Similarity reductions of the generalized Burgers equation \( u_t + u^n u_x + (\frac{j}{2^j} + \alpha)u + (\beta + \frac{\gamma}{x})u^{n+1} = u_{xx} \), where \( \alpha, \beta, \gamma \) are non-negative constants, \( n \) a positive integer and \( j = 0, 1, 2 \), are obtained by the direct method of Clarkson and Kruskal [1]. This is the first work to report the similarity variables as an incomplete gamma function and also as a power of \( x/\sqrt{t} \), and to provide a perturbation solution of an Euler–Painlevé transcendent.

1. Introduction

The Burgers equation

\[ u_t + uu_x = u_{xx}, \quad (1) \]

has been shown by Hopf [2] and Cole [3] to reduce to the linear heat conduction equation

\[ \phi_t = \phi_{xx}, \quad (2) \]
later Woodard [4], Ames [5], and Tajiri et al. [6] obtained similarity solutions of (1) using both classical and non-classical Lie’s method of infinitesimal transformations [7]. Benton and Platzman [8] have catalogued all the available solutions of (1).

A detailed similarity analysis of the following varigated generalizations of Burgers equations

\[ u_t + u^\beta u_x + \lambda u^\alpha = \frac{\delta}{2} u_{xx}, \quad (4) \]

\[ u_t + u^\alpha u_x + \frac{ju}{2t} = \frac{\delta}{2} u_{xx}, \quad (5) \]

\[ u_t + u^\beta u_x + f(t) u^\alpha = \frac{\delta}{2} g(t) u_{xx}. \quad (6) \]

have been carried out in a systematic manner by Sachdev and his collaborators [9–11]. In (4)–(6), \( g(t) \) is a smooth function of \( t; \alpha, \beta, \delta \in \mathbb{R}^+ \) and \( j = 1, 2 \). They reduced (4)–(6) to a class of nonlinear ordinary differential equations of the form

\[ yy'' + ay'^2 + f(x)yy' + g(x)y'^2 + by' + c = 0, \quad (7) \]

where \( a, b, \) and \( c \in \mathbb{R} \). Equations belonging to the class (7) are referred by Sachdev et al. to as Euler–Painlevé transcendent. The reason is: Equation (7) with \( b = c = 0 \) was introduced by Euler and Painlevé (cf. [12]). Equation (7) is exactly linearizable to

\[ v'' + f v' + (a + 1) g v = 0, \quad (8) \]

through

\[ y = v^{\frac{1}{a+1}}. \quad (9) \]

But for the Burgers equation and its generalizations it has been established that \( b \neq 0 \) and \( c \) is any constant. For the Burgers equation the solution is expressible in terms of a complementary error/exponential function. But solutions have not been found in terms of elementary functions for generalized Burgers equations.

Soewono and Debnath [13] studied

\[ u_t + u^\beta u_x = t^N u_{xx}, \quad (10) \]

where \( \beta > 0 \) and \( N \in \mathbb{R} \).
Doyle and Englefield [14] have determined similarity reductions of the generalized Burgers equation
\[ u_t + uu_x = \Delta(t)u_{xx}, \tag{11} \]
introduced by Lighthill [15], using the method proposed by Olver [16] for defining an optimal system of group-invariant solutions.

Mayil Vaganan [17], unlike in the previous works, employed the \textit{direct method} of Clarkson and Kruskal [1] to derive the similarity reductions of the Burgers equation (1) as well as the generalized Burgers equations:
\[ u_t + uu_x + \frac{j u}{2t} = u_{xx}, \tag{12} \]
\[ u_t + u^2 u_x + \frac{j u}{2t} = u_{xx}, \tag{13} \]
\[ u_t + u^2 u_x = u_{xx}, \tag{14} \]
\[ u_t + uu_x + f(x, t) = g(t)u_{xx}, \tag{15} \]
\[ u_t + u^\beta u_x + f(t)u^\alpha = g(t)u_{xx}. \tag{16} \]

It must be mentioned here that Clarkson and Kruskal [1] have themselves given an extensive analysis of similarity reductions of the Burgers equation (1).

In the present paper, we give a detailed account of the similarity solutions of the generalized Burgers equation
\[ u_t + uu_x + \left( \frac{j}{2t} + \alpha \right) u + \left( \beta + \gamma \frac{x}{x} \right) u^{n+1} = u_{xx}, \tag{17} \]
where \( \alpha, \beta, \) and \( \gamma \) are non-negative constants, \( n \) a positive integer and \( j = 0, 1, 2 \) by the \textit{direct method} of Clarkson and Kruskal [1]. To the best of our knowledge we are the first to obtain the similarity variable in terms of an incomplete gamma function and also determine a perturbation solution of an Euler–Painlevé transcendent.

