THE JOURNAL OF FINANCE • VOL. LVIII, NO. 6 • DECEMBER 2003

Model Misspecification and Underdiversification

RAMAN UPPAL and TAN WANG*

ABSTRACT

In this paper, we study intertemporal portfolio choice when an investor accounts explicitly for model misspecification. We develop a framework that allows for ambiguity about not just the joint distribution of returns for all stocks in the portfolio, but also for different levels of ambiguity for the marginal distribution of returns for *any* subset of these stocks. We find that when the overall ambiguity about the joint distribution of returns is high, then small differences in ambiguity for the marginal return distribution will result in a portfolio that is significantly underdiversified relative to the standard mean-variance portfolio.

TRADITIONAL RATIONAL EXPECTATIONS MODELS of portfolio choice assume that investors know perfectly the true probability law governing the stochastic processes of asset returns. However, in many situations agents are uncertain about the true model,[1] and hence any particular probability law used to describe the asset return processes would be subject to potential model misspecification. How do agents act in such situations? One possibility is that they summarize their uncertainty using a probability distribution. However, evidence from experimental economics and psychology (Ellsberg (1961)) suggests that in some situations agents' uncertainty cannot be expressed using a single probability distribution; that is, there is ambiguity. The objective of this paper is to develop a model of intertemporal portfolio choice in which investors account explicitly for this ambiguity.

*London Business School and CEPR, and University of British Columbia, respectively. We are extremely grateful for detailed discussions with Bernard Dumas and Larry Epstein. We also acknowledge comments and suggestions from an anonymous referee, Suleyman Basak, Murray Carlson, Greg Connor, Adlai Fisher, Francisco Gomes, Tim Johnson, Burton Hollifield, Kai Li, Pascal Maenhout, Jun Pan, Sinan Sarpca, Stephen Schaefer, Luis Viceira, Hongjun Yan, Bill Ziemba, and seminar participants at Lancaster University, London Business School, MIT Sloan School of Management, University of Amsterdam, University of British Columbia, University of North Carolina, University of Warwick, the 2002 meetings of the AFA, SIRIF Conference on Behavioral Finance, and the Conference on Recent Developments in Securities Valuation and Risk Management at London School of Economics.

[1] For instance, Merton (1980) discusses the difficulty in estimating the expected return of an asset, French and Poterba (1991) report that the standard error of the estimated mean annual return on the U.S. stock market (based on 60 years of data) is 200 basis points, and Gorman and Jorgensen (1999) report similar evidence for several non-U.S. equity markets.

To explore the implications of ambiguity for portfolio diversification, we need a model that allows for differences in the degree of ambiguity across the returns processes of the various assets in the portfolio. The main contribution of our work is to develop such a model. In our framework, an agent formulates a *reference model* of the probability law based on the data available, but recognizes that the reference model is only an approximation to the true model. We then use this framework to derive in closed form the optimal portfolio weights of an investor who accounts for ambiguity in asset returns.

There are two classes of models in the existing literature that capture how agents act in the presence of model uncertainty.[2] One class, including Epstein and Wang (1994) in discrete time and Chen and Epstein (2002) in continuous time, extends the Lucas (1978) model to incorporate the effect of Knightian uncertainty by allowing for multiple priors; Epstein and Miao (2003) present an application of this approach to international portfolio choice in a two-agent equilibrium setting in which agents differ in their ambiguity about returns. The second class of models, including Andersen, Hansen, and Sargent (1999), Hansen, Sargent, and Tallarini (1999), and Hansen and Sargent (2001b) introduces model misspecification and a preference for robustness into the Lucas model.[3] In these latter models, agents recognize the possibility of model misspecification, and account for it in their decisions; Maenhout (1999) applies this framework to study portfolio choice between a riskless and a single risky asset.

Our work is related to both of the above classes of models. In Andersen et al. (1999) and Maenhout (1999), uncertainty is described by a *single* parameter that reflects the overall level of ambiguity. However, to understand the effect of ambiguity on diversification, we need a model in which the ambiguity is not the same across assets. Thus, in contrast to these models, our framework allows for *differences* in the degree of ambiguity about the (marginal) probability laws for the returns of different assets.[4] Our formulation is sufficiently general to incorporate ambiguity about the joint distribution of returns for all stocks being considered, as well as different levels of ambiguity also for the return distribution of *any* subset of these stocks, with the subsets possibly overlapping. Moreover, our formulation is dynamically consistent.[5]

Under the Chen and Epstein (2000) approach, and its application to the home-bias puzzle in Epstein and Miao (2003), agents exhibit extreme pessimism with respect to their multiple priors. However, under our approach, even though

[2] There is an ongoing discussion about the exact relation between these two classes of models. The reader is referred to Epstein and Schneider (2002), Hansen and Sargent (2001a), and Hansen, Sargent, and Wang (2002) for this discussion.

[3] Hansen (2002a) provides an extensive discussion of the relation of the robust decision-making approach to Bayesian models, adaptive models, and models with filtering.

[4] As a by-product, we show that once one allows for differences in the level of ambiguity across assets, the investor's preferences are no longer observationally equivalent to recursive utility (Epstein and Zin (1989), Duffie and Epstein (1992)). Hence, the observational equivalence result in Andersen et al. (1999), Maenhout (1999) and Skiadas (2003) holds only under the extreme case where the level of ambiguity is the same for all assets.

[5] See Epstein and Schneider (2002) for a discussion of dynamic consistency in these models.

agents have multiple priors, they do not exhibit extreme pessimism; instead, they use the reference model to differentiate among the priors. This is an important conceptual difference; knowledge of the data and the economic environment, although not perfect (otherwise there would be no model uncertainty), is used by economic agents in discriminating among candidate priors for the true model of the economy. This difference also leads to a formulation that has the differentiability needed to derive the Bellman equation. Consequently, our characterization of the optimal portfolio is a straightforward extension of the standard Merton (1971) portfolio model without ambiguity.

To illustrate how one would apply our model, we calibrate it to data on international equity returns and compare the resulting portfolio to the Merton portfolio. We also describe how one can gauge whether the choice of the parameters determining the level of ambiguity is reasonable. The calibration shows that when the overall ambiguity about the joint distribution of returns is high, then small differences in ambiguity for the marginal return distribution will result in a portfolio that is significantly underdiversified relative to the standard mean-variance portfolio.

