Mathematics for Humans: Kant’s Philosophy of Arithmetic Revisited

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In Kant we find an old form of intuitionism, now almost completely abandoned, in which space and time are taken to be forms of conception inherent in human reason.

L.E.J. Brouwer

The concept natural number cannot itself be categorically characterized in pure logic. We can only say that the natural numbers are those which come in the sequence 1, 2, 3, . . . . We do have an intuition of this sequence. Perhaps, as Kant supposed, it is connected to the intuition of succession in time.

Ian Hacking

I. Introduction

According to Kant, mathematics is the pure formal science of quantity or magnitude. In turn, quantities or magnitudes are of two fundamentally different kinds: numerical and spatial. Arithmetic is the pure science of numbers, and geometry is the pure science of space. Whether arithmetic or geometry, however, mathematics for Kant is synthetic a priori, not analytic a priori – which is to say that it is a substantive or world-dependent science, not a purely logical science. But how can mathematics be at once a priori (i.e., experience-independent and necessary) and also substantive or world-dependent? As Brouwer correctly observes, for Kant mathematics is possible because it presupposes the innate human cognitive capacity for pure temporal and spatial representation, the innate human cognitive capacity for pure intuition (CPR A38–39/B55–56; P Ak. iv. 280–283). In turn, as the Transcendental Aesthetic shows, our pure intuitions of time and space are the non-empirical necessary subjective forms of inner and outer human sensibility.

In this essay I revisit Kant’s much-criticized views on arithmetic. In so doing I make a case for the claim that his theory of arithmetic is not in fact subject to the most familiar and forceful objection against it, namely that his doctrine of the dependence of arithmetic on time is plainly false, or even worse, simply unintelligible; on the contrary, Kant’s doctrine about time and arithmetic is highly original, fully intelligible, and with qualifications due to the inherent limitations of his conceptions of arithmetic and logic, defensible to an important extent.
My case has three stages. In the first stage, I reconstruct Kant’s argument for the synthetic apriority of arithmetic (section II). In the second stage, I develop a new account of his notorious doctrine of the dependence of arithmetic on time (section III). And finally in the third stage, I develop a correspondingly new account of the Kantian notion of arithmetical construction (section IV).

II. Why Arithmetic is Synthetic A Priori

By ‘elementary arithmetic’ I mean elementary logic (i.e., bivalent first-order quantified polyadic predicate calculus including identity) plus the five Peano axioms,

(1) 0 is a number.
(2) The successor of any number is a number.
(3) No two numbers have the same successor.
(4) 0 is not the successor of any number.
(5) Any property which belongs to 0, and also to the successor of every number which has the property, belongs to all numbers.,

taken together with the primitive recursive functions over the natural numbers – the successor function, addition, multiplication, exponentiation, etc. According to Gödel’s first incompleteness theorem, elementary arithmetic is incomplete, which is to say that there are sentences of elementary arithmetic that are true and unprovable. One way of making sense of this from a Kantian point of view is to say that elementary arithmetic is incomplete because it is synthetic and not analytic. Frege argued that elementary arithmetic is analytic because its truths are derivable from general logical laws together with ‘logical definitions’.

But Fregean logicism foundered on Russell’s set-theoretic paradox, the deep unclarity of Frege’s notion of a logical definition, and Gödel-incompleteness. Neo-logicians argue that if we drop the Fregean identification of numbers with sets of equinumerous sets and adopt second-order logic (i.e., elementary logic plus quantification over properties and functions) plus Hume’s principle, then elementary arithmetic is after all analytic. But neo-logicians must appeal to a logic stronger than elementary logic in order to show this. And even in view of the provability of elementary arithmetic in second-order logic plus Hume’s principle, still no one can deny that Gödel-incompleteness entails that elementary arithmetic is not analytic, on the assumption that the criterion for analyticity is provability in elementary logic.

Now to be sure Kant’s logic is very different from the logics used by logicians and neo-logicians, for his logic is significantly weaker than elementary logic. Kant’s logic includes only truth-functional logic, Aristotelian syllogistic, and a theory of (fine-grained, decomposable) monadic concepts, which is to say in more modern terms that it includes only monadic logic and a partial anticipation of higher-order intensional logic. And this may lead us to think, as Alan Hazen has put it, that ‘Kant had a terrifyingly narrow-minded and mathematically trivial,
conception of the province of logic'. Well, yes: Kant’s conception of the province of logic does not include polyadic predicate logic. But on the other hand, Kant’s logic certainly captures a fundamental fragment of elementary logic. Furthermore, since we already know from Gödel-incompleteness that elementary arithmetic is not analytic on the assumption that the criterion for analyticity is provability in elementary logic, and since Kant’s logic is weaker than elementary logic, there seems to be little or no reason to believe that Kant’s argument for the syntheticity of arithmetic will be vitiated by the limited character of his logic alone. For if a stronger logic shows that elementary arithmetic is not analytic, then Kant’s thesis that elementary arithmetic is not analytic surely cannot depend on the relative weakness of his logic.

That having been said by way of finessing familiar worries about the limited scope of Kant’s logic, in this section I will develop an argument for the synthetic apriority of arithmetic by unpacking Kant’s argument for a slightly weaker thesis: that a fundamental fragment of elementary arithmetic is synthetic a priori. There are two reasons for this. First, since Kant’s logic is only a monadic logic, and therefore contains no theory of either multiple first-order quantification or second-order quantification, he would not be able to formulate Peano’s axioms (2) through (5). So he would not be able to formulate classical first-order Peano arithmetic or PA, much less a second-order reading of the principle of mathematical induction, axiom (5). On the other hand however, even though lacking a general theory of quantification, presumably Kant would still be able to formulate primitive recursive arithmetic or PRA, the quantifier-free theory of the natural numbers and the primitive recursive functions. But second, if a fundamental fragment of elementary arithmetic is synthetic a priori, then obviously elementary arithmetic as a whole is also synthetic a priori. For convenience, I will call the fundamental fragment of elementary arithmetic that was studied by Kant ‘arithmetic*’. And to give my argument some theoretical bite, I assume that arithmetic* includes PRA and monadic logic but falls short of PA.

I turn now to Kant’s argument for the synthetic apriority of arithmetic*. The argument has four crucial background assumptions that we need to make explicit before surveying it step-by-step.

First, according to Kant a true proposition is analytic if and only if its denial leads to a logical contradiction:

The contrary of that which as a concept is contained and is thought in the cognition of the object, is always correctly denied, while the concept itself must necessarily be affirmed of it, since its opposite would contradict the object. Hence we must allow the principle of contradiction to count as the universal and completely sufficient principle of all analytic cognition. (CPR A151/B191)

Therefore the negative criterion of a synthetic proposition is that its denial is logically or analytically consistent.

