{h}_n = [n^p]$, with $2/r < \alpha < \beta < 1$, satisfies the assumptions required in theorem 3.

**Remark 9.** Concerning the validity of (23) for $\hat{H}_k^2$ in (22) we refer to the discussion in section 8.1 below. It will turn out that the conditions on $\{h_n\}$ and $\{\hat{h}_n\}$ have to be strengthened slightly in order to achieve the rate in (23) also under the alternatives.

**Proof.** We show that the null asymptotics of (12) and (13) remain valid when the parameters $\theta$ and $\eta$ are replaced by their corresponding estimates $\hat{\theta}_k$ and $\hat{\eta}_k$.

First note that, uniformly in $k = \hat{h}_n, \ldots, n$,

$$\hat{\theta}_k - \theta \leq \frac{W_0(k - h_n)}{h_n} + o \left( \left( \frac{h_n}{\log(n/h_n)} \right)^{1-1/(1-r)} \right),$$

under $H_0$: $k_n = n$, in view of (6), (7), and the law of the iterated logarithm (LIL) for the Wiener process.

So, since $\max_{k \leq k \leq n} Y_k = c_p(\log(n/h_n))$, but $\eta/\hat{\eta}_k = 1 + o_p(1/\log(n/h_n))$, we have

$$\hat{Y}_k = \frac{N(k) - N(k - h_n) - h_n \eta \theta}{\eta \hat{h}_n} - \hat{h}_n \theta - \frac{1}{\eta \hat{h}_n} + o_p \left( \frac{\hat{h}_n^\delta - \theta}{\eta \hat{h}_n} \right),$$

uniformly in $k = \hat{h}_n, \ldots, n$, where $\delta = \frac{1}{2} - (1/r) > 0$. 

Now, due to our assumptions on \( \{ \hat{h}_n \} \),
\[
a_n^{(1)} M_n^{(1-)} - b_n^{(1)} = a_n^{(1)} \max_{h_n \leq k \leq n} Y_k - b_n^{(1)} + o_p(1) + a_p \left( a_n^{(1)} h_n^{-d} \right).
\]
(28)

Since \( a_n^{(1)} = \sqrt{2 \log(n/h_n)} \), but \( n^{2/\gamma}/h_n \to 0 \), the second remainder term is also of order \( o_p(1) \), which, in view of the arguments leading to (12), suffices for the proof of (24). Note that
\[
\sqrt{\log(h_n/h_n)} - \sqrt{\log(n/h_n)} \to -\infty.
\]
(29)

Analogously, (25) can be derived along the lines preceding (13).

**Remark 10.** It is obvious from the above proof that an estimation corresponding to (27) cannot work for a stopping rule based on the \( Z_k \)'s, since the rate \( c_p(\sqrt{(h_n \log \log n)/h_n}) \) would have to be replaced by \( c_p(1) \), which is not negligible any longer in the desired extreme value asymptotics. So, it turns out that the stopping rules \( \hat{1}^+ \) and \( \hat{1} \) are the appropriate ones in case of unknown parameters \( \theta \) and \( \eta \).

6. Asymptotics under \( H_\theta^+ \): \( \theta, \eta \) known; the one-sided case

In this section we study the behaviour of our test statistics under the one-sided alternative
\[
H_\theta^+ : 0 < k_n^+ < n \quad \text{and} \quad \mu^+ = \mu - \Delta \ (0 < \Delta < \mu),
\]
(30)

where \( \Delta \) denotes a possible shift in the mean at some unknown time point \( k_n^+ \in (0, n) \).

Let the set-up of section 3 be given. In addition to the overall maxima defined above we also introduce, for \( 0 \leq i < j \leq n \), the incremental maxima,
\[
M_{i,j}^{(1+)} = \max_{i < k \leq j} Y_{k,n} \quad \text{and} \quad M_{i,j}^{(2+)} = \max_{i < k \leq j} Z_{k,n}.
\]
(31)

We shall consider two different approaches. In the first one we let \( n \to \infty \) in order to find the asymptotics of the overall maxima (as \( n \to \infty \)). In the second one we keep \( n \) fixed but “large” and introduce the first passage time relevant for the test in order to provide asymptotics for the power and the critical region.

6.1. Stopping rule \( \hat{1}^+ \)

In this subsection we wish to get hold of the quantity
\[
M_n^{(1+)} = \max_{h_n \leq k \leq n} Y_{k,n} = \max_{h_n \leq k \leq n} \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}}
\]
(32)

in order to determine the power of the test; since we observe increments of size \( h_n \) of the counting process it is tacitly assumed that we start observing at time \( h_n \) (recall section 4). Since a change-point, if it exists, cannot be detected before time \( h_n \), we must also assume \( h_n \leq k_n^+ \leq n \).

6.1.1. The asymptotic distribution of the maximum

For \( k \leq k_n^+ \) no modification is needed in the approximation of \( Y_{k,n} \) as given in section 4. For the case \( k_n^+ < k \leq k_n^+ + h_n \), the time points involved are located on opposite sides of the (hypothetical) change point. In this case,
\[ Y_{k,n} = \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} \]
\[ = \frac{N_0(k) - N_0(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} - \frac{N_0(k) - N_0(k^*_n) - (k - k^*_n) \theta}{\eta \sqrt{k - k^*_n h_n}} \sqrt{\frac{k - k^*_n}{h_n}} \]
\[ + \frac{N_1(k - k^*_n) - (k - k^*_n) \theta^*}{\eta^* \sqrt{k - k^*_n}} \sqrt{\frac{k - k^*_n}{h_n}} + \frac{\Delta 00^* (k - k^*_n)}{\eta^* \sqrt{h_n}} \]
\[ = A_{k,n}^{(11)} - B_{k,n}^{(11)} \sqrt{\frac{k - k^*_n}{h_n}} + C_{k,n}^{(11)} \frac{\eta^*}{\eta} \sqrt{\frac{k - k^*_n}{h_n}} + \frac{\Delta 00^* (k - k^*_n)}{\eta^* \sqrt{h_n}}. \quad (33) \]

For \( k > k^*_n + h_n \), finally,
\[ Y_{k,n} = \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} \]
\[ = \frac{N_1(k - k^*_n) - N_1(k - k^*_n - h_n) - h_n \theta^*}{\eta^* \sqrt{h_n}} \frac{\eta^*}{\eta} \sqrt{\frac{k - k^*_n}{h_n}} + \frac{\Delta 00^*}{\eta^*} \sqrt{\frac{k - k^*_n}{h_n}} \]
\[ = B_{k,n}^{(12)} \frac{\eta^*}{\eta} + \frac{\Delta 00^*}{\eta^*} \sqrt{\frac{k - k^*_n}{h_n}}. \quad (34) \]

In order to obtain precise asymptotics we estimate \( M_n^{(1^+)\prime} \) via the maximum of the maxima over subsets of the index set under consideration as follows,
\[ M_n^{(1^+)\prime} = \max \left\{ M_n^{(1^+)\prime}_{k^*_n - 1; k^*_n, k^*_n + h_n, k^*_n + h_n}; M_n^{(1^+)\prime}_{k^*_n + h_n}; M_n^{(1^+)\prime}_{k^*_n + h_n, h_n} \right\}, \quad (35) \]

and then consider each incremental maximum separately.