The rest of this paper is organized as follows. In Section I, we develop a utility function for an agent who recognizes the possibility of model misspecification. In Section II, we apply this utility function to study the problem of portfolio selection in the case of multiple risky assets, and we then analyze some special cases that convey the intuition underlying our framework and its implications for portfolio selection. In Section III, we report the results from the calibration exercise. We conclude in Section IV. Proofs are presented in the Appendix.

I. Preferences in the Presence of Model Misspecification

In the first part of this section, we explain how the standard time-separable preferences have been extended in the recent literature to allow for decision making in the presence of model misspecification. Our main contribution to the existing literature is in the second part, wherein we extend this basic framework to allow for differences in the degree of ambiguity about the various elements of the state vector process. While we will be using a model set in continuous time, we start by motivating the analysis in discrete time.

A. The Basic Model with a Single Source of Misspecification

In the standard rational expectations model of portfolio choice and asset pricing, the investor is typically assumed to have intertemporally additive expected utility of the form

$$V_t = u(c_t) + \beta E^P[V_{t+1}]. \tag{1}$$

A fundamental assumption underlying this model is that the investor knows precisely the true probability law of asset returns, P, when computing the expectation in the equation above. In the literature, however, many argue that this assumption is too strong and that an agent should be allowed to account for model misspecification in his decision process.[6]

To incorporate information about model misspecification into the agent's decision process, following Andersen et al. (1999) and Maenhout (1999), we extend (1) to

$$V_t = u(c_t) + \beta \inf_{\xi} [\psi(E_t^{\varepsilon}[V_{t+1}])\phi L(\xi) + E_t^{\xi}[V_{t+1}]]. \tag{2}$$

The intuition behind the model in (2) can be described as follows. Let the investor's knowledge about the uncertainty in the economy be described by a probability measure P, called the reference probability or *reference model.* It is often the case that P is the result of some estimation process and thus, it is subject to misspecification error. Because the investor is not sure if P is the right model, it is natural that he would consider alternative models. Let a possible alternative to the reference model P be described by a probability measure Q^{ξ} given by

$$dQ^{\xi} = \xi(X_{t+1})dP, \tag{3}$$

where X_t is the vector of state variables and $\xi(x)$ is a density function. Of course, there can be many possible alternatives. The investor's problem is to determine how to take into account the possible alternatives when making his decisions.

To evaluate the alternative models, the investor needs an index that tells him, given his information, how each alternative compares with the reference model. In equation (2) this is done through the penalty function

$$\phi L(\xi) = \phi E^{\xi}[\ln \xi], \tag{4}$$

where E^{ξ} is the expectation under Q^{ξ} and $\phi \geq 0$ is a parameter whose role is explained below.[7] Lastly, the term $\psi(E_t^{\varepsilon}[V_{t+1}])$ in (2) is a normalization function that converts the penalty to units of utility so that it is consistent with the units of $E_t^{\xi}[V_{t+1}]$; the particular functional form of $\psi(\cdot)$ is often chosen for analytical convenience.[8] The minimization over ξ in (2) reflects the agent's aversion to ambiguity, that is, model misspecification.

Putting everything together, the intuitive interpretation of (2) is that when faced with potential model misspecification, the investor ponders whether he should use model Q^{ξ} to evaluate his future utility. The term $\phi L(\xi)$ is used as a penalty function for rejecting the reference model P and accepting the alternative model Q^{ξ}. However, if one can easily distinguish an alternative model Q^{ξ} from the reference model P, then accepting Q^{ξ} will incur a penalty. The magnitude of the penalty depends on the level of ambiguity in the reference model P.

[6] For references to this literature, see Epstein and Wang (1994), Hansen et al. (1999), Andersen et al. (1999), Maenhout (1999), and Chen and Epstein (2002).

[7] One interpretation of the index in (4) is that it is an approximation to the empirical likelihood ratio adjusted for the level of ambiguity. See Andersen et al. (1999) and Hansen and Sargent (2001b) for other interpretations of the index.

[8] Note that the utility function $u(c)$ is unique only up to a positive linear transform. As will be seen, when ψ is a linear function, the preference defined by equation (2) remains unchanged when $u(c)$ is replaced with $au(c)$, $a > 0$.

In the extreme case where $\phi \approx \infty$ (i.e., the investor is extremely confident about P), any alternative model Q^{ξ} that deviates from the reference model will be penalized heavily, in which case, equation (2) reduces to the standard expected utility in equation (1). Thus the standard expected utility can be viewed as the special case of (2) where the investor knows the true model—rational expectations—and, hence, has no ambiguity about the reference model. On the other hand, considering models that the investor cannot clearly distinguish will result in only a small penalty. Among these models, due to his concern for model misspecification, the investor uses the one that gives the lowest expected utility. For the extreme case where $\phi \approx 0$ (i.e., the investor has no knowledge about P) equation (2) reduces to

$$V_t = u(c_t) + \beta \inf \; V_{t+1}. \tag{5}$$

In this case, the investor will consider the worst-case scenario as the only possible outcome.

In general, the investor balances his concern about model misspecification and the knowledge he has about the economy as represented by P. He does not wish to throw away information by setting $\phi \approx 0$ and guarding only against model misspecification, nor does he want to ignore his ambiguity about the information by setting $\phi \approx \infty$ and overlooking completely the possibility of model misspecification.

We conclude this section with remarks on the comparison between the Bayesian approach to dealing with estimation risk and our approach to model misspecification. Though one can always find a parameterization for which a model with estimation risk will yield the same weights as our model, there is a fundamental difference between the Bayesian approach to estimation risk and our approach. In the Bayesian approach, model misspecification often comes in the form of parameter uncertainty. To be specific, suppose that a model of the probability law for asset returns is estimated and there exists one parameter which cannot be estimated precisely. Let $P(X; \alpha)$ be the probability distribution function and α be the parameter about which one is uncertain. Given that the parameter α is unknown, the question for the investor is to determine how to incorporate the parameter uncertainty into his decision process. The critical assumption of the Bayesian approach is that this parameter uncertainty (i.e., model misspecification) can be represented by a prior distribution F, and thus the investor's utility can be computed by the following:[9]

$$V_t = u(c_t) + \beta E^F[E^{P(\alpha)}[V_{t+1}]]. \tag{6}$$

In contrast, under our framework, one need not restrict model misspecification to uncertainty regarding a particular parameter. More importantly, we do not assume that model misspecification, as a subjective matter, can be represented by a probability distribution. This difference between the Bayesian approach and ours is exactly the same as that between the Savagian and Knightian approaches to decision making under uncertainty. Analytically, this difference amounts to the fact that under the Knightian approach, the investor minimizes

[9] If learning (updating) is to be incorporated, then F is the posterior.

over the set of possible models subject to a penalty for large deviations from the reference model, whereas under the Bayesian approach, the investor imposes a prior over the set of possible models. For a more extensive discussion of this difference, see Ellsberg (1961).