Second, for Kant the positive mark of the syntheticity of a proposition is its

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semantic dependence on intuition. In turn, an intuition for Kant is an immediate or non-descriptive, non-discursive or non-conceptual and non-propositional, essentially singular representation of some actual spatial or temporal object, or of the underlying spatial or temporal form of all such objects (CPR A19–22/B33–36, A68/B93, B132, A320/B377; P Ak. iv. 280–281). More precisely then, a proposition is synthetic if and only if its objective validity and truth require an intuition in this special sense:

\[\text{All concepts, and with them all principles, however a priori they may be, refer nevertheless to empirical intuitions, i.e., to data for possible experience. Without this reference they have no objective validity. \ldots One need only take as an example the concepts of mathematics, and first, indeed, in their pure intuitions. \ldots Although all these principles, and the representation of the object with which this science occupies itself, are generated in the mind completely a priori, they would still not mean anything at all, if we could not always exhibit their meaning in appearances (empirical objects). (CPR A239-240/B299)}\]

If one is to judge synthetically about a concept, then one must go beyond this concept, and indeed go to the intuition in which it is given. (CPR A721/B749)

Third, on this Kantian picture of syntheticity as semantic intuition-dependence, synthetic a posteriori propositions are dependent on empirical intuitions, and correspondingly synthetic a priori propositions are dependent on pure intuitions:

This principle [of syntheticity] is completely unambiguously presented in the whole Critique, from the chapter on the schematism on, though not in a specific formula. It is: All synthetic judgments of theoretical cognition are possible only through the reference of a given concept to an intuition. If the synthetic judgment is an experiential judgment, the intuition must be empirical; if the judgment is a priori synthetic, there must be a pure intuition to ground it. (PC Ak. xi. 38)

Fourth and finally, according to Kant synthetic a priori propositions are necessary, but unlike analytic propositions they are not absolutely necessary or true in every logically possibly world. Rather they restrictedly necessary. This is to say that they are true in all and only the humanly experienceable worlds; in worlds that are not experienceable, they are objectively invalid or truth-valueless:

Here we now have one of the required pieces for the solution of the general problem of transcendental philosophy: how are synthetic a priori propositions possible? – namely, pure a priori intuitions, space and time, in which, if we want to go beyond the given concept in an a priori judgment,
we encounter that which is to be discovered a priori and synthetically
connected with it, not in the concept but in the intuition that corresponds
to it: but on this ground such a judgment never extends beyond the
objects of the senses and can hold only for objects of possible experience.
(CPR B73)

Our theoretical cognition never transcends the field of experience. . . . [I]f
there is synthetic cognition a priori there is no alternative but that it must
contain the a priori conditions of the possibility of experience. (RP Ak. xx.
274)

To be sure, this barely scratches the surface of an adequate discussion of Kant’s
analytic/synthetic distinction.14 But for our purposes here, the bottom line is this:
From these four points, it follows that a truth of arithmetic is synthetic a priori
just in case it is (1) consistently deniable, (2) semantically dependent on pure intu-
tion, and (3) necessarily true in the restricted sense that it is true in every experi-
enceable world and never false otherwise because it lacks a truth-value in every
unexperienceable world.

So much for the semantic framework. My reconstruction of Kant’s argument
for the synthetic apriority of arithmetic is based on the notorious ‘finger-count-
ing’ passage at B14–16, the all-too-brief definition of the concept NUMBER at
A142–143/B182, and §10 of the Prolegomena. For clarity’s sake, I will spell out the
argument step-by-step; and where it is relevant, I will also quote the texts upon
which my reconstruction is based.

A Reconstruction of Kant’s Argument for the Syntheticity of Arithmetic

(1) Assume the existence of arithmetic*. (Implicit premise.)

(2) It is a priori and thereby necessary that 7+5=12. (From (1) and the defi-
nition of apriority.)

It must . . . be noted that properly mathematical propositions are always a
priori judgments and are never empirical, because they carry necessity with
them, which cannot be derived from experience. (CPR B14)

(3) There is at least one logically possible world in which 7+5 ≠ 12.
(Premise.)

The concept of twelve is by no means already thought merely by my thinking
of that unification of seven and five. (CPR B15)

That 5 should be added to 7, I have, to be sure, thought in the concept of a
sum = 7+5, but not that this sum is equivalent to the number 12. (CPR
B15–16)

(4) So it is not necessary that 7+5=12. (From (3).)
To be sure, one might initially think that the proposition $7+5=12$ is a merely analytic proposition that follows from the concept of a sum of seven and five in accordance with the principle of contradiction. Yet if one considers it more closely, one finds that the concept of the sum of 7 and 5 contains nothing than the unification of both numbers in a single one, through which it is not at all thought what this single number is which combines the two of them. (CPR B15)

(5) If and only if the pure intuition of time is invoked, can (2) and (4) be made consistent with one another. For in any possible world representable by our pure intuition of time, there is a sufficiently large and appropriately-structured supply of ‘stuff’ (Stoff) – the total set of homogeneous temporal moments generated in the sempiternal successive synthesis of the sensory manifold – to constitute a truth-maker of the arithmetic proposition in question. That is, (2) is made true by all the experienceable worlds. And in some worlds that are not representable by our pure intuition of time, nothing suffices to be a truth-maker of the arithmetic proposition in question. That is, (4) is made true by some unexperienceable worlds. So (2) is consistent with (4), assuming pure intuition; otherwise they are inconsistent. (Premise.)

Number is a representation that summarizes the successive addition of one homogeneous unit to another. Number is therefore nothing other than the unity of the synthesis of the manifold of a homogenous intuition in general, because I generate time itself in the apprehension of the intuition. (CPR A142–143/B182)

Now, the intuitions which pure mathematics lays at the foundation of all its cognitions and judgments which appear at once apodeictic and necessary are space and time. For mathematics must first exhibit all its concepts in pure intuition. . . If it proceeded in any other way, it would be impossible to make a single step; for mathematics proceeds, not analytically by dissection of concepts, but synthetically, and if pure intuition is wanting there is nothing in which the stuff (Stoff) for synthetic a priori judgments can be given. (P Ak. iv. 283)

(6) Since the proposition that $7+5=12$ is both necessary and also consistently deniable, and its necessity is grounded on our pure intuitional representation of time, it follows that it is synthetic a priori. (From (2), (4), (5), and the definitions of syntheticity, apriority, and synthetic apriority.)

(7) The argument applied to the proposition that $7+5=12$ can be applied, mutatis mutandis, to any other truth of arithmetic involving larger numbers. (Generalization of (6).)

One becomes all the more distinctly aware of that if one takes somewhat larger numbers, for it is then clear that, twist and turn our concepts as we...
will, without getting help from intuition we could never find the sum by means of the mere analysis of our concepts. (CPR B16)

(8) Therefore all truths of arithmetic* are synthetic a priori. (From (7).)

Pretty obviously, the most controversial move in this argument occurs at step (3), in which the consistent deniability of truths of arithmetic* is asserted. But what sort of a logically possible world could fail to be a truth-maker for something as apparently unexceptionable as ‘7+5=12’? We must find a possible world that lacks at least one of the underlying structural features of arithmetic*. One sort of world that will do the job is a radically finite world – in particular, a world containing no structure rich enough to include 12 or more elements – i.e., a world containing less than 12 objects – which thereby lacks the requisite supply of ‘stuff’ for satisfying ‘7+5=12’. Another different sort of world that would do the same job is a radically unrecursive world, that is, a world in which even if there is the requisite supply of ‘stuff’ for assigning reference to all the number words, nevertheless there is no way of iteratively operating on that stuff. This would be a world with enough objects to satisfy arithmetic propositions but without any primitive recursive functions over those objects, including of course the successor function and addition.