Equation (17) has already been shown amenable to exact analysis by Sachdev et al. [18]. In their work, after a myriad of transformations and tedious calculations an exact representation of N-wave solution of (17) has been obtained.

Because Equation (17) contains general powers of \( u \), the direct method may not be applied to it. Therefore we transform (17) to
\[ vv_t + v_x - n \left( \frac{j}{2t} + \alpha \right) v^2 - n \left( \beta + \gamma \frac{x}{x} \right) v + \frac{n+1}{n} v_x^2 - vv_{xx} = 0, \tag{18} \]
through
\[ u(x, t) = [v(x, t)]^{-1/n}. \tag{19} \]
Because we already have made the inverse substitution to (17) via (18), if we apply the direct method to (19) the resulting ordinary differential equation is an Euler–Painlevé transcendent.

The rest of this paper is organized as follows: Section 2 gives several reductions of (18) using the direct method as well as the solutions of the reduced ordinary differential equations. The conclusion of the present study is set forth in Section 3.

2. Direct similarity analysis of (18)

Now we make the ansatz

\[ v(x, t) = A(x, t) + B(x, t)H(z), \quad B(x, t) \neq 0, \]  

(20)

where \( z = z(x, t) \) is the similarity variable.

Putting (20) into (18), we get

\[
AA_t + A_x - n \left( \frac{j}{2t} + \alpha \right) A^2 - n \left( B + \frac{\gamma}{x} \right) A \\
+ \frac{n + 1}{n} A_x^2 - AA_{xx} + \left( AB_t + BA_t + B_x - 2n \left[ \frac{j}{2t} + \alpha \right] AB \right) \\
- n(\beta + \gamma x)B + 2\frac{n + 1}{n} A_x B_x - AB_{xx} - BA_{xx} \right) H \\
+ \left( ABz_t + BA_x + 2\frac{n + 1}{n} BA_t z_x - 2AB_x z_t - AB_{xx} z_t \right) H' \\
+ \left( BB_t - n \left( \frac{j}{2t} + \alpha \right) B^2 + \frac{n + 1}{n} B_x^2 - BB_{xx} \right) H^2 \\
+ \left( B^2 z_t + \frac{2}{n} BB_x z_x - B^2 z_{xx} \right) HH' \\
+ \left( \frac{n + 1}{n} B^2 z_x^2 \right) H'^2 - ABz_x^2 H'' - B^2 z_x^2 HH'' = 0.
\]

(21)

For (21) to represent an ordinary differential equation governing \( H(z) \), we introduce functions \( \Gamma_n(z) \), \( n = 1, 2, \ldots, 7 \) in the following manner:

\[
AA_t + A_x - n \left( \frac{j}{2t} + \alpha \right) A^2 \\
- n \left( \beta + \frac{\gamma}{x} \right) A + \frac{n + 1}{n} A_x^2 - AA_x = -B^2 z_x^2 \Gamma_1(z),
\]

(22)
\[ AB_t + BA_t + B_x - 2n \left( \frac{j}{2t} + \alpha \right) AB \]
\[- n \left( \beta + \frac{\gamma}{x} \right) B + \frac{2(n+1)}{n} A_x B_x - AB_{xx} - BA_{xx} = -B^2 z_x^2 \Gamma_2(z), \] (23)
\[ ABz_t + Bz_x + \frac{2(n+1)}{n} B A_x z_x - 2 A B_x z_x - A B z_{xx} = -B^2 z_x^2 \Gamma_3(z), \] (24)
\[ BB_t - n \left( \frac{j}{2t} + \alpha \right) B^2 + \frac{n+1}{n} B_x^2 - BB_{xx} = -B^2 z_x^2 \Gamma_4(z), \] (25)
\[ B^2 z_t + \frac{2}{n} B B_x z_x - B^2 z_{xx} = -B^2 z_x^2 \Gamma_5(z), \] (26)
\[ \frac{n+1}{n} = -\Gamma_6(z), \] (27)
\[ A = B \Gamma_7(z). \] (28)

In view of (22)–(28), (21) takes the form
\[ \Gamma_1(z) + \Gamma_2(z) H + \Gamma_3(z) H' + \Gamma_4(z) H^2 \]
\[ + \Gamma_5(z) HH' + \Gamma_6(z) H^2 + \Gamma_7(z) H'' + HH'' = 0. \] (29)

The following remarks are used in the determination of \( \alpha, \beta, z, \) and \( \Gamma_n(z), n = 1, 2, \ldots, 7 \) from the system (22)–(28):

**Remark 1:** If \( A(x, t) \) has the form \( A(x, t) = \hat{A}(x, t) + B(x, t) \Gamma(z) \), then we put \( \Gamma(z) \equiv 0 \).