B. Extension: Different Levels of Ambiguity for Each State Variable

While the basic model in the previous section captures the investor's concern for model misspecification, it does not allow for different levels of ambiguity for different elements of the state vector process. In this section, we extend the basic model to allow for such differences in ambiguity, which distinguishes our work from that of Andersen et al. (1999) and Maenhout (1999). The basic intuition underlying this extended model is the same as that elucidated in the preceding section; the main change is the development of an appropriate penalty function.

We start by considering the problem in discrete time. Suppose that uncertainty is generated by more than a single state variable. Imagine an investor whose knowledge about the probability law for the state variables is limited and this information comes from separate sources such that the investor is more confident about some sources relative to others. Then take, for instance, a two-country universe in which each country has one large firm and one small firm. We would like to allow for knowledge about the joint distribution of returns for all four stocks from an analyst who covers a broad spectrum of stocks. We would also like to allow for additional information from analysts specializing in a subset of these four stocks: the sets consisting of only foreign stocks, only domestic stocks, only large stocks, only small stocks, and each of the individual stocks. We would like to develop a framework that is sufficiently general to allow for different levels of ambiguity for information from different sources and about different subsets of assets. To have a model capable of reflecting this feature of the investor's information, we extend the basic model described in the previous section by first generalizing the relative entropy index in (4) and then incorporating this more general index into the utility function in (2).

Let $X_t = (X_{1t}, \ldots, X_{nt})$ be the vector of all state variables. Let Q^{ξ} represent an alternative model as in the previous section, with

$$dQ^{\xi} = \xi dP, \tag{7}$$

where ξ is a scalar that perturbs P, the joint distribution of all the state variables. Let $J_i = \{j_1, \ldots, j_{n_i}\}$ be a subset of $\{1, \ldots, n\}$, and let $X_{J_i} = (X_{j_1}, \ldots, X_{j_{n_i}})$ be the corresponding subvector of X_t. Suppose that the investor has a separate source of information about each subset of state variables, X_{J_i}. Then, as in (4), we can use an index to describe this information. However, because the information is about the subset of state variables, the index is now calculated with respect to the marginal distribution of X_{J_i}:

$$\phi_i L(\xi_i) = \phi_i \int \left[\xi_i(X_{J_i,t+1}) \ln \xi_i(X_{J_i,t+1})\right] dP_{J_i}, \tag{8}$$

where P_{J_i} is the marginal distribution of the subvector X_{J_i} under the reference probability measure P, and $\xi_i = dQ^{\xi}_{J_i}/dP_{J_i}$. If there are K sources of information for the various subsets of state variables, then the overall index is taken to be the sum

$$\sum_{i=1}^{K} \phi_i L(\xi_i). \tag{9}$$

The investor's utility function is now given by the following recursive equation, which is similar to (2), but with the inclusion of the index (9) that allows for multiple sources of information about the vector of state variables:

$$V_t = u(c_t) + \beta \inf_{\xi} \left\{ \psi(E_t^{\varepsilon}[V_{t+1}]) \sum_{i=1}^{K} \phi_i L(\xi_i) + E^{\xi}[V_{t+1}] \right\}, \tag{10}$$

where, as before, $\psi(E_t^{\varepsilon}[V_{t+1}])$ is a normalization factor. The interpretation of (10) is essentially the same as in the previous section. The only difference is that if one source of information about a particular subset of the state variables is more reliable, the investor will assign a higher penalty for deviating from that information. For instance, if the investor has very reliable information about the return of a particular stock, he will put a high penalty for any alternative model whose marginal distribution for the return on this stock deviates from that of the reference model.

We now extend the utility function formulated for discrete time in equation (10) to a continuous time setting. Suppose that the state variables $X_t = (X_{1t}, \ldots, X_{nt})$ follow the process

$$dX_t = \mu_X(X_t, t)dt + \sigma_X(X_t, t)dw_t, \tag{11}$$

where w_t is a n-dimensional Brownian motion. Let

$$\mathcal{A}(f) = f_t + \mu_X f_X + \frac{1}{2}\mathrm{tr}\left(f_{XX}\sigma_X\sigma_X^{\top}\right) \tag{12}$$

be the differential operator associated with the diffusion process X_t. Denote by $[\sigma_{J_i}\sigma_{J_i}^{\top}]_n$ the $n \times n$ matrix whose element in the j_kth row and j_ℓth column, for j_k and j_ℓ in J_i, is equal to the element in the kth row and ℓth column of the matrix $[\sigma_{X_{J_i}}\sigma_{X_{J_i}}^{\top}]^{-1}$, the inverse of the variance–covariance matrix of X_{J_i} or zero otherwise.[10]

THEOREM 1: *The continuous-time version of (10) is*

$$0 = \inf_{v} \left\{ u(c) - \rho V + \mathcal{A}(V) + v^{\top} V_X + \frac{\psi(V)}{2} v^{\top}\Phi v \right\}, \tag{13}$$

where $v = (v_1, \ldots, v_n)^{\top}$ *and*

$$\Phi = \sum_i \phi_i [\sigma_{J_i}\sigma_{J_i}^{\top}]_n. \tag{14}$$

[10] Specific examples of $[\sigma_{J_i}\sigma_{J_i}^{\top}]_n$ can be seen in equations (26) and (31).

In equation (13), the first three terms correspond to the standard Hamilton-Jacobi-Bellman equation for the expected utility function under the reference probability P and ρ corresponds to β in (10). The fourth term, $v^\top V_X$, arises from the change of probability measure from P to Q^ξ in (10). By Girsanov's Theorem, the change of probability measure is equivalent to a change in the drift term of the process of X_t. The drift change is given by $v = (v_1, \ldots, v_n)$. That is, under the probability Q^ξ, the process for X_t is

$$dX_t = [\mu_X(X_t, t) + v_t]dt + \sigma_X(X_t, t)dw_t^\xi, \tag{15}$$

where w_t^ξ is a Brownian motion under Q^ξ. Observe that the effect of the change from the reference model P to the alternative model Q^ξ is completely captured by this term; this will be useful for understanding the results in the portfolio choice problem that we will consider in the next two sections. Finally, the last term in (13) corresponds to the penalty function in (10). The fact that the utility function of the agent can be characterized by the Hamilton–Jacobi–Bellman equation (13) indicates that our formulation of the agent's preference is dynamically consistent.[11]

In the next section, we apply this general framework to the Merton portfolio problem, and then consider examples based on specific parameterizations.