Now because, according to Kant, our pure intuitional representation of time yields an infinite given whole (CPR A32/B47–48) in which the members of the unidirectional series of homogeneous moments are successively summed up to the magnitude of any later moment, it follows that both the radically finite and radically unrecursive worlds just described will not conform to our pure intuition of time. Indeed, it seems plausible to believe that any countermodel to arithmetic* will also fail to conform to our pure intuition of time. And since our pure intuitional representation of the time-series is a necessary condition of the possibility of all sensory representation of objects (CPR A31/B46), it then follows that all the countermodels to arithmetic* will also be unexperienceable worlds. In turn, the recognition that all the countermodels to arithmetic* also violate the conditions of the possibility of human experience yields the further recognition that we must ground the necessary truth of arithmetic* directly on the innate human capacity for pure temporal intuition. For arithmetic* is true only in worlds that include either the time-structure itself or else something isomorphic to the time-structure. And in every temporally-structured world not only is arithmetic* true, but also its truth-maker is cognizable a priori, and furthermore it has a direct application to objects of human experience – neither of which is guaranteed in timeless worlds. So while there are some conceivable and therefore possible timeless or noumenal worlds that make arithmetic* true, none of them will count in favor of arithmetic*’s synthetic necessity, a priori cognizability, or applicability. Only temporally-structured worlds can count in favor of these; and only pure temporal intuition gives us direct cognitive and semantic access to all those special worlds.
III. The Meaning of the Concept NUMBER

At this point, you might wonder what precisely is going on. In arguing that arithmetic is synthetic a priori, is Kant arguing that arithmetic is the science of time as we represent it in pure intuition, just as he argues that geometry is the science of space as we represent it in pure intuition? No. In the ‘transcendental exposition’ of the representation of space in §3 of the B edition version of the Transcendental Aesthetic, Kant argues explicitly that our pure intuition of space is necessary and sufficient for the objective validity of geometry. But in the corresponding transcendental exposition of the representation of time Kant very pointedly does not focus on arithmetic but instead on the ‘general doctrine of motion’ (CPR B49) or universal Newtonian mechanics. I think that we may take this to be an indication of an important asymmetry between Kant’s theories of geometry and arithmetic, as Philip Kitcher points out: ‘Kant did not believe, as is often supposed, that arithmetic stands to time as geometry does to space’.17

In his classic Commentary on the first Critique, Norman Kemp Smith also makes a pertinent remark in this connection:

Though Kant in the first edition of the Critique had spoken of the mathematical sciences as based on the intuition of space and time, he had not, despite his constant tendency to conceive space and time as parallel forms of experience, based any separate mathematical discipline upon time.18

In one sense this is quite correct, but in another sense it is misleading. The problem lies in a certain ambiguity in Kemp Smith’s phrase ‘based on’. That phrase has both a logico-metaphysical sense and a semantic sense. According to the logico-metaphysical sense, X is based on Y if and only if Y is a necessary and sufficient condition of X (where this covers everything from identity to strong supervenience). But the semantic sense of ‘based on’ is different. According to the semantic sense, X is based on Y if and only if Y is the semantic value of X, i.e., Y determines the extension of X. So if X is a concept-term, and X is about Y, then Y determines what objects X applies to; if X is a propositional term, and X is about Y, then Y determines the truth-maker(s) of X; if X is a theory, and X is about Y, then Y determines the model(s) of X. It should be noted that Y certainly can (although it does not necessarily have to) determine the extension of X by being identical with the extension of X. But the crucial point is that once we have isolated the semantic sense of ‘based on’ as aboutness, it is then quite correct to say that Kant does not conceive of arithmetic as a science that is about time and its formal features in the way that geometry is about space and its formal features. Instead, arithmetic is about the natural numbers and their formal features, not about time. Still, as Michael Friedman aptly puts it, ‘there is no doubt that [for Kant] arithmetic involves time’.19 So how can it be true that for Kant arithmetic is not about time, yet arithmetic still presupposes time as a necessary and sufficient condition of its objective validity?
The interpretation I favor is that our pure intuition of the infinite unidirectional successive time-series supplies a fundamental semantic condition for arithmetic*, but does not fully determine the semantics of arithmetic* until it is combined with a second representational factor – i.e., a purely logical factor. I will discuss this purely logical factor at the end of this section. But right now we need to see how pure intuition manages to supply a fundamental semantic condition for arithmetic*. The answer is revealed in these texts, the first of which we have seen already:

[N]umber [is] a representation that summarizes the successive addition of one homogeneous unit to another. Number is therefore nothing other than the unity of the synthesis of the manifold of a homogenous intuition in general, because I generate time itself in the apprehension of the intuition. (CPR A142–143/B182)

Time is in itself a series (and the formal condition of all series). (CPR A411/B438, emphasis added)

Arithmetic attains its concepts of numbers by the successive addition of units in time. (P Ak. iv. 283)

Time [is] the successive progression as form of all counting and of all counting and of all numerical quantities; for time is the basic condition of all this producing of quantities. (PC Ak. xi. 208, emphasis added)

Here is what I think Kant is driving at: the pure intuition of time sharply constrains what can count as a model for arithmetic*, but does not itself determine the extension of number terms or arithmetic propositions. Arithmetic*, by means of number concepts, represents the natural numbers, their intrinsic and relational properties, and the recursive functions over them. But all models of arithmetic* are non-conceptually structurally restricted or limited by means of our pure intuitional representation of the infinite unidirectional successive time-series. So nothing will count as a model of arithmetic* unless it is at least isomorphic with the infinite unidirectional successive time series delivered by pure intuition. Pure intuition does not therefore tell us just what the intended or standard model of arithmetic* is – pure intuition does not tell us what the numbers are – but it does tell us what the numbers cannot be, and it lays down a basic condition for something’s being a referent of numerical terms or a truth-maker for arithmetic* propositions.

So to repeat, I am saying that Kant’s thesis about the role of our pure intuition of time in arithmetic* is that our pure intuition of the infinite unidirectional successive time-series supplies a fundamental semantic condition for the objective validity or meaningfulness of the concept NUMBER by partially determining what will count as a referent for numerical terms or a truth-maker for arithmetic* propositions: such terms and propositions cannot be about the natural numbers unless their extensions are isomorphic with time. A similar point is made by Charles Parsons:
Time provides a universal source of models for the numbers. . . . What would give time a special role in our concept of number which it does not have in general is not its necessity, since time is in some way necessary for all concepts, nor an explicit reference to time in numerical statements, which does not exist, but its sufficiency, because the temporal order provides a representative of the number which is present to our consciousness if any is present at all.20

Of course the pure intuition of time does more than merely constraining the class of models for arithmetic*. By virtue of the threefold fact that our pure intuition of the infinite unidirectional time-series (i) is built dispositionally into human representational capacities, (ii) is a necessary condition of all sensory experience of objects, and also (iii) picks out time, which is partially constitutive of the empirical world, it follows then that Kant can neatly explain not only (i*) how arithmetic* is synthetically necessary, or true in all experienceable worlds and never false otherwise (i.e., because time is included in every experienceable world and every model of arithmetic* is isomorphic with time), but also (ii*) how arithmetic* is cognizable a priori for creatures like us (i.e., because our capacity for pure temporal intuition is innate), and (iii*) how arithmetic* is guaranteed to have empirical application (i.e., because the representation of time is guaranteed to have empirical application). The dimension of applicability, moreover, is a particularly crucial factor, as Frege points out in Basic Laws of Arithmetic:

It is applicability that raises arithmetic from the rank of a game to that of a science. Applicability therefore belongs to it of necessity.21

All of this adds up to an important point. As Michael Potter has observed, two fundamental and intimately-related problems in the philosophy of arithmetic are (1) how to explain arithmetic’s necessity?, and (2) how to explain arithmetic’s empirical applicability?22 But there is also a second pair of similarly fundamental and intimately-related problems: (3) on the one hand a uniform semantics of natural language implies that numbers are humanly-knowable truth-makers of arithmetic truths, but on the other hand the reasonable assumption that numbers are causally inert abstract objects implies that they are unknowable when combined with the equally reasonable assumption that mathematical intuition is analogous to sense perception,23 and (4) what are the numbers?24 Paul Benacerraf has famously argued for the salience and interrelatedness of the third and fourth problems.25 The deep significance of Kant’s philosophy of arithmetic lies in the fact that he adumbrates a joint solution to the four problems.

Now I want to wrap up this section by taking a look at a very puzzling letter that Kant wrote to his friend and disciple Johann Schultz in 1788, i.e., one year after the publication of the second or B edition of the first Critique. Schultz was then working on the manuscript of a book entitled Prüfung der kantischen Kritik der reinen Vernunft (‘Examination of the Kantian Critique of Pure Reason’), which he had shown to Kant. In that manuscript, Schultz had anticipated Frege by defending the
idea that all the truths of arithmetic are purely logical or analytic. Here is the key part of Kant’s response to the manuscript:

Time, you correctly notice, has no influence on the properties of numbers (considered as pure determinations of quantity), as it may have on the character of those alterations (of quantity) that are possible only relative to a specific state of inner sense and its form (time). The science of numbers, notwithstanding the succession that every construction of quantity requires, is a pure intellectual synthesis, which we represent to ourselves in thought. But insofar as specific quantities (quanta) are to be determined according to this science of numbers, they must be given to us in such a way that we can grasp their intuition successively; and thus this grasping is subjected to the time condition. (PC Ak. x. 556–557)

Part of what Kant is doing here is simply reiterating his view that while arithmetic presupposes our pure intuition of the time-series, arithmetic is not itself the science of ‘alterations’ (Veränderungen) or events – that is, arithmetic is not about time. But for our purposes the crucial question is, what does Kant means by his remark that ‘the science of numbers . . . is a pure intellectual synthesis, which we represent to ourselves in thought’? What he seems to be saying is that arithmetic is grounded on pure conceptualization, which of course in his terms would make it purely logical or analytic in nature. So by 1788 has Kant quietly switched over to some version of logicism?

No. One way of seeing this is to return to a subtle point he makes in the B edition of the Critique of Pure Reason. There Kant explicitly commits himself to the thesis that all mathematics is strictly constrained by pure logic in that ‘the inferences of the mathematician all proceed in accordance with the principle of contradiction’ (CPR B14). But this constraint on inference and proof ‘is required by the nature of any apodictic certainty’ (CPR 14), so it is not special to mathematics. More generally, it does not follow that mathematics is essentially logic just because its proofs must meet some minimal pure logical requirements:

Since one found that the inferences of the mathematician all proceed in accordance with the principle of contradiction . . . , one was persuaded that the principles could also be cognized from the principle of contradiction, in which, however, they erred; for a synthetic proposition can of course be comprehended in accordance with the principle of contradiction, but only insofar as another synthetic proposition is presupposed from which it can be deduced, never in itself. (CPR 14)

This of course sets Kant’s view on mathematics sharply apart from the Leibnizian view, according to which all necessary truth is ultimately reducible to the logical principle of identity or non-contradiction (Leibniz regarded these as equivalent). But the crucial point is that it is a mistake to think that the admitted fact of strict logical constraints on mathematics entails a reduction of mathematics to logic.
Kant’s view, on the contrary, is that mathematics can essentially include logical elements without in any way undermining its syntheticity. Now it has also been sometimes suggested by commentators that Kant is saying at B14 that only the premises and conclusions of mathematical reasoning are non-logical, while also holding that all the inferential transitions or steps of proof are of a purely logical nature. So their idea is that while mathematics is indeed synthetic as regards its semantic content, its formal machinery of proof is purely logical. This I think is also a mistake, for reasons we will see in the next section.

Right now, the question on the table is whether Kant in the letter to Schultz in 1788 is intentionally or unintentionally backsliding towards some sort of logicism about arithmetic. And one reason for thinking that he is not backsliding, as we have just seen, is that in the B edition of the first Critique, published only a year before the letter to Schultz, he is explicitly committed to the idea that the presence of significant logical factors in mathematics is consistent with the denial of logicism. But the decisive reason for thinking that he is not has to do with his views on the role of logic in the semantic constitution of the concept NUMBER. In the letter to Schultz, Kant is saying, I think, that NUMBER does indeed have a purely logical source of representational content in our conceptual faculty, the understanding, but that this source of content does not exhaust the content of NUMBER.

So what, according to Kant, does NUMBER mean? Here is what he says explicitly in the first Critique:

No one can define the concept of a magnitude in general except by something like this: That it is the determination of a thing through which it can be thought how many units are posited in it. Only this how-many-times is grounded on successive repetition, thus on time and the synthesis of the homogeneous in it. (CPR A242/B300, emphasis added)

And here is what I think he means by that remark, when we combine it with what he says in the letter to Schultz. Kant’s view, it seems, is that NUMBER is necessarily partially based on the three ‘logical functions’ of quantification in judgments:

Universal (e.g., all Fs are Gs),
Particular (e.g., some Fs are Gs),
Singular (e.g., the F is G, or this F is G) (CPR A70/B95).

The logical functions of quantification, in turn, correlate one-to-one with the three categories of quantity:

Totality
Plurality
Unity (CPR A80/B106).28
Now in the Schematism, Kant says that ‘the pure image of all magnitudes (quantorum) . . . for all objects of the senses . . . is time’ and that ‘the pure schema of magnitude (quantitatis), however, as a concept of the understanding, is number’ (CPR A142/B182). As I understand it, what he means is that the concept NUMBER is what results if one takes the basic logical constants of quantity (all, some, the/this), maps them onto the corresponding metaphysical categories of quantity (totality, plurality, unity), and then systematically interprets those quantitative categories in terms of the pure intuition of time, as follows: (1) the logical function of universality in judgments, corresponding to ‘all Fs’, goes over into the infinite totality of successive moments of time, and so yields an exemplar or paradigm of the whole series of the natural numbers; (2) the logical function of particularity in judgments, corresponding to ‘some Fs’, goes over into any finite plurality of successive moments of time (i.e., a duration), and so yields exemplars or paradigms of any finite natural number; and (3) the logical function of singularity in judgments, corresponding to ‘the F’ or ‘this F’, goes over into any arbitrarily chosen single moment or unit of time, and so yields an exemplar or paradigm of the number 1. Now for Kant all empirical magnitudes or quantities are finite or infinite (CPR A430/B458), discrete or continuous (CPR A526–527/B554–555), and extensive or intensive (CPR A162–163/B203–204, A165–171/B208–212).29 And as we have just seen, in the Schematism Kant tells us that all appearances, as magnitudes or quantities, fall under the schematized concept NUMBER (CPR A161–176/B202–218). So NUMBER is directly applicable to all sorts of empirical magnitudes by virtue of its construal in terms of the pure intuition of time.