**Remark 2:** If \( B(x, t) \) is found to have the form \( B(x, t) = \hat{B}(x, t) \Gamma(z) \), then we may choose \( \Gamma(z) \equiv 1 \).

**Remark 3:** If \( z(x, t) \) is to be determined from the relation \( F(z) = \hat{z}(x, t) \), where \( F(z) \) is an invertible function, then without loss of generality, we may take \( F(z) \equiv z \).

Using Remark 1 in Equation (28), we obtain \( \Gamma_7(z) \equiv 0 \), and therefore
\[ A \equiv 0. \] (30)

Substituting (30) in Equations (22)–(24), we have \( \Gamma_1(z) = 0 \) and
\[ B_x - n \left( \beta + \frac{\gamma}{x} \right) B = -B^2 z_x^2 \Gamma_2(z), \] (31)
\[ B = z_x^{-1} \left( -\Gamma_3^{-1}(z) \right). \] (32)
By Remark 2, Equation (32) gives $\Gamma_3(z) = -1$ and therefore

$$B = z^{-1}. \quad (33)$$

Writing $\Gamma_2(z) = -\Lambda_2(z)/\Lambda_1(z)$ (31) and integrating twice with respect to $x$, we get

$$\Lambda_2(z) = M(t) \int x^{-n\gamma} e^{-n\beta x} \, dx + N(t), \quad (34)$$

where $M(t)$ and $N(t)$ are functions of integration.

Applying Remark 3 to (34), we may choose $\Lambda_2(z) = z$ to obtain the similarity variable,

$$z = M(t) \int x^{-n\gamma} e^{-n\beta x} \, dx + N(t). \quad (35)$$

Using $\Lambda_2(z) = z$ in $\Gamma_2(z) = -\Lambda_2''(z)/\Lambda_2'(z)$, we find that $\Gamma_2(z) = 0$.

We remark that the integral in (35) when integrated between the limits 0 and $x$ gives rise to the similarity variable in terms of the incomplete gamma function:

$$z = M(t) \gamma(1-n\gamma, X) (n\beta)^{1-n\gamma} + N(t), \quad (36)$$

where $\gamma(1-n\gamma, X)$, with $X = n\beta x$, is the incomplete gamma function. But the foregoing analysis becomes intractable if we proceed with the similarity variable as given by (36). So we consider three special cases of (35):

2.1. $\beta = 0$, $n\gamma \neq 1$

Carrying out the integration in (35), after putting $\beta = 0$, we find that

$$z = M(t) \frac{x^{1-n\gamma}}{1-n\gamma}, \quad (37)$$

where we have assumed that $N(t) = 0$.

On using (33) and (37) in (25), we get

$$\left[ \frac{M'}{M} + n \left( j/2t + \alpha \right) \right] \frac{x^{2n\gamma}}{M^2} - n\gamma(\gamma + 1) \frac{x^{2n\gamma-2}}{M^2} = \Gamma_4(z). \quad (38)$$

We write $x$ by solving (37) as

$$x = \left[ \frac{(1-n\gamma)z}{M} \right]^\frac{1}{1-n\gamma}. \quad (39)$$

Applying (39) in (38), we have

$$\left[ \frac{M'}{M \gamma^{n\gamma}} + n \left( \frac{j}{2t} + \alpha \right) \frac{1}{M \gamma^{n\gamma}} \right] (1-n\gamma)^{\frac{2n\gamma}{1-n\gamma}} z^{\frac{2n\gamma}{1-n\gamma}} - \frac{n\gamma(\gamma + 1)}{(1-n\gamma)^2} z^{-2} = \Gamma_4(z). \quad (40)$$
Equation (40) is meaningful only if we take
\[
\frac{M'}{M^{1-n\gamma}} + n \left( \frac{j}{2t} + \alpha \right) \frac{1}{M^{\frac{1}{1-n\gamma}}} = a, \tag{41}
\]
where \(a\) is a constant.