II. Portfolio Selection with Multiple Risky Assets

In this section, we study the portfolio choice problem of an investor who is concerned about model misspecification. The portfolio choice model we use is standard, following from Merton (1971, 1973), except for the preferences of the investor, which are the ones developed in the previous section.

A. Individual Investor's Portfolio Choice

The investor can consume a single good, invest in N risky stocks, and borrow and lend at an exogenously given riskless rate $r_{t=r(Y_t)}$. We let c denote the consumption rate of the investor, W the wealth of the investor, and π_j the share of the investor's wealth invested in the jth risky asset.

The return processes of the N stocks are given by

$$dR_t \equiv \mu_R(R_t, Y_t)dt + \sigma_R(R_t, Y_t)dw_t, \tag{16}$$

and

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dw_t. \tag{17}$$

These processes are viewed as the reference model. We assume that Y_t is a K-dimensional process and that the Brownian motion is $(N+K)$-dimensional.

[11] This implies that the criticism in Epstein and Schneider (2002) about the lack of dynamic consistency of the Hansen-Sargent formulation does not apply to our model.

The dynamics of the investor's wealth, for a given investment decision π and consumption decision c, is

$$dW_t = W_t\left[r_t + \pi_t(\mu_R - r_t) - \frac{c_t}{W_t}\right]dt + W_t\pi_t\sigma_R dw_t. \tag{18}$$

The investor wishes to maximize his intertemporal lifetime utility, subject to the budget equation (18), while taking into account model misspecification when making his decisions.

To use Theorem 1 appropriately to derive the Bellman equation corresponding to the investor's utility maximization problem, we need to distinguish between exogenous and endogenous state variables. As we know from Merton (1971), the investor's knowledge about the investment opportunities is described by the reference model given by (16) and (17). Thus, the state variables for the problem without model misspecification are R_t, Y_t, and the investor's wealth, W_t. However, the investor's wealth process (18) is derived from the stock returns. This can be seen by expressing the evolution of wealth in terms of stock returns:

$$dW_t = W_t(1 - \pi_t\mathbf{1})r_t dt - c_t dt + W_t\pi_t dR_t. \tag{19}$$

Thus, W_t itself is not a source of model misspecification. The wealth process, W_t, only inherits the model misspecification through R_t. Because of this difference, R_t and Y_t are called *exogenous* state variables, while W_t is an *endogenous* state variable. These two sets of state variables need to be treated differently. In particular, the drift adjustment for W is $v_W = W_t\pi_t v_R$, because the amount of wealth invested in the risky asset is $W_t\pi_t$, and, under probability Q^ξ, the drift adjustment for R is v_R.

We write the investor's indirect utility function as $V(W_t, R_t, Y_t, t)$. Applying Theorem 1 and using the appropriate drift adjustment for W as discussed above, the Hamilton–Jacobi–Bellman equation for the investor's utility maximization problem is

$$\begin{aligned}
0 = \sup_{c,\pi}\inf_{v_Y,v_R}\Bigg\{ & u(c) - \rho V + V_t + WV_W\left[r + \pi(\mu_R - r) - \frac{c}{W}\right] \\
& + \frac{W^2}{2}V_{WW}\pi^\top\sigma_R\sigma_R^\top\pi + V_R\mu_R \\
& + V_Y\mu_Y + \frac{1}{2}\mathrm{tr}\left[\begin{pmatrix} V_{RR} & V_{RY} \\ V_{YR} & V_{YY}\end{pmatrix}\begin{pmatrix}\sigma_R \\ \sigma_Y\end{pmatrix}\begin{pmatrix}\sigma_R \\ \sigma_Y\end{pmatrix}^\top\right] \\
& + W\pi\sigma_R\sigma_Y^\top V_{WY} + W\pi\sigma_R\sigma_R^\top V_{WR} \\
& + V_W W\pi v_R + V_Y v_Y + V_R v_R + \frac{\psi(V)}{2}v^\top\Phi v\Bigg\},
\end{aligned} \tag{20}$$

where Φ is defined by the expression (14). The terms in the first four lines of this equation are the same as those that would appear in the standard Bellman equation. The next three terms, $V_W W\pi v_R + V_Y v_Y + V_R v_R$, reflect the drift adjustment due to the change of probability measure. The last term, $(1/2)\psi(V)v^\top\Phi v$, is the penalty function.

Let I_R be the identity matrix of the same dimension as R, I be the identity matrix of the dimension of $(R+Y)$, and π^{Merton} be the optimal portfolio when there is no model misspecification. Then, substituting the first-order condition for the minimization problem in (20) into the first-order conditions for the maximization problem and solving for π gives the following result.

THEOREM 2: *The optimal portfolio of an investor is given by*

$$\pi = -\frac{1}{WV_{WW}}[\sigma_R\sigma_R^\top]^{-1}[V_W(\mu_R - r + v_R^*) + \sigma_R\sigma_Y^\top V_{WY} + \sigma_R\sigma_R^\top V_{WR}], \tag{21}$$

where

$$\begin{bmatrix} v_R^* \\ v_Y^* \end{bmatrix} = -\frac{1}{\psi(V)}\Phi^{-1}\begin{bmatrix} V_W W\pi_t + V_R \\ V_Y \end{bmatrix}. \tag{22}$$

Or, in closed-form,

$$\begin{aligned} \pi &= \frac{-1}{WV_{WW}}B\left[\sigma_R\sigma_R^\top\right]^{-1}\left[V_W(\mu_R - r) + \sigma_R\sigma_Y^\top V_{WY} + \sigma_R\sigma_R^\top V_{WR}\right] \\ &\quad + \frac{V_W}{WV_{WW}}B\left[\sigma_R\sigma_R^\top\right]^{-1}\begin{bmatrix} I_R & 0 \\ 0 & 0 \end{bmatrix}\Phi^{-1}\begin{bmatrix} V_R/\psi(V) \\ V_Y/\psi(V) \end{bmatrix} \\ &= B\pi^{\text{Merton}} + \frac{V_W}{WV_{WW}}B\left[\sigma_R\sigma_R^\top\right]^{-1}\begin{bmatrix} I_R & 0 \\ 0 & 0 \end{bmatrix}\Phi^{-1}\begin{bmatrix} V_R/\psi(V) \\ V_Y/\psi(V) \end{bmatrix}, \end{aligned} \tag{23}$$

with

$$B = \left(I - \frac{(V_W)^2}{\psi(V)V_{WW}}\left[\sigma_R\sigma_R^\top\right]^{-1}\begin{bmatrix} I_R & 0 \\ 0 & 0 \end{bmatrix}\Phi^{-1}\begin{bmatrix} I_R \\ 0 \end{bmatrix}\right)^{-1}. \tag{24}$$

Without the term v_R^*, equation (21) reduces to the standard Merton formula. As noted above, this term corresponds to the drift adjustment to the wealth process due to R. Hence, (21) can be viewed as the Merton formula with μ_R adjusted by v_R^*. Similarly, when the ϕs tend to infinity, $\Phi^{-1} \to 0$ and equation (23) reduces to the familiar Merton (1971) result.