In other words, according to Kant the concept NUMBER has a purely logical source of representational content; but that logical input does not exhaust its semantic content, since it also has a complementary non-logical source of its representational content – the pure intuition of the infinite unidirectional successive time-series. So the concept NUMBER is a partially logical but not wholly logical concept: it represents numbers in purely logical terms, but these logical terms alone do not suffice to fix its meaning or objective validity adequately. Its meaning is adequately fixed, however, when we supplement its purely logical content by combining it with a certain non-logical structure. That is, when we represent natural numbers by using and specifying the concept NUMBER, we must also invoke a supplementary pure intuition of the infinite unidirectional successive time-series, which supplies the other fundamental semantic condition for the objective validity of NUMBER in particular and for arithmetic more generally, by sharply constraining what will count as a model for the latter, and by securing the empirical applicability of the former.

If this interpretation of Kant’s response to Schultz is correct, then it brings us back to Hacking’s point in the second epigraph, to the effect that the concept of a natural number cannot be categorically characterized in elementary logic. This is closely connected to the fact (originally discovered by Thoralf Skolem) that elementary arithmetic has non-standard models.30 The Hacking-Skolem worry, then, is that by means of elementary logic alone we cannot determine just which
of the many models of elementary arithmetic is the intended or standard model that is to be identified with the natural numbers.

Kant’s view about the numbers, by contrast to that of any theory attempting to give a reduction of arithmetic to logic, is that something is a natural number if and only if it satisfies the purely logical categories of quantity and is isomorphic to some part of the infinite unidirectional successive time-series picked out by pure intuition. So a given number is how we collect or colligate all Fs, or some Fs, or the/this F, in a way that formally mimics the unidirectional successive synthesis of moments in time. The number 5, e.g., is how we collect or colligate whatever falls under the concept F (say, all the fingers on one hand including the thumb) in exactly the same way that we representationally generate just that many moments of time. And the number zero is how we collect or colligate no Fs at all in exactly the same way that we representationally generate no moments of time by representing the specious present in which nothing has yet happened – the ‘beginning, the pure intuition = 0’ (CPR A165/B208). Otherwise put, the representational generation of numbers by counting is the logical representation of all objects, some objects, the/this object, or even no objects (which is represented in terms of negation and the particular quantifier), under some first-order (typically, empirical) concept C, taken together with the representation of time.

Number concepts, in other words, are schematized concepts. That is, they are concepts whose meaningful content is partially determined by a non-logical structure – the structure of total infinite unidirectional time, as delivered by pure intuition. So the natural numbers are in effect nothing but positions in a logically-conceptually constrained intuitional time-structure, which is to say that Kant’s theory of the numbers is a highly original (and specifically non-platonistic) version of ante rem mathematical structuralism. But that is not to say that the natural numbers are really something other than the natural numbers. On the contrary, the numbers are what they are, and not some other things. Numbers are sui generis entities because they are fully determined by logical concepts with a sui generis semantic content, which is the same as to say that numbers are nothing but positions in an empirically applicable and humanly graspable (because intuitional) time-structure under special logical-conceptual constraints.

If my interpretation of Kant’s response to Schultz is correct, then Kant is saying along with Hacking that pure logic on its own underdetermines the meaning of the concept NUMBER. But where Kant goes well beyond Hacking is by saying that only our pure temporal intuition can do the further semantic job that logic fails to do on its own, and by saying that numbers are sui generis entities – therefore radically irreducible entities – with sui generis properties and relations. Arithmetic is about the natural numbers and their formal features, and requires both pure logic and the pure intuitional representation of time in order to be about such things. The natural numbers are the semantic values of numerical terms and among the semantic values of arithmetic propositions. But the natural numbers, in turn, are natural precisely because their special structuralist ontology is primitive and essentially bound up with human nature. So in a twist on Leopold Kronecker’s famous quip about number theory to the effect that God made the
integers and everything else was done by humans, we might say that for Kant human nature made the natural numbers and everything else was done by logic.

IV. Construction as Construal

No part of Kant’s philosophy of arithmetic is a walk in the park; but I have been saving the trickiest bit of it for last. This is Kant’s theory of mathematical ‘construction’ (Konstruktion) in its particular application to arithmetic.

In his all-too-brief discussion of the nature of mathematics in chapter I, section 1 of the Transcendental Doctrine of Method, Kant distinguishes between two sorts of rational or a priori cognition: philosophical cognition and mathematical cognition. He had been concerned to draw this distinction sharply since the Inquiry concerning the Distinctness of the Principles of Natural Theology and Morality of 1764, in order to explain what he regarded as a set of manifest differences in semantic and epistemic character between the two, despite their both falling into the realm of the a priori. In the Inquiry the distinction turned on a difference between two sorts of conceptual reasoning: philosophical reasoning is a priori analysis of metaphysical concepts, or a non-empirical advance from ‘given’ metaphysical concepts to their decompositional parts; whereas mathematical reasoning is a priori synthesis, or the non-empirical ‘making’ of new concepts by combining two given concepts. But this way of drawing the distinction has two important problems. First, on the side of philosophical cognition, it does not distinguish between the mere analysis of concepts and the specifically philosophical analysis of concepts and therefore does not show why the propositions of philosophy are synthetic a priori, not analytic. Second, on the side of mathematical cognition, it does not adequately discriminate between the making of new concepts by mere arbitrary decision or stipulation (CPR A729/B757) and specifically mathematical cognition. As a consequence, it threatens to make the distinction between analytic and synthetic a priori propositions wholly relative to the intentions of the judger, since according to it every putatively synthetic a priori proposition can be re-formulated as an analytic proposition whose predicate concept is contained in its subject concept by an act of sheer stipulation on the part of the judging subject.

So in the first Critique Kant thoroughly re-works the distinction between philosophical and mathematical cognition. For our purposes, we can leave aside the renovated notion of philosophical cognition. What is important for us is that the essence of mathematical cognition is now said to lie in ‘the construction of concepts’:

|M|athematical cognition [is cognition from] from the construction of concepts. But to construct a concept means to exhibit (darstellen) a priori the intuition corresponding to it. For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as intuition, is an
individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept. Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely a priori, without having had to borrow the pattern for it from any experience. The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition we have taken account only of the action (Handlung) of constructing the concept, to which many determinations, e.g., those of the magnitude of the sides and the angles, are entirely indifferent, and thus we have abstracted from these differences, which do not alter the concept of the triangle. . . . [M]athematical cognition considers the universal in the particular, indeed even in the individual, yet nonetheless a priori and by means of reason, so that just as this individual is determined under certain general conditions of construction, the object of the concept, to which this individual corresponds only as its schema, must likewise be thought as universally determined. (CPR A713–714/B741–742)