In view of (41), (40) reads as follows:
\[
\Gamma_4(z) = a(1 - n\gamma)^{\frac{2n\gamma}{1-n\gamma}} z^{\frac{2n\gamma}{1-n\gamma} - n\gamma(\gamma + 1)} \frac{1 - n\gamma}{(1 - n\gamma)^2} z^{-2}. \tag{42}
\]
Substituting (33) and (37) in (26), we obtain
\[
-(1 - n\gamma)^{\frac{2n\gamma}{1-n\gamma}} \frac{M'}{M^{\frac{1}{1-n\gamma}}} z^{\frac{1+n\gamma}{1-n\gamma}} - \frac{\gamma(2 + n)}{1 - n\gamma} z^{-1} = \Gamma_5(z). \tag{43}
\]
Equation (43) requires that
\[
\frac{M'}{M^{\frac{1}{1-n\gamma}}} = b, \tag{44}
\]
where \(b\) is a constant, and becomes
\[
\Gamma_5(z) = - \frac{1}{1 - n\gamma} \left[ b(1 - n\gamma)^{\frac{1+n\gamma}{1-n\gamma}} z^{\frac{1+n\gamma}{1-n\gamma}} + \gamma(2 + n)z^{-1} \right]. \tag{45}
\]
Solving (44) for \(M(t)\), we obtain
\[
M(t) = \left[ - \frac{2b}{1 - n\gamma} t \right]^{\frac{1-n\gamma}{2}}. \tag{46}
\]
Substituting (44) and (46) in (41), we get
\[
\alpha = 0, \tag{47}
\]
and
\[
a = \frac{b [n(\gamma + j) - 1]}{n\gamma - 1}. \tag{48}
\]
Applying (48) in (42), we find that
\[
\Gamma_4(z) = - \frac{1}{(1 - n\gamma)^2} \left[ b [n(\gamma + j) - 1] (1 - n\gamma)^{\frac{1+n\gamma}{1-n\gamma}} z^{\frac{2n\gamma}{1-n\gamma}} + n\gamma(\gamma + 1)z^{-2} \right]. \tag{49}
\]
On inserting (46), (37) becomes
\[ z = -(n\gamma - 1) \frac{ny+1}{r} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma}. \] (50)

It is strange to observe from (50) that the similarity variable of (18) and hence that of the generalized Burgers equation (17) is not \( \frac{x}{\sqrt{t}} \), but a power of it as given in (50). We caution the reader that if we remove the power of \( \frac{x}{\sqrt{t}} \) in (50) by the introduction of \( z \) as a new independent variable, then this approach only renders the remaining analysis intractable. Therefore we conclude that unlike for the other generalized Burgers equations, where the similarity variable is simply \( \frac{x}{\sqrt{t}} \), for the generalized Burgers equation (17), the similarity variable must be a power of \( \frac{x}{\sqrt{t}} \).

Applying (30), (33), and (50) into the assumed ansatz (20) for \( v(x, t) \), the similarity transformation (18) of (20), with \( \alpha = \beta = 0 \), namely,

\[ vv_t + v_x - \frac{nj}{2t} v^2 - \frac{n\gamma}{x} v + \frac{n + 1}{n} v_x^2 - vv_{xx} = 0, \] (51)

now takes the form

\[ v(x, t) = \frac{1}{1-n\gamma} \frac{x}{z} H(z), \] (52)

\[ z(x, t) = -(n\gamma - 1) \frac{ny+1}{r} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma}. \] (53)

Using (47) and \( \beta = 0 \) in (17) the \( u \) equation assumes the form

\[ u_t + uu_x + \left( \frac{j}{2t} \right) u + \left( \frac{\gamma}{x} \right) u^{n+1} = u_{xx}. \] (54)

Substituting for \( \Gamma_n(z) \)'s, \( n = 4, 5, 6 \) from (49), (45), and (27) and also using \( \Gamma_1(z) = \Gamma_2(z) = \Gamma_3(z) = 0, \Gamma_3(z) = -1 \) in (29), we get the ordinary differential equation for \( H(z) \) appearing in the similarity transformation (52) which is given by

\[ H' + \frac{1}{(1-n\gamma)^2} \left[ b(n\gamma + nj - 1)(1-n\gamma) \frac{ny+1}{r} \frac{2ny}{1-n\gamma} z \right] H^2 + \frac{n + 1}{n} H^2 \]

\[ + n\gamma(\gamma + 1)z^{-2} \right] H^2 + \frac{n + 1}{n} H^2 \]