B. Understanding the Portfolio Model

To gain some insight into the expression for the portfolio weight in (21), we consider an economy where the stock price processes are given by geometric Brownian motions, the riskless rate is constant with $r < \mu_R$, and the investor has power utility of the form $U(c) = c^{1-\gamma}/(1-\gamma)$ and is long lived ($T = \infty$). Following Maenhout (1999), we set $\psi(V) = [(1-\gamma)/\gamma]V$. Under these assumptions, $V(W) = \kappa_0 W^{1-\gamma}/(1-\gamma)$, where κ_0 is a constant that depends on the parameters of the economy. We first look at the case where there is a single risky asset (this is the case considered in Maenhout (1999)) and then consider the case in which there are two risky assets.

In the case in which there is only one risky asset, the explicit expression for the optimal portfolio simplifies to

$$\pi = \left(\frac{\phi}{1+\phi}\right) \underbrace{\frac{1}{\gamma}\,\frac{\mu_R - r}{\sigma_R^2}}_{\text{Merton weight}}. \tag{25}$$

The implications of model misspecification for portfolio choice can be observed from equation (25). For the case $\phi = \infty$, the expression for the portfolio is simply the optimal Merton weight. However, for values of $\phi < \infty$, the investment in the risky asset is less than what it would be in the absence of model misspecification. In the limit, as $\phi \to 0$, investment in the risky asset drops to zero and the investor holds only the riskless asset. Thus, in the context of the portfolio choice problem, the consequence of model misspecification is a reduction in the investment in the asset about whose process the investor is ambiguous. However, the adjustment is limited by the penalty for being too far away from the reference model; this concern keeps the investor from choosing the most pessimistic scenario.

Also observe that the portfolio weight is exactly the same as that in the Merton model when the investor's risk aversion is given by $\gamma(1+1/\phi)$. In other words, observationally the ambiguity parameter ϕ is not separable from risk aversion. This is the observational-equivalence result noted in Andersen et al. (1999) and Maenhout (1999). The intuition behind this result is that the possibility of model misspecification adds another source of uncertainty to the riskiness of the consumption process. If the investor is averse to model misspecification and pessimistic with regard to alternative models, he appears more risk averse in evaluating the given consumption process.

We now consider the case with two risky assets whose returns are driven by Brownian motions in order to illustrate how concern for model misspecification affects the optimal portfolio weights. For expositional convenience, we assume that the two assets have the same expected return (μ) and volatility (σ), and that their returns are uncorrelated. The more general case, in which both the expected returns and volatilities are different for each asset and the returns are correlated, is considered in the next section.

When there are two risky assets, there are three return distributions about which an investor may have knowledge: the joint distribution for the returns on assets 1 and 2, the marginal distribution for asset 1, and the marginal distribution for asset 2. We use ϕ_0 to denote the investor's knowledge of the joint distribution of returns, and ϕ_j, $j = \{1, 2\}$, to characterize the ambiguity about the marginal distribution for the individual asset j. Thus,

$$\Phi = \phi_0 \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}^{-1} + \phi_1 \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma^2} \end{pmatrix}. \tag{26}$$

Case 1: Equal ambiguity about both return processes. We first consider the case in which an investor has knowledge only about the joint process for the returns on assets 1 and 2, implying $\phi_0 > 0$, with no additional knowledge about the marginal distribution for returns on assets 1 or 2 ($\phi_1 = \phi_2 = 0$). The optimal portfolio weight is

$$\pi = \underbrace{\frac{1}{\gamma}\,\frac{\mu - r}{\sigma^2}}_{\text{Merton weights}} \begin{bmatrix} \frac{\phi_0}{(1+\phi_0)} \\ \frac{\phi_0}{(1+\phi_0)} \end{bmatrix}. \tag{27}$$

This is the expression we would get if we used the Maenhout (1999) formulation with multiple risky assets. As one can see, the adjustment factor for model misspecification to the Merton portfolio weights, $\phi_0/(1+\phi_0)$, is the same for both assets 1 and 2. Thus, under such a specification, ambiguity about the return distributions would not bias the portfolio toward a particular asset, and the adjustment to the Merton portfolio weights can be interpreted either as a change in risk aversion from γ to $\gamma(1+1/\phi_0)$ or as a change in the expected return from μ to $[\mu - (\mu - r)/(1+\phi_0)]$.

Case 2: Unequal ambiguity about the returns processes. To focus on the effect of differences in ambiguity about the returns processes for the two risky assets, we now assume $\phi_j \neq 0$, $j = \{1, 2\}$, with ϕ_0 set to zero to obtain a more transparent expression for the portfolio weights.

Under this specification, the optimal portfolio weights are

$$\pi = \underbrace{\frac{1}{\gamma}\,\frac{\mu - r}{\sigma^2}}_{\text{Merton weight}} \begin{bmatrix} \frac{\phi_1}{(1+\phi_1)} \\ \frac{\phi_2}{(1+\phi_2)} \end{bmatrix}. \tag{28}$$

We can interpret these weights as the Merton weights, adjusted by the factor $\phi_j/(1+\phi_j)$, $j = \{1, 2\}$. In the limit, as $\phi_j \to 0$, $\pi_j \to 0$; on the other hand, as $\phi_j \to \infty$, π_j approaches the Merton weight.