A few sentences later, Kant remarks that the form of mathematical cognition itself guarantees that it will pertain solely to quantities, because 'only the concept of magnitudes can be constructed, i.e., exhibited a priori in intuition' (CPR A714/B742). Then following up on that, in the context of a notoriously puzzling passage on the nature of algebra, Kant speaks of 'constructions of magnitude in general (numbers)' (CPR A717/B745). This obviously refers back to what Kant says in the Schematism about the pure concepts of magnitude or quantity, i.e., that the concept NUMBER is what results from the schematization of the pure concepts of quantity and that the representation of time is the 'pure image' or schema of all magnitudes for all objects of the senses in general (CPR A142–143/B182). In the Schematism, moreover, Kant directly ties the notion of a schema of a pure concept to the faculty of pure or productive imagination (CPR: A140–142/B180–181). These ideas are also carried beyond the first Critique. In the Prolegomena he says that 'mathematics must first exhibit all its concepts in intuition, and pure mathematics in pure intuition, i.e., it must construct them' (P Ak. iv. 283). And in a similar vein, in the letter to Schultz he speaks of 'construction, a single counting up in an a priori intuition' and of 'the construction of the concept of quantity' (PC Ak. x. 556).

Three things are immediately clear from these texts and many other similar ones: (a) that mathematics requires the construction of concepts, (b) that mathematical construction of concepts is carried out by means of pure intuition together with the pure imagination, and (c) that arithmetic in particular requires the construction of numerical concepts, or concepts of magnitudes. But that is where immediate clarity runs out. Now our goal is to understand (c); but obviously that is intelligible if and only if (a) and (b) are intelligible. So what we need to know
are answers to these three questions, in sequence: (a*) what, generally speaking, is the construction of a concept?, (b*) how, specifically, does one construct a concept by means of pure intuition together with the pure imagination?, and (c*) what, precisely, does it mean to construct a numerical concept?

(1) What, generally speaking, is the construction of a concept? The German abstract noun ‘Konstruktion’ and its associated verb ‘konstruieren’, just like the corresponding English terms ‘construction’ and ‘construct’, are ambiguous. On the one hand, they express the notion of putting something together or building something new, by the assembly of diverse concrete or abstract materials, or by repeated operations on diverse concrete or abstract materials – as in ‘the construction of a house’ or ‘the construction of a formal system’. And on the other hand, they express the notion of grammatical parsing or semantic interpretation – as in ‘I chose to put a certain construction on that sentence’ or ‘The judge constructs the law’. Let us call the first sense ‘construction as creation’ and the second sense ‘construction as construal’. Given Kant’s well-known interest in jurisprudence and his equally well-known fondness for using legal metaphors and analogies in metaphysical, epistemic, logical, and semantic contexts, it seems obvious that thinking of construction as construal would be as natural to him as thinking of construction as creation.

Nevertheless I think it is almost universally assumed by readers of Kant that mathematical construction should be read as a some sort of creation of formal objects. But Kant explicitly says that in mathematical construction it is concepts that are constructed by means of pure intuition. Now there is certainly a sense in which, for Kant, concepts are created from diverse cognitive materials by assembly or repeated operations, i.e., by synthetic mental processes involving comparison, reflection, and abstraction. Kant calls this the ‘generation’ of concepts (JL Ak. ix. 94–95). But the generation of concepts does not seem to be what Kant has in mind in the case of mathematics, since he makes no mention of this sort of mental activity in that context. Moreover pure intuition plays no special role in the generation of concepts.

For these reasons, I think, most readers of Kant typically take a quick and unacknowledged interpretative slide from ‘the construction of concepts’ to ‘the construction of objects falling under concepts’, and then, taking into account the fact that Kant is talking about some sort of mental process involving pure intuition and pure imagination, hastily conclude that Kantian mathematical construction is the mental creation of mathematical objects. This, e.g., is precisely the sort of mathematical construction that is at work in Brouwer’s intuitionism. According to Brouwer, natural numbers are generated by the infinitely iterated application of formal operations to the conscious contents of the diachronic stream of an individual’s mental states. So Brouwer posits an original (infinitist) creation of mathematical objects in inner sense.

But why should we allow Brouwerian intuitionism – with its psychologistic implications – to drive our interpretation of Kant? Although Kant is a cognitivist, he explicitly rejects psychologism of any sort. More generally, if reading the construction of concepts as the mental creation of mathematical objects not only
does violence to Kant’s views, but also imports many of the problems of a philosophy of mathematics whose motivations and rationale are more or less foreign to Kant’s, then why read it that way? Surely it is more charitable to Kant to try out the hypothesis that by the construction of mathematical concepts Kant means the construal of mathematical concepts and not the mental creation of mathematical objects. So let us do just that.

(2) How, specifically, does someone construct a concept by means of pure intuition together with the pure imagination? My working hypothesis is that, in general, to construct a concept for Kant is to parse or to semantically interpret it. We know from the crucial text at CPR A713–714/B741–742 that to construct a pure mathematical concept of the understanding is for the pure imagination to ‘exhibit’ an instance of that concept in pure intuition. And we also know from the Schematism that for the pure imagination to exhibit an instance of a pure concept in pure intuition is to produce a schema of that concept. Because of its relative richness of detail, it is useful to quote from the Schematism at some length:

We will call this formal and pure condition of the sensibility, to which the use of the concept of the understanding is restricted, the schema of this concept of the understanding. . . . The schema is in itself always only a product of the imagination; but since the synthesis of the latter has as its aim no individual intuition but rather only the unity in the determination of sensibility, the schema is to be distinguished from the image. Thus, if I place five points in a row, . . . , this is an image of the number five. On the contrary, if I only think number in general, which could be five or a hundred, this thinking is more the representation of a method for representing a multitude (i.e., a thousand) in accordance with a certain concept than the image itself, which in this case I could survey and compare with the concept only with difficulty. Now this representation of a general procedure of the imagination for providing a concept with its image is what I call the schema for this concept.

In fact it is not images of objects but schemata that ground our pure sensible concepts. . . . [T]he image (Bild) is a product of the empirical faculty of productive imagination, [but] the schema of sensible concepts (such as figures in space) is a product and as it were a monogram of pure a priori imagination, through which and in accordance with which the images first become possible. . . . The schema of a pure concept of the understanding . . . is something that can never be brought to an image at all, but rather is only the pure synthesis, in accord with a rule of unity according to concepts in general, which the category expresses, and is a transcendental product of the imagination, which concerns the determination of inner sense in general, in accordance with conditions of its form (time). (CPR A140–142/B180–181)

So what does all this tell us? According to Kant a mathematical concept is shown to be objectively valid or empirically meaningful just insofar as it can be
supplied with a corresponding schema, in pure intuition, by means of the pure
imagination. This is the same as to construct that concept. But the schema is not
itself an object in the strict or narrow sense (i.e., a Gegenstand), namely an empir-
icial substance or an object of experience; nor is it an empirical image of an empir-
icial object. Instead, it is an object only in the loose or broad sense (i.e., an Objekt),
which can include representational targets or intentional objects of all sorts. More
precisely, a schema is a quasi-object since it is no more than a sort of rule or
method, pattern, or template, whose sole function is to illustrate the form and
content of the relevant mathematical concept. Quasi-objects are both ontically
incomplete (i.e., they lack some properties required for objecthood in the strict
and narrow sense) and partially indeterminate (i.e., there exist properties for
which it is neither true nor false that they apply to the quasi-object). Most
precisely of all, a schema is a quasi-objective exemplary or paradigmatic instance
of a concept, produced by the pure imagination, such that it encodes the relevant
conceptual content or conceptual information in a specifically spatial or temporal
format. In the terms of contemporary cognitive science, a schema is a ‘mental
model’. The schema always bears spatial or temporal structure because it is
generated with reference to pure intuition, and therefore has a direct bearing on
the possibility of sensory experience. So the act of construction does not create a
mathematical object, but instead only construes a mathematical concept by imag-
inatively producing a pure spatial or temporal schematic exemplar of it (i.e., a
mental model), which is only a quasi-object. The pure spatial or temporal
schematic exemplar, in turn, exhibits or illustrates the content of the concept by
providing a constraint on all possible models of any proposition or theory into
which that concept enters, namely that the model in question has to be at least
isomorphic with the pure spatial or temporal exemplar that is used to construe
that concept.