By the arguments in the proof of theorem 1 it first follows that
\[ M_n^{(1^+)\prime}_{k^*_n - 1; k^*_n, k^*_n + h_n} = \max_{h_n, k < k^*_n} \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} = C_p \left( \sqrt{\log(n/h_n)} \right) \text{ as } n \to \infty. \quad (36) \]

By analogously applying proposition 1 to the first and second contribution below, and theorem 2 to the third one, keeping in mind that \( 0 < k - k^*_n \leq h_n \leq k^*_n \) in this portion of the estimates, we obtain
\[ M_n^{(1^+)\prime}_{k^*_n; k^*_n + h_n} = \max_{k^*_n < k \leq k^*_n + h_n} \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} \]
\[ = \max_{k^*_n < k \leq k^*_n + h_n} \left\{ A_{k,n}^{(11)} - B_{k,n}^{(11)} \sqrt{\frac{k - k^*_n}{h_n}} + C_{k,n}^{(11)} \frac{\eta^*}{\eta} \sqrt{\frac{k - k^*_n}{h_n}} + \frac{\Delta 00^* (k - k^*_n)}{\eta^* \sqrt{h_n}} \right\} \]
\[ = C_p (1) + C_p \left( \sqrt{\log \log h_n} \right) + C_p \left( \sqrt{\log \log h_n} \right) + C_p \left( \sqrt{\log h_n} \right). \quad (37) \]

Finally, by the obvious analogue of theorem 1,
\[ M_n^{(1^+)\prime}_{k^*_n + h_n, n} = \max_{k^*_n + h_n < k \leq n} \frac{N(k) - N(k - h_n) - h_n \theta}{\eta \sqrt{h_n}} \]
\[ = \max_{k^*_n + h_n < k \leq n} B_{k,n}^{(12)} \frac{\eta^*}{\eta} + \frac{\Delta 00^*}{\eta^*} \sqrt{h_n} \]
\[ = C_p \left( \sqrt{\log((n - k^*_n)/h_n)} \right) + C_p \left( \sqrt{h_n} \right). \quad (38) \]
A more detailed examination of the three estimates shows that the overall maximum is attained in the interval \(k^*_n < k \leq n\) asymptotically. Following the proof of theorem 1 we next find that

\[
M_n^{(1)^+} = \max_{k^*_n + h_n < k \leq n} \frac{N(k) - N(k - h_n) - h_n\theta}{\eta\sqrt{h_n}} \quad \text{as } n \to \infty.
\]

where \(\{W_i(t): t \geq 0\}\) and \(\{\tilde{W}_i(t): t \geq 0\}\) are Wiener processes.

From the above discussion the following asymptotic under \(H^+_1\) is now immediate.

**Theorem 4**

Let \(\{h_n\}\) and \(M_n^{(1)^+}\) be as in theorem 1, with the additional assumption that

\[
\frac{\log((n - k^*_n)/h_n)}{\log\log n} \to \infty \quad \text{as } n \to \infty,
\]

and set

\[
a^*_n = \sqrt{2\log((n - k^*_n)/h_n)},
\]

\[
b^*_n = 2\log((n - k^*_n)/h_n) + \frac{1}{2}\log\log((n - k^*_n)/h_n) - \frac{1}{2}\log \pi.
\]

Assuming (6)–(7) for some \(r > 2\), we have the following asymptotic distribution under \(H^+_1\):

\[
a^*_n \eta \left( \frac{M_n^{(1)^+}}{\eta} \sqrt{h_n} \right) - b^*_n \overset{d}{\to} E^+,
\]

as \(n \to \infty\), where \(E^+\) is as in theorem 1.

**Proof.** The proof of (42) follows from a combination of (35)–(40) with a corresponding analogue of theorem 1. Observe that
as \( n \to \infty \).

Remark 11. In view of theorem 4, the test based on stopping rule \( 1^+ \) is of (asymptotic) power 1, since, under \( H_1^+ \),

\[
M_n^{(1^+)} \sim \frac{\Delta \theta \eta}{\eta} \sqrt{h_n} \implies c_n^+ = \mathcal{C}(h_n^{(1)}/a_n^{(1)}) = \mathcal{C}\left(\sqrt{\log(n/h_n)}\right) \quad \text{as } n \to \infty.
\]

6.1.2. Stopping times and the power

With regard to (40) we introduce the stopping time

\[
T(x) = \inf \left\{ t : \frac{\eta^*}{\eta} \{W_i(t) - W_i(t - 1)\} + \frac{\Delta \theta \eta}{\eta} \sqrt{h_n} \geq x \right\}
\]

\[
= \inf \left\{ t : \{W_i(t) - W_i(t - 1)\} \geq \frac{\eta}{\eta^*} x - \frac{\Delta \theta \eta}{\eta^*} \sqrt{h_n} \right\}, \quad x > 0,
\]

in order to determine the (asymptotic) power of the test, which becomes \( P_{\Delta}(T(x) \leq (n - k_n^*)/h_n) \), with \( x = c_n^+ \) from remark 3.

For some facts concerning \( \sup_{t \leq s} \{W_i(s) - W_i(s - 1)\} \), \( t > 0 \), we refer to section A.2.

A more tractable case is stopping rule \( 2^+ \), which rather than dealing with increments of the counting process, concerns the counting process itself.

6.2. Stopping rule \( 2^+ \)

In this section we focus on

\[
M_n^{(2^+)} = \max_{k_n \leq k \leq n} \frac{N(k) - k \theta}{\eta \sqrt{k}}.
\]

Here, too, we start observing at \( k_n \), where \( k_n \) satisfies the initial assumptions in theorem 2. The discussion follows the general pattern of the previous subsection.

6.2.1. The asymptotic distribution of the maximum

Once again, when \( k \leq k_n^* \) nothing changes compared to section 4. For \( k > k_n^* \), however,

\[
Z_{k,n} = \frac{N(k) - k \theta}{\eta \sqrt{k}}
\]

\[
= \frac{N_0(k_n^*) - k_n^* \theta}{\eta \sqrt{k_n^*}} \sqrt{\frac{k_n^*}{k}} + \frac{N_1(k - k_n^*) - (k - k_n^*) \theta^*}{\eta^* \sqrt{k - k_n^*}} \sqrt{\frac{k - k_n^*}{k}} + \frac{\Delta \theta \eta}{\eta^*} \sqrt{k}
\]

\[
= A_n^{(2)} \sqrt{\frac{k_n^*}{k}} + B_n^{(2)} \sqrt{\frac{k - k_n^*}{k}} + \frac{\Delta \theta \eta}{\eta} \sqrt{k}.
\]

\( \text{C} \) Board of the Foundation of the Scandinavian Journal of Statistics 2002.
In order to estimate the quantity in (45) precisely, we split the maximum as follows:

\[ M_n^{(2^+)} = \max \left\{ M_{k_{n-1},k_n}^{(2^+)}, M_{k_n,n}^{(2^+)} \right\}. \]  

(47)

As for the first maximum, we have, similar to theorem 2, as \( n \to \infty \),

\[ M_{k_{n-1},k_n}^{(2^+)} = \max_{k_{n-1} < k_n \leq n} \frac{N(k) - k\theta}{\eta \sqrt{k}} = C_p \left( \sqrt{\log \log(k_n/k_n)} \right). \]  

(48)

Analogously,

\[ M_{k_n,n}^{(2^+)} = \max_{k_n < k_n \leq n} \frac{N(k) - k\theta}{\eta \sqrt{k}} \]
\[ = \max_{k_n < k_n \leq n} \left\{ A_n^{(2)} \sqrt{\frac{k_n}{k}} + B_n^{(2)} \sqrt{\frac{k_k - k_k^*}{k_k} \eta} + \frac{\Delta \theta^* k - k_k^*}{\eta} \right\} \]
\[ = C_p(1) + C_p \left( \sqrt{\log \log(n - k_n^*)} \right) + \frac{\Delta \theta^*}{\eta} \frac{n - k_n^*}{\sqrt{n}}. \]  

(49)

By comparing the two estimates we first mention the following analogue of (39):

\[ \frac{\sqrt{n}}{n - k_n^*} M_n^{(2^+)} = \frac{\sqrt{n}}{n - k_n^*} \max_{k_n < k_n \leq n} \frac{N(k) - k\theta}{\eta \sqrt{k}} \frac{\Delta \theta^*}{\eta} \text{ as } n \to \infty. \]  

(50)

A closer inspection shows that the maximum in (47) is, asymptotically, as \( n \to \infty \), attained in the second interval, and, furthermore, that we must distinguish between the cases

\( k_n^* = o(n), \quad k_n^* \sim \lambda n \quad (0 < \lambda < 1), \quad \text{ and } \quad k_n^* = n - o(n). \)

**Theorem 5**

Let \( M_n^{(2^+)} \) be as in theorem 2.