In this setting, in which the agent has knowledge of the marginal distributions for the returns on the two risky assets but no knowledge of the joint distribution ($\phi_0 = 0$), the adjustment to the Merton portfolio weights can no longer be interpreted in terms of a change in the agent's risk aversion. Rather, the appropriate interpretation is that of an *asset-specific* change in the expected return from μ to $\left[\mu - (\mu - r)/(1+\phi_j)\right], j = \{1, 2\}$. Thus, the observational-equivalence result noted in Andersen et al. (1999) and Maenhout (1999) between ambiguity aversion and Stochastic Differential Utility is valid only if the agent is equally ambiguous about the distributions of returns for all assets.

Finally, we note that the ratio of the portfolio weight for asset 1 to asset 2 is given by $\phi_1(1+\phi_2)/\phi_2(1+\phi_1)$, which is greater than unity if and only if $\phi_1 > \phi_2$. Thus, for $\phi_1 > \phi_2$ the portfolio that accounts for model misspecification will appear biased toward asset 1 relative to the Merton (1971) portfolio that ignores model misspecification and the Maenhout (1999) model wherein there is a single parameter governing ambiguity toward all risky assets. In the next section, we examine the magnitude of this bias in the context of international portfolio choice.

III. Calibration to International Equity Returns

In this section, we illustrate how one can apply the model developed above by exploring its implications for underdiversification. Motivated by the evidence in Tversky and Heath (1991) that shows individuals behave as though unfamiliar gambles are riskier than familiar ones (even though they assign identical probability distributions to the two gambles), we calibrate the portfolio model to data on domestic and foreign stock returns, and explore how the portfolio weights change as agents exhibit a greater ambiguity about the return distribution for foreign stocks relative to domestic stocks. We would like to emphasize that the goal of this exercise is not to reproduce the weights documented in the literature on the "home-bias" puzzle, but rather (i) to illustrate how one can apply the model, (ii) to understand the conditions under which the model will yield a portfolio that is underdiversified, and (iii) to show how one can evaluate whether the parameter values chosen are reasonable.[12]

In Section III.A, we describe the choice of parameter values, in Section III.B we report the portfolio weights for a range of ambiguity levels, and in Section III.C we explain how one can assess whether the values chosen for the parameters that determine the level of ambiguity are reasonable.

A. Choice of Parameter Values

We examine the problem from the perspective of a U.S. investor under the assumption that asset prices are geometric Brownian motions and the investment opportunity set is constant. To compare our results to the findings of French and Poterba (1991), we use the same moments of asset returns as they do. These moments are based on data for the period 1975 to 1989 and consist of CPI-adjusted real returns, where the investor is assumed to use 3-month forward contracts to fully hedge the amount of the initial investment. We look at a universe with three "countries": the United States, Japan, and Europe (consisting of France, West Germany, Switzerland, and the United Kingdom). The three countries are indexed by $i = \{1, 2, 3\}$. The volatilities of the real rates of return for the United States (US), Japan (JP), and Europe (EU) are 0.1650, 0.1825, and 0.2000, respectively and the US–JP, US–EU, and EU–JP correlations are 0.53, 0.55, 0.40, respectively. These volatility and correlation estimates are taken from French and Poterba (1990), with the numbers for Europe being averages of the reported estimates for France, West Germany, Switzerland, and the United Kingdom. The expected real rate of returns on U.S., Japanese, and European equities, are 0.0464, 0.0430, and 0.0460, respectively, and are computed from the estimates in Table 2 of French and Poterba (1991) under the assumption that the investor's degree of risk aversion is 3.

[12] A survey of the papers attempting to provide various explanations for this puzzle can be found in Stulz (1995) and Lewis (1999).

In the calibration exercise, we compute the portfolio weights for our model under the assumption that the investor has knowledge of two return distributions. The first is the joint distribution of the stock returns of *all* three countries; the investor's ambiguity about the joint distribution is represented by the parameter $\phi_0 = \phi$. The investor is also assumed to have some knowledge about the marginal distribution of U.S. stock returns; the ambiguity about this is denoted by ϕ_1. To constrain the additional knowledge that the investor has about the marginal distribution of U.S. returns, we specify that $\phi_1 = m\phi_0 = m\phi$. Finally, we assume that the U.S. agent has no additional knowledge of European and Japanese stock returns over and above what is known about the joint distribution.

From Theorem 2, the portfolio weights for this specification are

$$\pi = B\frac{1}{\gamma}[\sigma_R\sigma_R^\top]^{-1}(\mu_R - r), \tag{29}$$

$$B = \left(I + [\sigma_R\sigma_R^\top]^{-1}\Phi^{-1}\right)^{-1}, \tag{30}$$

and

$$\Phi = \phi\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}^{-1} + m\phi\begin{pmatrix} \sigma_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

where σ_{ii} is the variance of the stock returns for country i, and σ_{ij}, $i \neq j$ is the covariance between the stock returns of i and j. Thus, for the calibration we need to specify values for two additional parameters: ϕ and m. We first report the portfolio weights and the total investment in risky assets based on (29) for a range of values for the two parameters, ϕ and m. We then explain how one can assess whether the values chosen for these parameters are reasonable.

B. Portfolio Weights

The portfolio weights from our model are compared to those from the Merton (1971) model, in which agents are assumed to have no ambiguity about the returns process ($\phi = \infty$, so that $B = I$), and also to those from the Maenhout (1999) model, in which the agent has some ambiguity about the joint process for the returns of *all* risky assets in the portfolio ($0 < \phi < \infty$), but no additional knowledge about the returns process of any individual asset ($m = 0$).

The first three columns of numbers in Table I give the weights allocated to U.S., Japanese, and European equities in a portfolio consisting of *only* these three risky assets. In the next two columns, we report the total proportion of wealth invested in the three risky assets and the proportion invested in the riskless asset. The last three columns of the table give the adjustment to the expected return. The first row (from Table 1 of French and Poterba (1991)) reports the "Observed weight" for the U.S. economy, which exhibits a strong bias toward U.S. equities. The second row (also from Table 1 in French and Poterba (1991)) gives the value-

Table I

Portfolio Weights and Adjustment to Expected Returns

This table compares the portfolio weights of a U.S. investor who accounts for model misspecification to those from the Merton (1971) and Maenhout (1999) models. The first three columns of numbers report the weights allocated to the U.S., Japanese (JP), and European (EU) equities in the risky-asset portfolio. The next two columns give the proportion of total wealth invested in the three risky assets and the weight in the risk-free asset. The last three columns report the adjustments to expected returns implied by the choice of ϕ and m, which are defined below. The three panels correspond to different levels of overall ambiguity, indexed by ϕ. The first row in *each* panel, "$m=0$", corresponds to the case in equation (31), where one has knowledge about the joint distribution of asset returns for the three indexes (US, JP, and EU) but no additional knowledge about any of the marginal distributions; this matches the model in Maenhout (1999). The rows titled "$m=1$" to "$m=5$" correspond to the case in which the investor is less ambiguous about the marginal distribution of U.S. returns. As in French and Poterba (1991), the investor is assumed to have a risk aversion of three, while the vector of expected real rates of return on U.S., Japanese, and European equities is $\{0.0464, 0.0430, 0.0460\}$, the volatility vector for U.S., Japanese, and European markets is $\{0.1650, 0.1825, 0.2000\}$, and the US–JP, US–EU, and EU–JP correlations are $\{0.53, 0.55, 0.40\}$. The table shows that the bias toward U.S. equities increases as ambiguity about the marginal distribution for U.S. stock returns decreases (measured by an increase in m); this effect is larger when the overall level of ambiguity is high (low ϕ).