(3) What, precisely, does it mean to construct a numerical concept? To construct a
concept, I have said, is to use the pure imagination to create a schema of that
concept, or a pure spatial or temporal mental model of it. This mental model
encodes conceptual information in a spatial or temporal format. In the case of
numerical concepts, the mental model is always temporal in character – not in the
sense that it represents an event of some sort, but rather in the sense that it is itself
a model of something that is among the natural numbers.

Unfortunately, Kant says very little indeed about how a schematized numeri-
cal concept enters into arithmetic* propositions. In the controversial remarks
about algebra, he says that algebra ‘exhibits every procedure (Behandlung)
through which magnitude is generated and altered in accordance with certain
rules in intuition’ (CPR A717/B745), by which he clearly means that arithmetic*
especially includes operations as subtraction, addition, division, multiplication,
exponentiation, extraction of roots, and so-on. So the general idea seems to be that
an arithmetic* proposition is a logical complex consisting of schematized numeri-
cal concepts and some arithmetic* operations on those concepts. In the Axioms
of Intuition, Kant says explicitly that arithmetic* propositions are ‘propositions of
numerical relation (Zahlverhältnis)’ or ‘numerical formulas’ (Zahlformeln) (CPR:
A165/B206) that are neither general in form or content (but in fact singular) nor logically derivable from more general axioms. This is presumably because every atomic arithmetical proposition – say, ‘7+5=12’ or ‘3+4=7’ – expresses an operation on concepts of numbers, and each of those concepts can be construed in only one way, i.e., in terms of its own particular schema, in relation to the pure intuition of time:

That 7+5=12 is not an analytic proposition. For I do not think the number 12 either in the representation of 7 nor in that of 5 nor in the representation of the combination (Zusammensetzung) of the two... Although it is synthetic, however, it is still only a singular proposition. Insofar as it is only the synthesis of that which is homogeneous (of units) that is at issue here, the synthesis can take place only in a single way, even though the subsequent use of these numbers is general... The number 7 [in the proposition 7+5=12]... is possible in only a single way, and likewise the number 12, which is generated through the synthesis of the former with 5. Such propositions must therefore not be called axioms (for otherwise there would be infinitely many of them) but rather numerical formulas. (CPR A164–165/B205–206).

In the problem, conjoin 3 and 4 in one number, the number 7 must arise not out of a decomposition of the constituent concepts by rather by means of a construction, that is, synthetically. This construction, a singular counting up in an a priori intuition, exhibits the concept of the conjunction of two numbers. (PC Ak. x. 556)

This unusual doctrine of arithmetic* propositions, based directly on Kant’s idea that number concepts are constructed by means of the imaginative introduction of exemplary quasi-objects (schemata, mental models), has important implications for the question of whether inferential steps in arithmetic* proofs are purely logical in character. Take the following simple arithmetic* argument:

(1) 7+5=12
(2) 3+4=7
(3) Therefore 3+4+5=12.

If Kant is right, then the logical substitution of ‘3+4’ for ‘7’ under the extensional law of identity requires the constructions of the concepts THREE and FOUR, the operation-concept PLUS, and their synthesis. Arithmetic* identity is not a purely logical relation.40 So the inference-step of substitution would not have been valid unless pure temporal intuition and pure imagination had contributed representational content to the numerical concepts. This appears to be generally true of logical inferences in arithmetic* proofs. Therefore logical inferences in arithmetic* proofs require intuitional andimaginational supplementation and are not purely logical in character.41
V. Conclusion

As I see it, Kant is not asserting that arithmetic is the pure science of time. Rather, as Hacking suggests, Kant is asserting a highly original two-part doctrine about the cognitive semantics of the concept NUMBER: (a) that the content of the concept NUMBER requires our pure formal intuition of the sempiternal (or infinite unidirectional) series of successive moments of time as a non-logical necessary condition of that concept’s objective representational content; and (b) that the content of the concept NUMBER equally requires the logical functions of quantity and their corresponding categories. If Kant is right about this, then arithmetic is essentially the result of combining the formal ontology of our human intuitional representation of time with the conceptual resources of logic in Kant’s sense. That Kant’s own conception of arithmetic comprehends at most the primitive recursive fragment of elementary arithmetic, and that Kant’s own conception of logic comprehends at most the monadic fragment of elementary logic, are ultimately far less important than his deep insight into the essentially two-sided temporal/intuitional and logical/conceptual structure of the pure science of numbers. This dual structure is at once irreducibly anthropocentric and also strictly constrained. So arithmetic for Kant is not only an exact science, but also and perhaps most fundamentally, a human science.42

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NOTES

1 Brouwer (1964: 67).
2 Hacking (1979: 316).
3 For convenience, I cite Kant’s works infratextually in parentheses. The citations normally include both an abbreviation of the English title and the corresponding volume and page numbers in the standard ‘Akademie’ (Ak) edition of Kant’s works: Kant’s gesammelte Schriften, 29 vols, hrsg. Königlich Preussischen (now Deutschen) Akademie der Wissenschaften (Berlin: G. Reimer [now de Gruyter], 1902). For references to the first Critique, however, I follow the common practice of giving page numbers from the A (1781) and B (1787) German editions only. Here is a list of the relevant abbreviations and translations:


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4 Frege (1953).
5 Benacerraf (1981).
7 Hume’s principle says that the number of Fs = the number of Gs if and only if there are just as many Fs as Gs.
8 Monadic logic is a restricted form of elementary logic that permits quantification into one-place predicates only. Interestingly, monadic logic is not only consistent and complete but also effectively decidable. See Boolos and Jeffrey (1989: chs. 22, 25).
10 And we certainly should not undervalue the fact that Kant’s logic partially anticipates higher-order intensional logic; indeed, this may be part of the key to understanding his theory of analyticity. See Hanna (2001: 80–83).
11 This is a standard complaint against Kant’s argument for the synthetic apriority of mathematics, going at least as far back as Russell’s Principles of Mathematics. See Friedman (1992: 55–135).
14 Hanna (2001: chs. 3–5).
15 Similar points about finite countermodels for arithmetic are made by Parsons (1983: 131) and Shapiro (1998: 604). This is not to say that appeals to such countermodels are uncontroversial, however. Indeed, there are at least two big worries about radically finite worlds (and I am grateful to an anonymous referee at EJP for raising these worries): (1) it is well known that there are inferential gaps between imaginability and conceivability, and also between conceivability and possibility; and (2) radically finite worlds fail to verify what seem to be obvious truths based on the extensional law of identity, e.g., 12 ≠ 13. Obviously I cannot adequately rebut these objections in a footnote; but here are very brief indications of possible replies. First, the imaginability-conceivability and conceivability-possibility gaps are alike double-edged swords, in the sense that both the critics and the defenders of the radically finite worlds thesis must appeal to conceivability arguments. Indeed it seems to me that the resistance to the possibility of radically finite worlds depends mostly on the challengeable thesis – challengeable because, presumably, justified by the step from the inconceivability of its denial to its necessity – that the natural numbers exist necessarily. Second, on my interpretation of Kant’s modal theory, possible worlds are formal constructions on concepts (see Hanna [2001: 85, 241–242]), so for Kant the step from conceivability to possibility is automatically guaranteed. And third, for Kant extensional identity is not a purely logical notion (see Hanna [2001:142, n. 57]), and if he is right then it is not surprising that the extensional law of identity fails in some logically possible worlds.
16 A third sort of world would be a quas-world, i.e., a world in which some sort of non-Peano addition-function holds. See Shapiro (2000: 89–90) and Kripke (1982: ch. 2).
18 Kemp Smith (1992: 133).
19 Friedman (1992: 105, n. 16).
20 Parsons (1983: 140). Nevertheless Parsons thinks that “Kant did not reach a stable position on the place of the concept of number in relation to the categories and the forms of intuition” (1992: 152). If I am right about the interplay between intuitional and logical factors in Kant’s analysis of the meaning of concept NUMBER, then Kant’s account is in fact more stable and cogent than Parsons supposes.


22 Potter (2000).

23 Benacerraf (1972).

24 This problem arises in several ways. In its most general form it is Quine’s problem of ‘what there is’ (1953: 14–15); in the context of first-order logic it is Hacking’s problem about categoricity (1979); and in the context of second-order logic it is Frege’s Caesar problem (1953: 68) about identifying the numbers with objects and Benacerraf’s problem of the indeterminacy of the reference of number terms (1965).

25 Benacerraf (1965), (1972), and (1996).

26 To be sure, Leibniz had already anticipated this idea. But Schultz was apparently the first philosopher to float it after the publication of the Critique of Pure Reason. Significantly, the published version of the Prüfung does not contain this thesis. This could simply be a matter of Schultz’s deferring to his teacher and master. But it could also be a matter of Schultz’s believing that Kant’s reply adequately handled his objection.

27 This view has been defended by L.W. Beck and G. Martin. The opposite view – that pure intuition enters even into the inferential transitions of arithmetic proofs – has been defended by Russell, Hintikka, and Friedman. See Friedman (1992: 80–95). As I indicated in note 11 above, Friedman is also committed to the Russellian view that the weakness of Kant’s logic is responsible for his doctrine that arithmetic is synthetic a priori. But it is of course perfectly consistent to hold that intuition enters even into the inferential transitions of arithmetic* proofs and also that the weakness of Kant’s logic is not responsible for his doctrine that arithmetic* is synthetic a priori.

28 Kant sometimes reverses ‘totality’ and ‘unity’; but for a good defense of the claim that Kant’s real intention is to put them in the order I have used in the text, see Longuenesse (1998: 248–249).

29 The extensive continuum has a magnitude equal to the natural numbers, and the intensive continuum has a magnitude equal to the real numbers. So, given Kant’s conception of pure intuition, together with the schematized ‘mathematical’ categories, it follows that the empirical world is both an extensive and intensive continuum. And in this way, it seems, Cantor’s continuum hypothesis is determinately true in every experienceable world.


31 See Shapiro (2000: ch. 10); and Shapiro (1997). Unfortunately it is hard to find a clear or widely-accepted statement of what is meant by saying that something (e.g., a universal) is ante rem. In any case, for me something is ante rem if and only if it is not uniquely located in spacetime and its existence does not logically require the existence of actual things. So, roughly speaking, for me something is ante rem if and only if it is abstract and neither de re nor in re. And Kant’s logically-constrained pure intuitional representation of the infinite unidirectional time-series is ante rem in precisely this sense (CPR A30–36/B46–53, A291–292/B347–348).


34 I think that Lisa Shabel is correct in holding that ‘in a Kantian context “algebra” cannot be taken simply to denote the arithmetic of indeterminate or variable numeric quantities but must be recognized as a method applied to the solution of arithmetic and geomet-
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ric problems, resulting in a geometric construction of “magnitude in general”: a line segment expressing either a number, or the determinate size of a quantum’ (1998: 617). In other words, Kantian algebra is a general science of magnitude that comprehends both geometry and arithmetic and indeed requires the fusion of geometry and arithmetic. So since this paper focuses on Kant’s theory of arithmetic, I will say nothing specifically about symbolic construction or algebra.

36 The mental creation of mathematical objects should be carefully distinguished from the mental creation of empirical objects. Kant holds that by constructing mathematical concepts a priori we also (partially) create empirical objects by determining basic elements of their form (CPR A723/B751). That follows directly from the thesis of transcendental idealism. But an empirical object is not a mathematical object, except insofar as mathematical concepts apply to empirical objects. My general point here is that for Kant there really are no mathematical objects – where such objects are taken to be ontically independent of number concepts – even though there are entities that are numbers. This apparent paradox is resolved when we recognize that numbers are nothing but positions in a logically-conceptually constrained intuitional time-structure.

37 See note 1 above, Brouwer (1952), and Shapiro (2000: ch. 7).
40 It follows that arithmetic* equations cannot be entered into proofs as instances of the extensional law of identity. This presumably is why Kant thinks that arithmetic* truths must be entered into proofs as primitively true or unprovable premises (indemonstrabilia), premises that depend on no assumptions, and are logically based on the empty set of premises. Such premises cannot be properly speaking called “axioms” because they are not general in form or content (CPR A164/B205), although otherwise they function just like axioms in the sense that axioms are all primitively true or unprovable premises in arithmetic* arguments (CPR: A733/B761). In (1953: 5–6), Frege criticizes Kant for appealing to arithmetic* indemonstrabilia, because Frege thinks that there must be only as many first principles as can be comprehended in a compact rational survey. Frege’s criticism is odd for two reasons. First, Kant thinks that pure temporal intuition guarantees that arithmetic* will be cognitively accessible, thus satisfying the requirement of a compact rational survey. Frege’s criticism is odd for two reasons. First, Kant thinks that pure temporal intuition guarantees that arithmetic* will be cognitively accessible, thus satisfying the requirement of a compact rational survey – so Frege’s worry depends entirely on the question-begging assumption that pure intuition is not our rational mode of access to arithmetic truth. Second, Frege himself thinks that logical reasoning depends on unprovable logical laws, but has no way of showing that there are not infinitely many such logical indemonstrabilia.

41 This too is perfectly consistent with Gödel-incompleteness and suggests the Kantian thesis that there are true unprovable sentences in elementary arithmetic precisely because arithmetic* proof is not purely logical but on the contrary always requires intuitional and imaginational supplementation.
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