Assuming (6)–(7) for some \( r > 2 \), we have, under \( H_1^+ \), as \( n \to \infty \),

(a) if \( k_n^* = o(n) \),

\[ M_n^{(2^+)} - \frac{\Delta \theta^* n - k_n^*}{\eta} \frac{d}{\sqrt{n}} \to N(0, \eta^2/\eta^2); \]  

(51)

(b) if \( k_n^* \sim \lambda n \) (for some \( 0 < \lambda < 1 \)),

\[ M_n^{(2^+)} - \frac{\Delta \theta^* n - k_n^*}{\eta} \frac{d}{\sqrt{n}} \to N(0, \lambda + (1 - \lambda)\eta^2/\eta^2); \]  

(52)

(c) if \( k_n^* = n - o(n) \),

\[ M_n^{(2^+)} - \frac{\Delta \theta^* n - k_n^*}{\eta} \frac{d}{\sqrt{n}} \to N(0, 1), \]  

(53)

provided, in addition, that \( (n - k_n^*)/\sqrt{n \log \log n} \to \infty \) as \( n \to \infty \).

**Remark 12.** Similar to remark 11, theorem 5 shows that the test based on stopping rule \( 2^+ \) is of (asymptotic) power 1.
Proof. (a) For the proof of (51) we first observe that, under $H_+^+$, as $n \to \infty$,

$$M_{k_n}^{(2+)} - \frac{\Delta \theta \sigma^2}{\eta} \frac{n - k_n^*}{\sqrt{n}} p \to -\infty. \quad (54)$$

Similarly, since $(k - k_n^*)/\sqrt{k}$ is strictly increasing in $k$ for $k_n^* < k < n$, we have, for any $0 < a < 1$,

$$M_{k_n[na]}^{(2+)} - \frac{\Delta \theta \sigma^2}{\eta} \frac{n - k_n^*}{\sqrt{n}} p \to -\infty, \quad as \ n \to \infty. \quad (55)$$

On the other hand,

$$M_{k_n[na],n}^{(2+)} - \frac{\Delta \theta \sigma^2}{\eta} \frac{n - k_n^*}{\sqrt{n}} \leq \frac{N_0(k_n^*) - k_n^*}{\eta \sqrt{k_n^*}} \sqrt{k_n^*} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}}. \quad (56)$$

Now, via proposition 1, the right-hand side of (56) tends in distribution to

$$\frac{\eta^*}{\eta} \sup_{0 < t < 1} \frac{\bar{W}(t)}{\sqrt{t}},$$

which, in turn, tends to a $N(0, \eta^*^2/\eta^2)$-random variable as $a \to 1$. Since

$$M_n^{(2+)} - \frac{\Delta \theta \sigma^2}{\eta} \frac{n - k_n^*}{\sqrt{n}} = \frac{N(n) - n0}{\eta \sqrt{n}} - \frac{\Delta \theta \sigma^2}{\eta} \frac{n - k_n^*}{\sqrt{n}}, \quad (57)$$

a corresponding decomposition shows that the right-hand side of (57) also tends in distribution to a $N(0, \eta^*^2/\eta^2)$ random variable. This completes the proof of (51).

(b) The only difference to the proof of (a) is that now

$$\frac{N_0(k_n^*) - k_n^*}{\eta \sqrt{k_n^*}} \sqrt{k_n^*} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}}.$$

in (46) gives an additional contribution to the asymptotics which, however, is independent of that of

$$\frac{N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}} \frac{\max_{[na] < k < n} N_1(k - k_n^*) - (k - k_n^*)0^* \sqrt{k - k_n^*}}{\eta \sqrt{k - k_n^*}}.$$

Thus, by obvious modifications of the arguments in part a), we arrive at (52).

(c) The difference to the proofs above is that $A_n^{(2)} \sqrt{k_n^*/k}$ in (46) is the dominating term in this case; note that the conclusion corresponds to $\lambda = 1$ in part (b).

6.2.2. Stopping times and the power

In view of the asymptotic normality of $A_n^{(2)}$ in (46), strong approximation (e.g. Steinebach, 1988), and the scaling property of the Wiener process, the asymptotic behaviour of $M_n^{(2+)}$ is the same as that of
Theorem 6

The following result.

Summarizing our findings in this subsection, and recalling proposition 5, we obtain the implicit description of the distribution of the order error is of the order $o_p(n^{1/2} \sqrt{k_n^*})$.

The case $k_n^* = o(n)$

Set $e_n = k_n^*/n \to 0$ as $n \to \infty$. The first step is to move to continuous time ("replacing" $k/k_n^*$ by $t$). The last expression then is asymptotically equivalent in distribution to

$$
\sup_{1 < t \leq 1/\epsilon_n} \left\{ W_0(t) \sqrt{ t \over 1 + \eta^* W_1(t - 1) \sqrt{ t \over 1 + \sqrt{n e_n} \Delta \theta^* \over \eta \sqrt{1 - \eta^* t} } } \right\},
$$

An application of Csörgő & Révész (1981, lem. 1.2.1) shows that the additional approximation error is of the order $o_p(\sqrt{(\log n)/k_n^*})$.

Now, let

$$
T(x) = \inf \left\{ t : W_0(t) \sqrt{1 \over t + 1 + \eta^* W_1(t) \over \eta \sqrt{t + 1} + \sqrt{n e_n} \Delta \theta^* \over \eta \sqrt{t + 1} } \geq x \right\}
$$

$$
= \inf \left\{ t : W_0(t) + \eta^* W_1(t) \sqrt{n e_n} \Delta \theta^* \over \eta \sqrt{t + 1} \geq x \sqrt{t + 1} \right\}, \quad x > 0.
$$

By invoking (105) with $\alpha = \eta^*/\eta$, $\beta = \sqrt{n e_n} \Delta \theta^* / \eta$, $\gamma = 1$, and $\tau^2 = 1$, we obtain the following implicit description of the distribution of $T(x)$.

$$
E \exp \left\{ s \sqrt{T(x)} + 1 - \left( s \sqrt{n e_n} \Delta \theta^* \over \eta + {1 \over 2} s^2 \eta^2 \right) T(x) \right\} = \exp (s^2/2), \quad s \geq 0.
$$

Summarizing our findings in this subsection, and recalling proposition 5, we obtain the following result.

Theorem 6

Assume (6) and (7) for some $r > 2$, and let $k_n$ be as in theorem 5.

Under the alternative

$$
H^*_1: \ 0 < k_n^* < n \quad \text{and} \quad \mu^* = \mu - \Delta,
$$

where $0 < \Delta < \mu$, and $k_n^*/n = e_n \to 0$ as $n \to \infty$, the power of the test is, for large $n$, approximately equal to $P_{\Delta} (T(x) < e_n/(1 - e_n))$, with $T(x)$ as defined in (60) and $x = d_n^*$ from remark 6, and where $\{W_0(t): t \geq 0\}$ and $\{W_1(t): t \geq 0\}$ are independent Wiener processes, and $\theta = 1/\mu$, $\eta^2 = \sigma^2/\mu^3$, $\theta^* = 1/\mu^*$, and $\eta^2 = \sigma^2/\mu^3$.
The error in the (distributional) approximation of

\[
\max_{k_0 \leq k \leq n} \frac{N(k) - k\theta}{\eta \sqrt{k}} \text{ by}
\]

\[
\sup_{0 < \tau \leq (1 - \varepsilon_n)/\eta} \left\{ W_0(1) \sqrt{\frac{1 - \frac{1}{t + 1}}{t + 1}} + \frac{\eta^* W_i(t)}{\eta \sqrt{t + 1}} + \sqrt{n \varepsilon_n \Delta \theta^* \frac{t}{\sqrt{t + 1}}} \right\}
\]

is of the order \( o_p(n^{1/r} / \sqrt{k_n^*}) + O_p(\sqrt{\log n}) \) \( = o_p(n^{1/r} / \sqrt{k_n^*}) \) as \( n \to \infty \).

The distribution of the stopping time \( T(x) \) is implicitly given in (61). Moreover, as \( x \to \infty \),

\[
\frac{n \varepsilon_n T(x)}{x^2} \to \left( \frac{\eta}{\Delta \theta^*} \right)^2 \text{ a.s. and in } L^r, \text{ for every } r > 0,
\]

and

\[
\frac{T(x) - \frac{\eta^2}{4n \varepsilon_n (\Delta \theta^*)^2} x^2}{\frac{2\eta^*}{n \varepsilon_n (\Delta \theta^*)^2}} = \frac{n \varepsilon_n (\Delta \theta^*)^2}{2\eta^*} T(x) - \frac{\eta}{2\eta^*} x \overset{d}{\to} N(0, 1).
\]

Remark 13. Under the null hypothesis the asymptotics are different in that \( T(x)/x^2 \) converges in distribution to the asymmetric stable distribution with index 1/2.