\[ + \frac{1}{1-n\gamma} \left[ b [(1-n\gamma)z] \frac{ny+1}{r} + \gamma(2+n)z^{-1} \right] H H' - HH'' = 0. \] (55)
To facilitate the derivation of a perturbation solution of (55), we first change it to
\[ e^\theta F' + \frac{1}{n(1-n\gamma)^2} \left( b \left[ n^2 j(1-n\gamma)^{\frac{1+4n\gamma}{1-4n\gamma}} - n(1-n\gamma)^{\frac{2}{1-4n\gamma}} \right] e^{\frac{2\theta}{1-4n\gamma}} 
\]
\[ + n^3 \gamma^2 + n^2 \gamma^2 - 2n^2 \gamma - 2n\gamma + n - 1 \right) F^2 + \frac{n+1}{n} F'^2 
\]
\[ + \frac{1}{n(1-n\gamma)} \left[ b n(1-n\gamma)^{\frac{1+4n\gamma}{1-4n\gamma}} e^{\frac{2\theta}{1-4n\gamma}} + 2n\gamma + n \right] FF' - FF'' = 0, \quad (56) \]

via the transformation
\[ \frac{H(z)}{z} = e^{-\theta} F(\theta), \quad (57) \]
\[ \theta(x, t) = \log z. \quad (58) \]

Following the work of Sachdev and his collaborators [9–11], we call (56) an Euler–Painlevé transcendent, and proceed to determine a perturbation solution of it by taking \( b = \epsilon \), a very small positive parameter.

Writing
\[ F(\theta) = F_0(\theta) + \epsilon F_1(\theta) + \epsilon^2 F_2(\theta) + \cdots \]
\[ \approx F_0(\theta) + \epsilon F_1(\theta), \quad (59) \]

into the Equation (56), with \( b = \epsilon \), and equating the coefficients of \( \epsilon^0 \) and \( \epsilon \) to zero, we have
\[ e^\theta F_0' + \frac{1}{n(1-n\gamma)^2} \left( n^3 \gamma^2 + n^2 \gamma^2 - 2n^2 \gamma - 2n\gamma + n - 1 \right) F_0^2 
\]
\[ + \frac{n+1}{n} F_0'^2 + \frac{2\gamma + 1}{1-n\gamma} F_0 F_0' - F_0 F_0'' = 0, \quad (60) \]
\[ e^\theta F_1' + (nj + n\gamma - 1)(1-n\gamma)^{\frac{3n\gamma-1}{1-4n\gamma}} e^{\frac{2\theta}{1-4n\gamma}} F_0^2 
\]
\[ + 2 \left( n^3 \gamma^2 + n^2 \gamma^2 - 2n^2 \gamma - 2n\gamma + n - 1 \right) F_0 F_1 + \frac{2(n+1)}{n} F_0 F_1' 
\]
\[ + (1-n\gamma)^{\frac{2n\gamma}{1-4n\gamma}} e^{\frac{2\theta}{1-4n\gamma}} F_0 F_1' + \frac{2\gamma + 1}{1-n\gamma} F_0 F_1' 
\]
\[ + \frac{1}{n(1-n\gamma)} F_0' F_1 - (F_0 F_1'' + F_1 F_0'') = 0. \quad (61) \]

A solution of (60) is sought in the form \( F_0(\theta) = Ae^\theta \); upon substituting this into (60) it is found that \( A \) is given by
\[ A = \frac{-(1-n\gamma)^2}{n^3 \gamma^2 - (3n+2)\gamma + 2}. \quad (62) \]
With $F_0(\theta) = Ae^\theta$, Equation (61) changes to

$$F''_1 + \frac{[-n^3\gamma^2 + 2n^2\gamma - (n + 2)]}{n(1 - n\gamma)^2} F'_1 = \frac{2(n^5 + n^4)\gamma^3 - 6(n^4 + n^3)\gamma^2 + (6n^3 + n^2)\gamma + (3n - 2n^2 - 1)}{n(1 - n\gamma)} F_1$$

$$= \frac{jn(1 - n\gamma)^{1 + n\gamma} e^{\frac{3 - n\gamma}{2n\gamma}}}{-n^2\gamma^2 + (3n + 2)\gamma - 2}. \quad (63)$$