	Risky-asset Portfolio			Weight in Risky Assets	Weight in Riskless Asset	Drift Adjustment		
	US	JP	EU			US	JP	EU
Observed weights	0.938	0.031	0.031					
Market weights	0.496	0.276	0.228					
Merton weights	0.504	0.278	0.218	0.700	0.300			
Panel A: $\phi=1.00$								
$m=0$	0.504	0.278	0.218	0.350	0.650	– 0.0232	– 0.0215	– 0.0230
$m=1$	0.610	0.219	0.171	0.444	0.556	– 0.0155	– 0.0170	– 0.0178
$m=2$	0.647	0.198	0.155	0.492	0.508	– 0.0116	– 0.0147	– 0.0153
$m=3$	0.666	0.187	0.146	0.520	0.480	– 0.0093	– 0.0133	– 0.0137
$m=4$	0.678	0.181	0.141	0.539	0.461	– 0.0077	– 0.0124	– 0.0127
$m=5$	0.686	0.176	0.138	0.553	0.447	– 0.0066	– 0.0118	– 0.0120
Panel B: $\phi=0.50$								
$m=0$	0.504	0.278	0.218	0.233	0.767	– 0.0309	– 0.0287	– 0.0307
$m=1$	0.647	0.198	0.155	0.328	0.672	– 0.0232	– 0.0241	– 0.0255
$m=2$	0.699	0.169	0.132	0.385	0.615	– 0.0186	– 0.0214	– 0.0224
$m=3$	0.726	0.154	0.120	0.423	0.577	– 0.0155	– 0.0196	– 0.0204
$m=4$	0.743	0.144	0.113	0.450	0.550	– 0.0133	– 0.0183	– 0.0189
$m=5$	0.754	0.138	0.108	0.470	0.530	– 0.0116	– 0.0173	– 0.0178
Panel C: $\phi=0.25$								
$m=0$	0.504	0.278	0.218	0.140	0.860	– 0.0371	– 0.0344	– 0.0368
$m=1$	0.678	0.181	0.141	0.216	0.784	– 0.0309	– 0.0308	– 0.0327
$m=2$	0.743	0.144	0.113	0.270	0.730	– 0.0265	– 0.0282	– 0.0297
$m=3$	0.776	0.125	0.098	0.310	0.690	– 0.0232	– 0.0262	– 0.0275
$m=4$	0.797	0.114	0.089	0.342	0.658	– 0.0206	– 0.0247	– 0.0258
$m=5$	0.811	0.106	0.083	0.367	0.633	– 0.0186	– 0.0235	– 0.0244

weighted market weights. The third row gives the weights determined from the Merton model.[13]

Panels A, B, and C of Table I give the portfolio weights of an investor who accounts for model misspecification. The three panels correspond to different levels of overall ambiguity, as indexed by ϕ; recall that a lower ϕ corresponds to a higher level of ambiguity. Within each panel, the portfolio weights are reported for m ranging from zero to five, where zero corresponds to the case where the investor has no additional knowledge about the marginal distribution of U.S. stock returns. Studying the effect of the parameter m, we see that for the case $m=0$, the weights in the risky-asset portfolio are the same as the Merton portfolio weights. This is true across all panels. Thus, a model with only a single parameter controlling the concern for model misspecification, as is the case in Andersen et al. (1999) and Maenhout (1999), cannot generate the limited diversification we observe in the data. As m increases, the investor's portfolio is increasingly biased toward U.S. equities relative to the Merton portfolio. This effect is smaller in Panel A, in which the overall level of ambiguity is given by $\phi=1$, and increases as ϕ decreases. For instance, relative to the Merton weight in U.S. equities of 0.504, the weight is 0.647 for $\phi=1$ and $m=2$ in Panel A, and increasing to 0.699 for $\phi=0.5$ and $m=2$ in Panel B, and to 0.743 for $\phi=0.25$ and $m=2$ in Panel C.

The first plot in Figure 1 shows the share allocated to U.S. equities in the portfolio of only risky assets for a broader range of values of ϕ than displayed in the table. The horizontal solid line shows that for the case $m=0$, the share allocated to U.S. equities does not change with ϕ. However, as m increases, the bias toward U.S. equities increases. The figure shows that the bias is highest when ϕ is low and m is high. The second plot in this figure shows the the total proportion of wealth invested in the three risky assets. The third plot shows the corresponding adjustment to expected returns, which is discussed in the next section.

C. Appropriate Choice of ϕ and m

Ideally, one would like to have a priori information about the appropriate range for both ϕ and m. Since we do not have direct information about ϕ and m, we infer their values indirectly by examining the adjustment to the drift of the returns process implied by the different levels of ϕ and m; this is useful because it is easier to interpret an adjustment to the expected return—for instance, by comparing it to the standard error in estimations of expected returns—than to assess whether m and ϕ are reasonable.

The adjustments to the U.S., Japanese, and European expected returns arising from model misspecification are given in the last three columns of Table I. Again focusing on the case $m=2$, we see from the first row of Panel A that the adjustment to U.S. expected returns is -0.0155, which is less than the standard error of

[13] That the market portfolio and the Merton portfolio are so close to one another is not typical, and the reader should not infer that the two sets of weights will always be close.