The case \( k_n^* \sim n\lambda^* \) \( (0 < \lambda < 1) \)

Now, set \( \varepsilon_n = k_n^*/n - \lambda^* \). By modifying the arguments of the previous subsection it follows, in a first step, that the approximation (58) of \( M_n^{(2^*)} \) now is asymptotically equivalent (in distribution) to

\[
\sup_{1 < \tau \leq (1/\lambda + \varepsilon_n)} \left\{ W_0(1) \sqrt{\frac{1}{t + 1}} + \eta^* W_i(t) \frac{1}{\sqrt{t + 1}} + \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* \frac{t}{\sqrt{t + 1}}} \right\},
\]

\[
\overset{d}{=} \sup_{0 < \tau \leq (1 - \varepsilon_n)/(\lambda + \varepsilon_n)} \left\{ W_0(1) \sqrt{\frac{1}{t + 1}} + \eta^* W_i(t) \frac{1}{\sqrt{t + 1}} + \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* \frac{t}{\sqrt{t + 1}}} \right\},
\]

the error being \( o_p(n^{1/r} / \sqrt{k_n^*}) + O_p(\sqrt{\log n}) = o_p(n^{1/r} - \frac{1}{2}) \) as \( n \to \infty \).

Next we define

\[
T(x) = \inf \left\{ t : W_0(1) \sqrt{\frac{1 - \frac{1}{t + 1}}{t + 1}} + \frac{\eta^* W_i(t)}{\eta \sqrt{t + 1}} + \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* \frac{t}{\sqrt{t + 1}}} \geq x \right\}
\]

\[
= \inf \left\{ t : W_0(1) + \frac{\eta^*}{\eta} W_i(t) + \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* \frac{t}{\sqrt{t + 1}}} \geq x \right\}, \quad x > 0.
\]

By invoking (105) with \( \alpha = \eta^*/\eta, \beta = \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* / \eta}, \gamma = 1, \) and \( \tau^2 = 1 \), we obtain the following analogue of (61);

\[
E \exp \left\{ sx \sqrt{T(x)} - 1 \left( s \sqrt{n(\lambda^* + \varepsilon_n) \Delta \theta^* / \eta} + \frac{1}{2} \frac{s^2 \eta^*}{\eta^2} T(x) \right) \right\} = \exp(s^2/2), \quad s \geq 0.
\]

Following is the counterpart of theorem 6.
Theorem 7
Assume (6) and (7) for some $r > 2$, and let $k_n$ be as in theorem 5.

Under the alternative

$$H_1^+: 0 < k^*_n < n \quad \text{and} \quad \mu^* = \mu - \Delta,$$

where $0 < \Delta < \mu$, and $\epsilon_n = k^*_n / n - \lambda$, with $0 < \lambda < 1$, the power of the test is, for large $n$, approximately equal to $P_A(T(x) \leq (1 - \lambda - \epsilon_n) / (\lambda + \epsilon_n))$, with $T(x)$ as defined in (63) and $x = d^+_n$ from remark 6, and where \{W_0(t): t \geq 0\} and \{W_1(t): t \geq 0\} are independent Wiener processes, $\theta = 1 / \mu$, $x = \sigma^2 / \mu^2$, $\theta^* = 1 / \mu^*$, and $\eta^2 = \sigma^2 / \mu^2$.

The error in the (distributional) approximation of

$$\max_{k_n \leq k \leq n} \frac{N(k) - k \theta}{\eta \sqrt{k}}$$

by

$$\sup_{0 < t \leq (1 - \lambda - \epsilon_n) / (\lambda + \epsilon_n)} \left\{ W_0(1) \sqrt{\frac{1}{t + 1} + \frac{\eta^* W_1(t)}{\eta / \sqrt{t + 1}} + \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} \frac{t}{\sqrt{t + 1}}} \right\}$$

is of the order $O_p(n^{(1/r) - 1/2}) + O_p(\sqrt{(\log n) / n}) = O_p(n^{(1/r) - 1/2})$ as $n \to \infty$.

The distribution of the stopping time $T(x)$ is implicitly given in (64). Moreover, as $x \to \infty$,

$$\frac{n(\lambda + \epsilon_n) T(x)}{x^2} \to \left( \frac{\eta}{\Delta \theta^*} \right)^2$$

a.s. and in $L^r$, for every $r > 0$.

and

$$\frac{T(x) - \frac{\eta^2}{2 \eta^*}}{n(\lambda + \epsilon_n)(\Delta \theta^*)^2 x^2} = \frac{n(\lambda + \epsilon_n)(\Delta \theta^*)^2}{2 \eta^*} T(x) \to \frac{\eta}{2 \eta^*} x \sim N(0, 1).$$

The case $k^*_n = n - o(n)$

Define $\epsilon_n$ via the relation $k^*_n = n(1 - \epsilon_n)$, that is, $\epsilon_n = 1 - k^*_n / n (\to 0$ as $n \to \infty$). By the usual approximation procedure it follows that (58) is asymptotically equivalent (in distribution) to

$$\sup_{1 < t \leq (1 - \epsilon_n)} \left\{ W_0(1) \sqrt{\frac{1}{t + 1} + \frac{\eta^* W_1(t - 1)}{\eta / \sqrt{t + 1}} + \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} \sqrt{t} \left( \frac{1}{t + 1} \right)} \right\},$$

$$\overset{d}{=} \sup_{0 < t \leq \epsilon_n / (1 - \epsilon_n)} \left\{ W_0(1) \sqrt{\frac{1}{t + 1} + \frac{\eta^* W_1(t)}{\eta / \sqrt{t + 1}} + \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} \frac{t}{\sqrt{t + 1}}} \right\}. \quad (65)$$

Once again, the error rate is $O_p(n^{(1/r) / \sqrt{k^*_n}}) + O_p(\sqrt{(\log n) / k^*_n}) = O_p(n^{(1/r) - 1/2})$ as $n \to \infty$.

The relevant stopping time is

$$T(x) = \inf \left\{ t: W_0(1) \sqrt{\frac{1}{t + 1} + \frac{\eta^* W_1(t)}{\eta / \sqrt{t + 1}} + \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} \frac{t}{\sqrt{t + 1}}} \geq x \right\},$$

$$= \inf \left\{ t: W_0(1) + \frac{\eta^* W_1(t)}{\eta / \sqrt{t + 1}} + \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} t \geq x \sqrt{t + 1} \right\}, \quad x > 0, \quad (66)$$

and the analogue of (61) and (64) becomes

$$E \exp \left\{ sx \sqrt{T(x)} + 1 - \left( s \sqrt{n(1 - \epsilon_n)} \frac{\Delta \theta^*}{\eta} + \frac{1}{2} \frac{s^2 \eta^2}{\eta^2} \right) T(x) \right\} = \exp \left( \frac{s^2}{2} \right), \quad s \geq 0. \quad (67)$$
The following counterpart of theorems 6 and 7 emerges.

**Theorem 8**

Assume (6) and (7) for some \( r > 2 \), and let \( k_n \) be as in theorem 5.

Under the alternative

\[
H_1^* : 0 < k_n^* < n \quad \text{and} \quad \mu^* = \mu - \Delta,
\]

where \( 0 < \Delta < \mu \), and \( \varepsilon_n = 1 - k_n^*/n \), the power of the test is, for large \( n \), approximately equal to

\[
P_\theta(T(x) \leqslant \varepsilon_n / (1 - \varepsilon_n)), \quad \text{with} \quad T(x) \quad \text{as defined in (66) and} \quad x = d_n^* \quad \text{from remark 6, and where}
\]

\(
\{W_0(t) : t \geqslant 0\} \quad \text{and} \quad \{W_1(t) : t \geqslant 0\} \quad \text{are independent Wiener processes,} \quad \theta = 1/\mu, \quad \eta^2 = \sigma^2/\mu^3,
\)

\( 0^* = 1/\mu^* \), and \( \eta^{*2} = \sigma^2/\mu^* \).

Theorem 8

The following counterpart of theorems 6 and 7 emerges.