The general solution of the inhomogeneous second order linear Equation (63) is

$$F_1(\theta) = c_1 e^{m_1\gamma} + c_2 e^{m_2\gamma}$$

$$+ \frac{jn^2(1 - n\gamma)^{2(1 - n\gamma)} e^{\frac{3 - n\gamma}{2n\gamma}}}{[(3n + 2)\gamma - n^2\gamma^2 - 2]} \left[ \gamma^5 - b\gamma^4 + c\gamma^3 - a_1\gamma^2 + b_1\gamma + c_1 \right]. \quad (64)$$

In (64), $a = 2(n^7 + n^6)$, $b = 10(n^6 + n^5)$, $c = 20n^5 + 15n^4$, $a_1 = 14n^4 + 3n^3 + n^2$, $b_1 = 4n^3 - 14n^2 + 4n$, and $c_1 = -2n^2 + 3n - 7$, and the constants $m_1$ and $m_2$ are

$$m_1 = \frac{1}{2n(-1 + n\gamma)^2} \left[ 2 + n - 2n^2\gamma + n^3\gamma^2 - ((2 + n - 2n^2\gamma + n^3\gamma^2)^2$$

$$+ 4n(-1 + n\gamma)^3((-1 + 3n + n^2(-2 + \gamma) - n^3(-6 + \gamma)\gamma$$

$$+ 2n^5\gamma^3 + n^4\gamma^2(-1 + 2\gamma)))/2 \right]. \quad (65)$$

$$m_2 = \frac{1}{2n(-1 + n\gamma)^2} \left[ 2 + n - 2n^2\gamma + n^3\gamma^2 + ((2 + n - 2n^2\gamma + n^3\gamma^2)^2$$

$$+ 4n(-1 + n\gamma)^3((-1 + 3n + n^2(-2 + \gamma) - n^3(-6 + \gamma)\gamma$$

$$+ 2n^5\gamma^3 + n^4\gamma^2(-1 + 2\gamma))/2 \right]. \quad (66)$$

The perturbation solution of (56), with $b = \epsilon$, is obtained by inserting $H_0(\theta) = Ae^\theta$, (64), and (59) in (57):

$$F(\theta) = A + \epsilon c_1 e^{(m_1 - 1)\gamma} + \epsilon c_2 e^{(m_2 - 1)\gamma}$$

$$+ \epsilon \frac{jn^2(1 - n\gamma)^{2(1 - n\gamma)} e^{\frac{3 - n\gamma}{2n\gamma}}}{(3n + 2)\gamma - n^2\gamma^2 - 2} \left[ \gamma^5 - b\gamma^4 + c\gamma^3 - a_1\gamma^2 + b_1\gamma + c_1 \right]. \quad (67)$$
The corresponding solution of (18) is now obtained from (57), (58), (67), and (52) as

\[
v(x, t) = \frac{1}{1 - n\gamma} x \left[ A + \epsilon c_1 \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{(m_1-1)} \right.
\]

\[
+ \epsilon c_2 \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{(m_2-1)}
\]

\[
\left. +\frac{j n^2 (1 - n\gamma)^{\frac{2(2-ny)}{1-n\gamma}} \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{\frac{2}{1-n\gamma}}} \right]
\]

\[
\left. -\frac{1}{(3n+2)\gamma - n^2\gamma^2 - 2} \left[ a\gamma^5 - b\gamma^4 + c\gamma^3 - a_1\gamma^2 + b_1\gamma + c_1 \right] \right]^{1/n}
\]

(68)

Thus the following solution of Equation (54) is derived from (19) and (68):

\[
u(x, t) = \left\{ \frac{1}{1 - n\gamma} x \left[ A + \epsilon c_1 \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{(m_1-1)} \right.
\]

\[
+ \epsilon c_2 \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{(m_2-1)}
\]

\[
\left. +\frac{j n^2 (1 - n\gamma)^{\frac{2(2-ny)}{1-n\gamma}} \left\{ -\left( n\gamma - 1 \right)^{\frac{ny+1}{2}} \left( \frac{x}{\sqrt{2bt}} \right)^{1-n\gamma} \right\}^{\frac{2}{1-n\gamma}}} \right]
\]

\[
\left. -\frac{1}{(3n+2)\gamma - n^2\gamma^2 - 2} \left[ a\gamma^5 - b\gamma^4 + c\gamma^3 - a_1\gamma^2 + b_1\gamma + c_1 \right] \right]^{1/n}
\]

(69)

2.2. \(\beta = 0, \ n\gamma = 1\)

We assume now that

\[
\beta = 0, \quad n\gamma = 1, \quad M(t) = 1.
\]

(70)