The error in the (distributional) approximation of

\[
\max_{k_n^* \leqslant k \leqslant n} \frac{N(k) - k\theta}{\eta \sqrt{k}} \quad \text{by}
\]

\[
\sup_{0 \leq r \leq \varepsilon_n / (1 - \varepsilon_n)} \left\{ \frac{W_0(1)}{t+1} \sqrt{1 + \eta^* W_1(t)} + \sqrt{n(1 - \varepsilon_n)} \frac{\Delta \theta^*}{\eta} \frac{t}{\sqrt{t+1}} \right\}
\]

is of the order \( o_p(n^{1/r-\frac{1}{2}}) \) + \( O_p(\sqrt{(\log n)/n}) = o_p(n^{1/r-\frac{1}{2}}) \) as \( n \to \infty \).

The distribution of the stopping time \( T(x) \) is implicitly given in (67). Moreover, as \( x \to \infty \),

\[
\frac{n(1 - \varepsilon_n)T(x)}{x^2} \to \left( \frac{\eta}{\Delta \theta^*} \right)^2 \quad \text{a.s. \ and \ in \ } L^r, \quad \text{for every} \ r > 0,
\]

and

\[
\frac{T(x) - \eta^2}{2 \eta^*} = \frac{n(1 - \varepsilon_n)(\Delta \theta^*)^2 x^2}{2 \eta^*} - \frac{n(1 - \varepsilon_n)(\Delta \theta^*)^2 T(x)}{2 \eta^* x} \quad \to \quad \frac{\eta}{2 \eta^*} \quad \text{as} \quad N(0, 1).
\]

**7. Asymptotics under \( H_1 ; \theta, \eta \) known; the two-sided case**

In this section we study the behaviour of the test statistics under the two-sided alternative

\[
H_1: 0 < k_n^* < n \quad \text{and} \quad \mu^* = \mu - \Delta, \quad -\infty < \Delta < \mu.
\]

In this case the relevant incremental maxima are

\[
M^{(1)}_{i,j} = \max_{i < k \leq j} |Y_{k,n}| \quad \text{and} \quad M^{(2)}_{i,j} = \max_{i < k \leq j} |Z_{k,n}|.
\]

**7.1. Stopping rule 1**

**Theorem 9**

Let \( \{a_n\} \) and \( M^{(1)}_n \) be as in theorem 1, suppose, in addition, that (41) holds, and let \( a_n^* \) and \( b_n^* \) be given as in theorem 4.

By assuming (6)–(7) for some \( r > 2 \), we have the following asymptotic distribution under \( H_1 : \)

\[
\sqrt{n} \left( \frac{M^{(1)}_{i,j}}{\eta} \right) \quad \text{as} \quad \chi^2_r.
\]

\( \chi^2_r \) is the \( r \)-dimensional chi-squared distribution.
as \( n \to \infty \), where \( E \) is as in theorem 1.

Remark 14. By arguing as in remark 11, the theorem shows that the two-sided test based on stopping rule 1 is of (asymptotic) power 1.

Proof. The proof of (68) is very similar to that of (42) in theorem 4. First note that, by the triangle inequality,

\[
\frac{|N(k) - N(k - h_n) - h_n\theta|}{\eta \sqrt{h_n}} - \frac{|A\theta^0 \sqrt{h_n}|}{\eta} \leq \left| \frac{N(k) - N(k - h_n) - h_n\theta}{\eta \sqrt{h_n}} - \frac{\Delta \theta^0 \sqrt{h_n}}{\eta} \right|.
\]

So, in correspondence with (43)–(44), we have, as \( n \to \infty \),

\[
a_n^* \frac{\eta}{\eta^*} \left( M_{n,h_n}^{(1)} - \frac{|A\theta^0 \sqrt{h_n}|}{\eta} \right) - b_n^* \xrightarrow{p} - \infty,
\]

\[
a_n^* \frac{\eta}{\eta^*} \left( M_{n,h_n}^{(1)} - \frac{|A\theta^0 \sqrt{h_n}|}{\eta} \right) - b_n^* \xrightarrow{p} - \infty.
\]

On the other hand, for \( \Delta > 0 \), say, (otherwise consider \( \{(N(k) - N(k - h_n) - h_n\theta)\} \) instead of \( \{N(k) - N(k - h_n) - h_n\theta)\} \), we obtain

\[
P \left( \min_{k^*_n+h_n<k\leq n} \frac{N(k) - N(k-h_n) - h_n\theta}{\eta \sqrt{h_n}} < 0 \right) = P \left( \frac{\eta^*}{\eta} \inf_{1 < t \leq (n-k^*_n)/h_n} \left\{ \hat{W}_i(t) - \hat{W}_i(t-1) \right\} + \frac{\Delta \theta^0 \sqrt{h_n}}{\eta} \right)
\]

\[
+ o\left( n^{1/r} / \log(n/h_n) / h_n \right) < 0 \xrightarrow{p} 0 \quad \text{as} \quad n \to \infty,
\]

since \( \mathcal{O}(\sqrt{h_n}) \) is the dominating order.

This means that, eventually,

\[
\max_{k^*_n+h_n<k\leq n} \frac{|N(k) - N(k-h_n) - h_n\theta|}{\eta \sqrt{h_n}}
\]

can be replaced by

\[
\max_{k^*_n+h_n<k\leq n} \frac{N(k) - N(k-h_n) - h_n\theta}{\eta \sqrt{h_n}},
\]

which, along the lines leading to (42), proves (68).

The relevant approximation with a Wiener process is the absolute value of the one-sided analogue in section 6, which, for \( \Delta > 0 \), eventually can be replaced by that of section 6; cf. (40). Similarly for the corresponding stopping time, that is, for the case \( \Delta > 0 \) it coincides eventually with that of section 6. In general it becomes the smallest of that one and the obvious analogue for the case \( \Delta < 0 \).

7.2. Stopping rule 2

Theorem 10

Let \( M_{n,2} \) be as in theorem 2.

Assuming (6)–(7) for some \( r > 2 \), we have, under \( H_1 \), as \( n \to \infty \),
(a) if \(k^*_n = o(n)\),

\[
M_n^{(2)} - \frac{|A|\theta^2 n - k^*_n}{\eta} \xrightarrow{d} N(0, \eta^2/\eta^2);
\]  

(b) if \(k^*_n \sim \lambda n\) (for some \(0 < \lambda < 1\)),

\[
M_n^{(2)} - \frac{|A|\theta^2 n - k^*_n}{\eta} \xrightarrow{d} N(0, \lambda + (1 - \lambda)\eta^2/\eta^2);
\]  

(c) if \(k^*_n = n - o(n)\),

\[
M_n^{(2)} - \frac{|A|\theta^2 n - k^*_n}{\eta} \xrightarrow{d} N(0, 1),
\]

provided, in addition, that \((n - k^*_n)/\sqrt{n \log \log n} \to \infty\) as \(n \to \infty\).

Remark 15. As in remark 12, the two-sided test based on stopping rule 2 also turns out to be of (asymptotic) power 1.

Proof. (a) For the proof of (72) we only need to observe that, in analogy with (71) in the proof of theorem 10,

\[
P\left(\min_{|\eta| \leq k \leq n} \frac{N(k) - k\theta}{\eta \sqrt{k}} < 0\right) \to 0 \quad \text{as} \quad n \to \infty,
\]

for \(A > 0\). Thus, \(| \cdot |\) can be dropped in the asymptotics of \(M_n^{(2)}\), which proves (72) along the lines of the proof of (51).

(b, c) The difference to the proof of (a) is the same as in the proof of theorem 5.

As for the Wiener process approximation and the corresponding stopping times, the remark made at the end of the previous subsection (properly modified) applies here too.

8. Asymptotics under \(H_1^+\) and \(H_1\): \(\theta, \eta\) unknown

Before we discuss some limiting properties of our test statistics under the alternatives \(H_1^+\) or \(H_1\), we first have to return to a possible (sequential) estimation of \(\eta^2\) as indicated in section 5.

8.1. Variance estimation

Let us first consider \(\hat{\eta}^2_k\) as in (22), i.e.

\[
\hat{\eta}^2_k = \frac{1}{K} \sum_{i=1}^{K} \left( \frac{N(ih_n) - N((i - 1)h_n) - h_n\hat{\theta}_{kh_n}}{h_n} \right)^2, \quad k = h_n, \ldots, n,
\]

where \(K = \lceil k/h_n \rceil\) and \(\hat{\theta}_{kh_n} = (N(Kh_n) - N(0))/(Kh_n)\). Via the embedding of proposition 1 we obtain the following consistency result under \(H_0\):

Proposition 2

Let \(\{h_n; n = 1, 2, \ldots\}\) and \(\{\hat{h}_n; n = 1, 2, \ldots\}\) be as in theorem 3, but suppose that

\[
n^{2/r} \log^2 (n/h_n)\{\log(n/h_n) + \log \log n\}/h_n = o(1),
\]

and
\[ \hat{h}_n / \{hn \log^2(n/h_n) \log \log(n/h_n)\} \rightarrow \infty \text{ as } n \rightarrow \infty. \]

Then, assuming (6) and (7) for some \( r > 2 \), we have, under \( H_0 \),
\[
\max_{hn < k < n} |\hat{\eta}_k^2 - \eta^2| = o_p(1/\log(n/h_n)) \text{ as } n \rightarrow \infty.
\] (76)

**Remark 16.** (a) Proposition 2 proves that condition (23) of theorem 3 is satisfied for a choice of \( \hat{\eta}_k^2 \) as in (22)/(75) under slightly stronger conditions on the sequences \( \{h_n\} \) and \( \{\hat{h}_n\} \).

(b) A choice of \( h_n = [n^\alpha] \) and \( \hat{h}_n = [n^\beta] \), with \( 2/r < \alpha < \beta < 1 \), satisfies the assumptions required in proposition 2.

**Proof.** We modify the arguments in the proof of th. 2.1 of Steinebach (1995). First observe that, under (6) and (7), with \( k_n^* = n \),
\[
\max_{i=1,...,K} |N(ih_n) - N((i-1)hn) - h_n \theta_{Khn} - (N_i - \bar{N}_K)| \overset{a.s.}{=} o(n^{1/r}),
\] (77)

where \( N_i = N_{n,i} = \eta(W_{n,i} - W_{0,(i-1)hn}) \), \( i = 1, \ldots, K \), and \( \bar{N}_K = \sum_{i=1}^{K} N_i/K \).

Furthermore, by th. 1.2.1 of Csörgő & Révész (1981), together with the law of the iterated logarithm (LIL),
\[
\max_{i=1,...,K} |N_i - \bar{N}_K| \leq \max_{i=1,...,K} |N_i| + |\bar{N}_K| = O(\sqrt{\log(n/h_n)} + \log \log n),
\] (78)
as \( n \rightarrow \infty \), uniformly in \( k = \hat{h}_n, \ldots, n \).

So, with
\[
\hat{\eta}_k^2 = \frac{\eta^2}{K} \sum_{i=1}^{K} \frac{(N_i - \bar{N}_K)^2}{h_n}, \quad k = \hat{h}_n, \ldots, n,
\] (79)

we conclude from (77)-(78), that
\[
\max_{hn < k < n} |\hat{\eta}_k^2 - \eta^2| \overset{a.s.}{=} o(n^{1/r} \sqrt{\log(n/h_n)} + \log \log n/\sqrt{h_n}) = O(1/\log(n/h_n)),
\] (80)

where the last estimate follows from our assumptions on \( \{h_n\} \).

Now, for each \( n \),
\[
\{\hat{\eta}_k^2: k = \hat{h}_n, \ldots, n\} \overset{d}{=} \{\eta^2_{\lambda_{K-1}}/K: k = \hat{h}_n, \ldots, n\},
\] (81)

where \( \lambda_{K-1} \) are central \( \chi^2 \)-distributed random variables with \( K - 1 \) degrees of freedom.

So, as a consequence of the LIL,
\[
\max_{hn < k < n} |\hat{\eta}_k^2 - \eta^2| = o_p\left(\sqrt{\log(n/h_n)/\hat{h}_n}\right) = o_p\left(1/\log(n/h_n)\right),
\] (82)

by our assumptions on \( \{h_n\} \) and \( \{\hat{h}_n\} \); recall that \( K = [k/h_n] \).

A combination of (80) and (82) completes the proof of (76).

Proposition 2 can even be extended to hold under the alternative \( H_1 \), i.e. we have:

**Proposition 3**

Let \( \{h_n\} \) and \( \{\hat{h}_n\} \) be as in proposition 2, assume that \( \hat{h}_n \leq k_n^* \leq n \), and set
\[
\eta_k^2 = \lambda_k \eta^2 + (1 - \lambda_k) \eta^2 + \lambda_k (1 - \lambda_k) (\theta - \theta')^2,
\] (83)

\[
\lambda_k = \begin{cases} 1 & \text{for } k = \hat{h}_n, \ldots, k_n^*, \\ K^*/K & \text{for } k = k_n^* + 1, \ldots, n, \end{cases}
\] (84)
where $K = [k/h_n]$ and $K^* = [k^*/h_n]$.
Then, under (6) and (7) for some $r > 2$,
\[
\max_{h_n < k < n} |\hat{\eta}_k^2 - \eta_k^2| = o_P\left(1/\log(n/h_n)\right) \quad \text{as } n \to \infty.
\]  
\tag{85}

Proof. In view of proposition 2 we only need to discuss $\max_{n \leq k \leq n} |\cdots|$ in (85). Moreover, if we change one summand in the definition of $\hat{\eta}_k^2$ in (75), this causes an error of maximum order
\[
C_P(1/K) = o_P\left(1/\log(n/h_n)\right)
\]  
by our assumptions on $\{h_n\}$ and $\{\hat{h}_n\}$. So, we can assume w.l.o.g. that $k^*_n = K^*h_n$.
Now, similar to (77),
\[
\max_{i=1, \ldots, K} |N(ih_n) - N((i - 1)h_n) - h_n \tilde{\theta}_{K} - (N_i - \tilde{\eta}_K)| \overset{a.s.}{=} o(n^{1/r}),
\]  
\tag{87}

but here with
\[
N_i = \begin{cases} 
  h_n \theta + \eta \{W_0(ih_n) - W_0((i - 1)h_n)\}, & \text{for } 1 \leq i \leq K^*, \\
  h_n \theta^* + \eta^* \{W_1((i - K^*)h_n) - W_1((i - K^* - 1)h_n)\}, & \text{for } K^* + 1 \leq i \leq K,
\end{cases}
\]
and, for $K^* < K \leq N = [n/h_n],
\[
\tilde{\eta}_K = \tilde{\eta}_K - (1 - \lambda_K^*)\tilde{\eta}_{K^*},
\]
where $\tilde{\eta}_{K^*} = \sum_{i=K^*+1}^{K} N_i/(K - K^*)$.

By combining (87) with th. 1.2.1 of Csörgő & Révész (1981) and the LIL, it follows that assertion (80) retains for $h_n \leq k \leq n$ i.e.
\[
\max_{h_n \leq k \leq n} |\hat{\eta}_k^2 - \tilde{\eta}_k^2| = o_P\left(1/\log(n/h_n)\right).
\]  
\tag{88}

Observe that here, after some calculations,
\[
\tilde{\eta}_K^2 = \frac{1}{K} \sum_{i=1}^{K} \frac{(N_i - \tilde{\eta}_K)^2}{h_n} = \lambda_K^2 \tilde{\eta}_{K^*}^2 + (1 - \lambda_K^*)\tilde{\eta}_{K^*}^2 + \lambda_K^* (1 - \lambda_K^*)\tilde{\eta}_{K^*}^2/h_n,
\]  
\tag{89}

for $K^* < K \leq N$, with
\[
\tilde{\eta}_K^2 = \frac{1}{K} \sum_{i=1}^{K} \frac{(N_i - \tilde{\eta}_K)^2}{h_n} \quad \text{and} \quad \tilde{\eta}_{K^*}^2 = \frac{1}{K - K^*} \sum_{i=K^*+1}^{K} \frac{(N_i - \tilde{\eta}_{K^*})^2}{h_n}.
\]

Now, from the underlying $\chi^2$-distributions in combination with the LIL,
\[
\lambda_K^* \tilde{\eta}_{K^*}^2 - \eta^2 \overset{a.s.}{=} C \left(\frac{K^*}{K} \sqrt{\frac{\log \log(n/h_n)}{K^*}}\right) = C \left(\sqrt{\frac{\log \log(n/h_n)}{h_n^2}}\right),
\]  
\tag{90}

\[
(1 - \lambda_K^*) \tilde{\eta}_{K^*}^2 - \eta^2 \overset{a.s.}{=} C \left(\frac{K - K^*}{K} \sqrt{\frac{\log \log(n/h_n)}{K - K^*}}\right) = C \left(\sqrt{\frac{\log \log(n/h_n)}{h_n^2}}\right),
\]  
\tag{91}

\[
\lambda_K^* (1 - \lambda_K^*) \frac{(N_i - \tilde{\eta}_K)^2}{h_n} - (\theta - \theta^*)^2 \overset{a.s.}{=} C \left(\sqrt{\frac{\log \log(n/h_n)}{h_n^2}}\right).
\]  
\tag{92}
Since $k_n^* \geq \hat{h}_n$, the latter rates are of order $o_p(1/\log(n/h_n))$ by our assumptions on $\{h_n\}$ and $\{\hat{h}_n\}$.