In view of (70), (35) gives

\[
z(x, t) = \log x + N(t).
\]

(71)
Substituting (33) and (71) in (25), we obtain

\[ B = x. \]  

(72)

Equation (25) implies that

\[ n \left( \frac{j}{2t} + \alpha \right) x^2 - \frac{n + 1}{n} = \Gamma_4(z). \]  

(73)

Replacing \( x \) in (73) from (71) results in

\[ n \left( \frac{j}{2t} + \alpha \right) e^{2z - 2N(t)} - \frac{n + 1}{n} = \Gamma_4(z). \]  

(74)

Equation (74) requires that

\[ \alpha = 0 \quad \text{and} \quad \frac{e^{-2N(t)}}{t} = r, \]  

(75)

where \( r \) is a constant, and reduces to

\[ \Gamma_4(z) = r \frac{n j}{2} e^{2z} - \frac{n + 1}{n}. \]  

(76)

Application of (33) and (71) in (26) gives

\[ -N' e^{2(z-N)} - \frac{n + 2}{n} = \Gamma_5(z), \]  

(77)

which forces us to take

\[ N' e^{-2N} = s, \]  

(78)

where \( s \) is another constant, to give

\[ \Gamma_5(z) = -s e^{2z} - \frac{n + 2}{n}. \]  

(79)

The second equation of (75) and (78) together determine

\[ N(t) = \log \left( \frac{1}{\sqrt{rt}} \right). \]  

(80)

On inserting (80) into (71), we find that

\[ z(x, t) = \log \left( \frac{x}{\sqrt{rt}} \right). \]  

(81)

Substituting (30) and (72) in (20), the similarity transformation, in this case, is

\[ v(x, t) = x H(z), \]  

(82)

\[ z(x, t) = \log \left( \frac{x}{\sqrt{rt}} \right). \]  

(83)
Applying for $\Gamma_n(z)$'s, $n = 4, 5, 6$ from (76), (79), and (27) and also using $\Gamma_1(z) = \Gamma_2(z) = \Gamma_7(z) = 0$, $\Gamma_3(z) = -1$ in (29), we get

\[
H' + \left(1 + \frac{1}{n} - \frac{rn}{2} je^{2z}\right) H^2 + \left(1 + \frac{2}{n} - \frac{r}{2} e^{2z}\right) HH'
+ \frac{n + 1}{n} H' - HH'' = 0. \tag{84}
\]

Equation (17), with $\alpha = \beta = 0$ and $n \gamma = 1$, is

\[
u_t + u^n u_x + \left(\frac{j}{2t}\right) u + \frac{1}{nx} u^{n+1} = u_{xx}. \tag{85}
\]

And putting (82) in (19), we obtain the similarity transformation of (85) as

\[
u(x, t) = [x H(z)]^{-1/n}, \tag{86}
\]

\[
z(x, t) = \log \left(\frac{x}{\sqrt{rt}}\right). \tag{87}
\]

When $n = 1$, a solution of (84) is

\[
H(z) = \frac{4}{r(2 - j)} e^{-2z}. \tag{88}
\]

2.3. $\beta \neq 0$, $\gamma = 0$

Corresponding solution of (85) with $n = 1$, namely,

\[
u_t + uu_x + \left(\frac{j}{2t}\right) u + \frac{1}{x} u^2 = u_{xx}, \tag{89}
\]

is then obtained by inserting (88) in (86) and (87) as

\[
u(x, t) = \frac{2 - j x}{4t}. \tag{90}
\]

We in this case assume that

\[
\beta \neq 0, \quad \gamma = 0, \quad N(t) = 0. \tag{91}
\]

Using (91) in (35), we obtain

\[
z = -\frac{1}{n \beta} M(t) e^{-n \beta_x}. \tag{92}
\]

Inserting (92) in (25), we get

\[
\Gamma_4(z) = \frac{1}{n^2 \beta^2} \left[\frac{M'}{M} + n \left(\frac{j}{2t} + \alpha\right) - n \beta^2\right] z^{-2}. \tag{93}
\]
An expression for $\Gamma_4(z)$ may be obtained from (93) as
\[ \Gamma_4(z) = \frac{(c - n\beta^2)}{n^2\beta^2} z^{-2}, \] (94)
provided we take
\[ \frac{M'}{M} + n \left( \frac{j}{2t} + \alpha \right) = c. \] (95)

In (95), $c$ is a constant.