By combining (88)–(92) the proof of (85) is complete.

In view of proposition 3 we suggest replacing $\hat{\eta}_k^2$ by

$$\hat{\eta}_k^2 = \min \{ \eta_j^2 : \hat{h}_n \leq j \leq k \}, \quad \hat{h}_n \leq k \leq n,$$

which satisfies the following asymptotics.

**Corollary 1**

*Under the assumptions of proposition 2 or 3 we have, for $\hat{\eta}_k^2$ and $\lambda_k$ as in (93) and (84), respectively,*

$$\max_{k_n \leq k \leq n} | \hat{\eta}_k^2 - \min \{ \eta_j^2, \lambda_k \eta_j^2 + (1 - \lambda_k)\eta_j^2 \} |$$

$$= o_p(1/\log(n/h_n)) \quad \text{as} \quad n \to \infty. \quad (94)$$

*Proof.* Note that, for $\eta_k^2$ from (83),

$$\min_{h_n \leq j \leq k} \eta_j^2 = \min \{ \eta_j^2, \lambda_k \eta_j^2 + (1 - \lambda_k)\eta_j^2 \},$$

and obviously,

$$| \min_{h_n \leq j < k} \eta_j^2 - \min_{h_n \leq j \leq k} \eta_j^2 | \leq \max_{h_n \leq j \leq k} | \eta_j^2 - \eta_k^2 |.$$

So, (94) is an immediate consequence of (85).

**Remark 17.** (a) Note that the estimators $\hat{\eta}_k^2$ and $\hat{\eta}_k^2$ are always uniformly bounded away from zero and infinity (in probability). The estimators $\hat{\eta}_k^2$ are likely to be more stable than $\hat{\eta}_k^2$.

For example, if $\eta_k^2 \gg \eta_j^2$, then $\min_{h_n \leq j \leq k} \eta_j^2 = \eta_k^2$ for all $k = h_n, \ldots, n$. (b) Again, a choice of $h_n = [n^r]$ and $\tilde{h}_n = [n^\beta]$, with $2/r < \alpha < \beta < 1$, satisfies the assumptions required in propositions 2 and 3.

### 8.2. Asymptotics

In view of the possible fluctuations of our (sequential) variance estimators $\hat{\eta}_k^2$ and $\hat{\eta}_k^2$, we cannot generally expect proper limiting distributions under alternatives. However, we obtain the following results on the rates of consistency of our test statistics $\hat{M}_n^{(1)}$ and $\hat{M}_n^{(1)}$.

In order to formulate our results, we define for notational convenience that

$$X_n \overset{P}{\leq} r_n, \quad \text{if} \quad P(X_n \geq (1+\varepsilon)r_n) \to 0 \quad \forall \varepsilon > 0,$$

$$X_n \overset{P}{\geq} r_n, \quad \text{if} \quad P(X_n \leq (1-\varepsilon)r_n) \to 0 \quad \forall \varepsilon > 0,$$

as $n \to \infty$, for any sequence of random variables $\{X_n\}$ and positive constants $\{r_n\}$.

**Theorem 11**

*Let $\{h_n\}$ and $\{\hat{h}_n\}$ be as in corollary 1, and replace $\hat{\eta}_k^2$ by $\hat{\eta}_k^2$ in the definition of $\hat{M}_n^{(1)}$ and $\hat{M}_n^{(1)}$ from theorem 3.*
Assuming (6) and (7) for some \( r > 2 \), and (41), we have, under \( H^+_1 \) and \( H_1 \), respectively,

\[
\sqrt{2 \log(n/h_n)} \overset{p}{\leq} \max_{h_n \leq k \leq n} \left( \hat{Y}_k - \sqrt{h_n \hat{\lambda}_k} \frac{\Delta \theta^*}{\hat{\eta}_k} \right)^p \overset{p}{\leq} \frac{\eta^*}{\min(\eta, \eta^*)} \sqrt{2 \log(n/h_n)},
\]

(95)

\[
\sqrt{2 \log(n/h_n)} \overset{p}{\leq} \max_{h_n \leq k \leq n} \left( |\hat{Y}_k| - \sqrt{h_n \hat{\lambda}_k} \frac{|\Delta \theta^*|}{\hat{\eta}_k} \right)^p \overset{p}{\leq} \frac{\eta^*}{\min(\eta, \eta^*)} \sqrt{2 \log(n/h_n)},
\]

(96)

where

\[
\hat{\lambda}_k = \begin{cases} 1 & \text{for } 1 \leq k \leq k^*_n + h_n, \\ \frac{k^*_n}{k - h_n} & \text{for } k^*_n + h_n < k \leq n. \end{cases}
\]

Moreover, from the above relations,

\[
\sqrt{h_n} \frac{\Delta \theta^*}{\eta} \overset{p}{\leq} \hat{M}_n^{(1)} \overset{p}{\leq} \sqrt{h_n} \frac{\Delta \theta^*}{\min(\eta, \eta^*)},
\]

(98)

\[
\sqrt{h_n} \frac{|\Delta \theta^*|}{\eta} \overset{p}{\leq} \hat{M}_n^{(1)} \overset{p}{\leq} \sqrt{h_n} \frac{|\Delta \theta^*|}{\min(\eta, \eta^*)}.
\]

(99)

Remark 18. Similar to remarks 11 and 14, the tests based on stopping rules \( \hat{I}^+ \) and \( \hat{I} \) have an (asymptotic) power 1.

Proof. In view of (6) and (7) we have, uniformly in \( \hat{h}_n \leq k \leq k^*_n + h_n \),

\[
\hat{\theta}_k - \theta \overset{a.s.}{=} \frac{W_0(k - h_n)}{k - h_n} + o(h_n^{-(1 - \frac{1}{r})}) \overset{a.s.}{=} \mathcal{C} \left( \sqrt{\log \log n}/h_n \right).
\]

Similarly, uniformly in \( k^*_n + h_n < k \leq n \),

\[
\hat{\theta}_k - \hat{\lambda}_k \theta - (1 - \hat{\lambda}_k) \theta^* \overset{a.s.}{=} \mathcal{C} \left( \sqrt{\log \log n}/h_n \right),
\]

with \( \hat{\lambda}_k \) as in (97). Now, for \( \hat{h}_n \leq k \leq n \),

\[
N(k) - N(k - h_n) - \hat{h}_n \hat{\theta}_k - h_n \hat{\lambda}_k \Delta \theta^* \overset{a.s.}{=} N(k) - N(k - h_n) - h_n \theta + h_n (\theta - \{ \hat{\lambda}_k \theta - (1 - \hat{\lambda}_k) \theta^* \}) - h_n \hat{\lambda}_k (\theta^* - \theta)
+ \mathcal{C} \left( \sqrt{h_n \log \log n} \right)
\overset{a.s.}{=} N(k) - N(k - h_n) - h_n \theta - h_n \Delta \theta^* + \mathcal{C} \left( \sqrt{h_n \log \log n} \right),
\]

(100)

where the \( \mathcal{C} \)-rates in (100) hold uniformly.

A combination of the latter relation with theorem 4 and corollary 1 proves (95).

The additional arguments for (96) are similar to those in the proof of theorem 9 and therefore omitted.

Finally, upon observing that \( \max_{h_n \leq k \leq n} \hat{\lambda}_k = 1 \) and \( \sqrt{\log(n/h_n)} = o(\sqrt{h_n}) \), (98)–(99) are immediate from (95)–(96).