Again using (92) in (26), we find that
\[ \Gamma_5(z) = -\left[ \frac{M'}{n^2\beta^2 M} - \frac{2 + n}{n} \right] z^{-1}. \] (96)

We must now set
\[ \frac{M'}{n^2\beta^2 M} - \frac{2 + n}{n} = p, \] (97)
where $p$ is another constant, to obtain $\Gamma_5(z)$ from (96) as
\[ \Gamma_5(z) = -\frac{p}{z}. \] (98)

Comparison of (95) and (97) lead to
\[ j = 0, \quad p = -\frac{1}{n\beta^2} \left[ \alpha - \frac{c}{n} + (2 + n)\beta^2 \right]. \] (99)

Using (99) in (95), and solving the latter for $M(t)$ results in
\[ M(t) = e^{-(n\alpha - c)t}. \] (100)

The second equation of (99) and (98) lead to
\[ \Gamma_5(z) = \frac{1}{n\beta^2} \left[ \alpha - \frac{c}{n} + (2 + n)\beta^2 \right] z^{-1}. \] (101)

On using (100), (92) yields the similarity variable:
\[ z(x, t) = -\frac{1}{n\beta} e^{-n[\beta x + (\alpha - \frac{c}{n})t]}. \] (102)

Inserting $\Gamma_n(z)$’s, $n = 4, 5, 6$ from (94), (101), and (27) and also using $\Gamma_1(z) = \Gamma_2(z) = \Gamma_7(z) = 0$, $\Gamma_3(z) = -1$ in (29), we obtain
\[ H' - \frac{(c - n\beta^2)}{n^2\beta^2} z^{-2} H^2 - \frac{1}{n\beta^2} \left[ \alpha - \frac{c}{n} + (2 + n)\beta^2 \right] z^{-1} H H' \]
\[ + \frac{n + 1}{n} H'^2 - HH'' = 0. \] (103)
Substituting (30), (33), and (102), Equation (20) gives the similarity transformation of (18), with $\gamma = j = 0$, namely,

$$vv_t + v_x - n\alpha v^2 - n\beta v + \frac{n + 1}{n} v_x^2 - v v_{xx} = 0,$$

is

$$v(x, t) = -\frac{1}{n\beta} \frac{H(z)}{z},$$

$$z(x, t) = -\frac{1}{n\beta} e^{-n[\beta x + (\alpha - \frac{1}{n}) t]}.$$

A solution of the second order nonlinear Equation (103) is easily found to be

$$H(z) = -\frac{1}{n\beta}.$$

Equation (17), with $j = \gamma = 0$, is

$$u_t + u^n u_x + \alpha u + \beta u^{n+1} = u_{xx},$$

whose similarity transformation is given by putting (105) in (19) as

$$u(x, t) = \left[-\frac{1}{n\beta} \frac{H(z)}{z}\right]^{-1/n},$$

$$z(x, t) = -\frac{1}{n\beta} e^{-n[\beta x + (\alpha - \frac{1}{n}) t]}.$$

Therefore the corresponding solution of (108) is obtained by inserting (106) and (107) in (109)

$$u(x, t) = \left(-\frac{l}{n\beta}\right)^{-\frac{1}{n}} e^{-\beta x - (\alpha - \frac{1}{n}) t}.$$

\[3. \text{ Conclusion}\]

The generalized Burgers equations (17) is subjected to direct similarity analysis to obtain its intermediate asymptotics [19]. The reduced ordinary differential equations named as Euler–Painlevé transcendents by Sachdev and his collaborators [9–11] are solved by perturbation technique; to the best of our knowledge, this is the first work reporting perturbation solutions of Euler–Painlevé transcendents.

To our astonishment, the similarity variable of (17), in general, has come out in terms of incomplete gamma function. Here we would like to point out
the work of Freeman [20] and Sachdev and collaborators [21, 22] who found
the similarity solutions in terms of incomplete beta function describing free
surface flows under gravity. Again this work is the first to report the similarity
variable in terms of functions such as incomplete gamma function.

Another significant deviation of the present paper is an introduction of the
transformation (19) to transform the generalized Burgers equation (17) to (18)
to facilitate the application of the direct method. As a consequence, the resulting
ordinary differential equations themselves are Euler–Painlevé transcendents.

Because a large number of Euler–Painlevé transcendents have been extensively
studied analytically and numerically by Sachdev and his collaborators [9–11]
we naturally do not repeat them in the present study. However a few simple
analytic solutions of the Euler–Painlevé transcendent, in addition to the
perturbation solution, have been determined by ad hoc methods.

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