9. Concluding remarks

In this paper we have suggested and analysed some new kinds of truncated sequential change-point procedures based on renewal counting processes. It has been assumed that information on the (typically continuous time-parameter) stochastic process under observation is only
partially available (say) at discrete time-points. Making use of a (weak or strong) embedding of the observed process into a Gaussian framework, asymptotic distributions of the suggested testing procedures have been obtained under the null as well as (one- or two-sided) alternative hypotheses. The proposed stopping rules may be viewed as analogues of the well-known Shewhart and CUSUM control charts from the sequential analysis of i.i.d. observations. Other stopping rules, such as e.g. an EWMA control chart, can be handled by similar methods and will be discussed elsewhere.

It is worthwhile mentioning that our approach is solely based on the existence of an invariance principle for the observed renewal process. Such embeddings are available for wide classes of stochastic processes and thus provide us with a rather general method in sequential change-point analysis. If necessary, they also allow for suitable sequential estimation of unknown parameters in the underlying model.

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References


Appendix

Here we collect some results concerning certain stopping times related to the Wiener process. The proofs follow essentially standard methods, such as exponential martingales, optional sampling, dominated convergence. We therefore confine ourselves to statements, and, at times, a sketch. For proofs and further details we refer to Gut & Steinebach (2000).

Let $f \sim W(t)$, $t \geq 0$ be a Wiener process, and let $T_0(x) = \inf\{t: W(t) \geq x\} = \inf\{t: W(t) = x\}, \ x > 0,$

be the usual first passage time. Set $L_4(s) = E\{sW(t)\}, s \geq 0$. It is well known that one can use the fact that

$$\frac{\exp\{sW(t)\}}{L_4(s)} = \exp\{sW(t) - s^2t/2\}, \ t \geq 0, \text{ is a martingale},$$

in order to determine the distribution of $T_0(x)$. This can similarly be done for the first passage time $T_\mu(x) = \inf\{t: W(t) + \mu t \geq x\}, \ x > 0$, of a Wiener process with linear drift $\{W(t) + \mu t, \ t \geq 0\}, \mu > 0$, see e.g. Borodin & Salminen (1996), and/or Gut (1974, sect. 4).

A.1. First passage times related to the Wiener process

As a first step we need extensions of some results in Gut (1974, sect. 4), concerning the existence of moment-generating functions, and, hence, moments of stopping times.

Let $\alpha, \beta, \gamma > 0$, set $X(t) = \alpha W(t) + \beta t$ for $t \geq 0$, and put

$$T_1(x) = \inf\{t: X(t) \geq x\sqrt{t + \gamma}\} = \inf\{t: X(t) = x\sqrt{t + \gamma}\}, \ x > 0.$$  \hspace{1cm} (103)

Since $\sqrt{t + \gamma}$ is regularly varying at infinity (with exponent 1/2), all the results and methods of Gut (1974, sect. 4) apply, in particular, $T_1(x)$ possesses a moment-generating function.

Next, let $W^*$ be a normal random variable with mean 0 and variance $\tau^2$, independent of the Wiener process (and, hence, of $\{X(t)\}$), and set

$$T_2(x) = \inf\{t: W^* + X(t) \geq x\sqrt{t + \gamma}\}, \ x > 0,$$  \hspace{1cm} (104)

Note that with $t$ restricted to the positive integers this stopping time corresponds to the first passage time of a “delayed” random walk.

Proposition 4

\[ E \exp\{sT_2(x)\} < \infty, \quad -\infty < s < s_2, \quad \text{for some } s_2 > 0. \]

In particular, \( E(T_2(x)^r) < \infty \) for all \( r > 0 \).

For a proof one follows the technique in Gut (1974) by first considering the first passage time of a related random walk across a horizontal barrier:

\[ T_2^*(x; \alpha, \beta) = \min\{n : W^* + X(n) \geq x\}, \quad x > 0, \]

and domination via

\[ T_2(x; \alpha, \beta) = \inf\{t : W^* + X(t) \geq x\} \leq T_2^*(x; \alpha, \beta), \]

and a final observation that

\[ T_2(x) \leq T_2(A, \alpha, \beta - \epsilon), \]

for \( t \) large, \( \epsilon < \beta \), and some positive constant \( A \).

For \( \gamma = 0 \) the situation is radically different in that \( T_1(x) \xrightarrow{a.s.} 0 \) in view of the law of the iterated logarithm.

Next we note that by the martingale property, Doob’s optional sampling theorem, dominated convergence, and the fact that \( zW(T_1) + \beta T_1 = x\sqrt{T_1 + \gamma} \), we have

\[ E \exp\{sx\sqrt{T_1 + \gamma} - (s\beta + s^2x^2/2)T_1\} = 1. \]

The same procedure, together with the fact that if the elements of a martingale are multiplied with the same independent random variable, then the resulting family remains a martingale, and the inequality

\[ W^* + zW(T_2 \wedge m) \leq W^* + zW(T_2 \wedge m) + \beta(T_2 \wedge m) \leq \max\{0, W^*\} + x\sqrt{T_2 + \gamma} \]

proves the analogous inequality for \( T_2 \), in particular,

\[ E \exp\{sx\sqrt{T_2 + \gamma} - (s\beta + s^2x^2/2)T_2\} = \exp(s^2r^2/2), \quad s \geq 0. \]  

(105)

Proposition 5

For \( i = 1, 2 \), we have

\[ \frac{T_i(x)}{x^2} \xrightarrow{a.s.} \frac{1}{\beta^2} \quad \text{as } x \to \infty, \]

\[ \frac{T_i(x) - x^2/\beta^2}{2ax/\beta^2} = \frac{\beta^2 T_i(x)}{2x} \frac{x}{2x} \xrightarrow{d} N(0,1) \quad \text{as } x \to \infty, \]

\((\{T_i(x)/x\}, x \geq 1) \) is uniformly integrable for all \( r > 0 \).

Almost sure and distributional convergence are proved (essentially) as in Gut (1974, sect. 4). Uniform integrability for \( i = 1 \), follows from the strong law, the fact that \( T_1(x) \) possesses moments of all orders, and the inequality \( T_1(x)/x \leq T_1([x] + 1)/[x] \leq 2T_1([x] + 1)/[x + 1] \) for all \( x > 1 \). For \( i = 2 \) one conditions on \( W^* \), and exploits the independence between \( W^* \) and \( \{X(t)\} \) to obtain \( E(T_2(x)^r) \leq C(1 + E|W^*|^r + x^r) \), for any \( r > 0 \).

For the two-sided case we let \( \beta \neq 0 \) and define

\[ T_3(x) = \inf\{t : |X(t)| \geq x\sqrt{t + \gamma}\} = \inf\{t : |X(t)| = x\sqrt{t + \gamma}\}, \quad x > 0, \]

\[ T_4(x) = \inf\{t : |W^* + X(t)| \geq x\sqrt{t + \gamma}\}, \quad x > 0. \]

If \( \beta > 0 \), then, clearly, \( T_3(x) \leq T_1(x) \) and \( T_4(x) \leq T_2(x) \). By a symmetry argument it follows that the moment-generating functions of \( T_3(x) \) and \( T_4(x) \) both exist.

A.2 First passage times related to increments of the Wiener process

Let $Y_k = \sup_{t \leq k+1} (W(t) - W(t-1))$, and $Z_k = W(k) - W(k-1)$, $k \geq 1$, and set

$T_5(x) = \inf \{t \geq 1 : W(t) - W(t-1) \geq x\}$, $x > 0$,

$T_6(x) = \min \left\{ n \geq 1 : \sup_{1 \leq t \leq n} (W(t) - W(t-1)) \geq x \right\}$

$= \min \left\{ n \geq 1 : \max_{1 \leq k \leq n-1} Y_k \geq x \right\}$, $x > 0$.

Because of dependencies, the problems analogous to those in section A.1 are much harder, and we can only provide some minor facts.

- The sequence $\{Y_k, k \geq 1\}$ is one-dependent;
- $T_5(x) \leq T_6(x)$ for all $x > 0$;
- $\sup_{1 \leq t \leq n} (W(t) - W(t-1)) \geq \max_{1 \leq k \leq n} (W(k) - W(k-1)) = \max_{1 \leq k \leq n} Z_k$;
- The moment-generating function of $T_6(x)$ (and therefore of $T_5(x)$) exists.