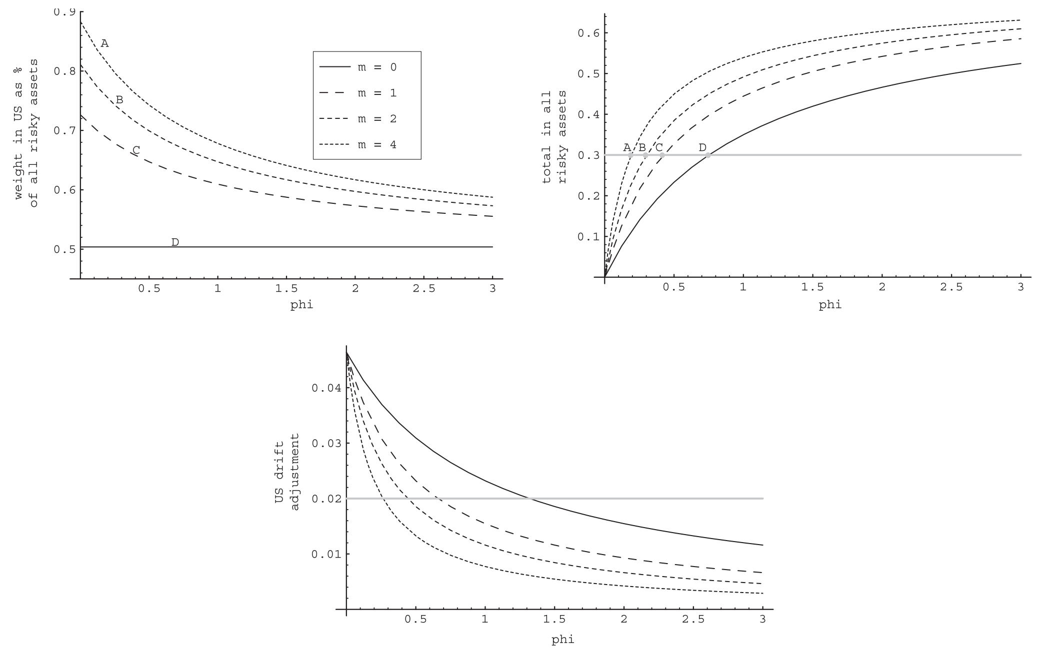

Figure 1. Portfolio weights and adjustment to expected returns. The three panels plot as a function of the agent's ambiguity (ϕ): (i) the portfolio weight allocated to U.S. equities relative to the total investment in risky assets, $\pi_{US}/(\pi_{US}+\pi_{JP}+\pi_{EU})$; (ii) the total investment in risky assets, $(\pi_{US}+\pi_{JP}+\pi_{EU})$; and (iii) the adjustment to U.S. expected returns (drift). The investor is assumed to have a risk aversion of three, while the vector of expected real rates of return on U.S., Japanese, and European equities is $\{0.0464, 0.0430, 0.0460\}$, the volatility vector for U.S., Japanese, and European markets is $\{0.1650, 0.1825, 0.2000\}$, and the US–JP, US–EU, and EU–JP correlations are $\{0.53, 0.55, 0.40\}$. In each panel, four cases are plotted. In the first case, an investor's knowledge of the joint distribution for the returns on U.S., Japanese, and European equities is given by ϕ, but there is no additional knowledge about the marginal distribution of U.S. returns ($m=0$). The three other cases correspond to $m=1$, $m=2$, and $m=4$, where m, ϕ measures the additional knowledge that the investor has about the marginal distribution for U.S. equity returns. From the first panel, we see that the holding of the U.S. assets increases with m, and is particularly pronounced for low values of ϕ, while the second panel shows that the total investment in risky assets increases with ϕ and with m. The third panel shows the adjustment to expected returns implied by different combinations of ϕ and m.

200 basis points reported in French and Poterba (1991). This is also illustrated in the bottom plot of Figure 1. Note also that the adjustment to the mean return *decreases* as m increases; this is important because an increase in m corresponds to an increase in the bias toward U.S. equities. Thus, a small absolute adjustment to the expected returns is sufficient to generate a large bias in portfolio holdings.

We conclude this section by summarizing the main observation: Differences in the levels of ambiguity about the returns distributions leads to portfolios that are underdiversified relative to the Merton (1971) model in which there is no ambiguity, and also to the Maenhout (1999) model in which there is a single parameter measuring the agent's ambiguity toward the distribution of all the assets returns. The underdiversification effect is strongest when the overall level of ambiguity is high.

IV. Conclusion

In this paper, we have developed a model which formalizes the problem of investors who are concerned about model misspecification because they understand that the distributions of assets returns are not estimated with perfect precision. Our model allows agents to have different levels of ambiguity for the distribution of returns for each of the stocks in the portfolio. The model shows that when the overall degree of ambiguity is high, small differences in ambiguity about the marginal distribution of asset returns will lead to a strong bias in portfolio holdings.

Traditional models of portfolio choice predict that investors should hold diversified portfolios. However, there is substantial evidence of a bias toward familiar assets in both international and domestic portfolios of institutions and individual households. International equity portfolios are strongly biased toward domestic stocks as reported by Cooper and Kaplanis (1994) and French and Poterba (1991). Of the limited foreign investments by U.S. and Canadian investors, a disproportionate share is invested across the common border, even though the correlation between U.S. and Canadian returns is higher than the correlations of either with Japanese and European equity returns as shown by Tesar and Werner (1995). Evidence on domestic portfolios reveals a similar lack of diversification: Huberman (2001) finds that U.S. households are more likely to invest in their local U.S. Regional Bell Operating Companies rather than some other Regional Bell Operating Company; Schultz (1996) documents that workers tend to hold their own company's stock in their retirement accounts; and, Grinblatt and Keloharju (2001) report that Finnish households are more likely to invest in firms that are located close to them and that communicate in the investor's native language (Swedish vs. Finnish). At the institutional level, Coval and Moskovitz (1999) find that U.S. mutual fund managers exhibit a preference for local companies. The model we develop can be viewed as offering at least a partial explanation for the observed underdiversification and bias toward familiar securities.

Appendix: Proofs of Theorems

Proof of Theorem 1

Let Q be an alternative model. According to Girsanov's Theorem, $dQ/dP = \xi_T$ is given by

$$\xi_t = \exp\left\{-\int_0^t a_s^\top dw_s - \frac{1}{2}\int_0^t |a_s|^2\, ds\right\}, \tag{A1}$$

for some appropriately adapted process a_t. The result of this change of probability is a drift adjustment to the process of X_t, given by $-\sigma_X a_t$. In other words, a_t can be chosen of the following form:

$$a_t^\top = -v_t^\top [\sigma_X \sigma_X^\top]^{-1} \sigma_X. \tag{A2}$$

Applying Girsanov's Theorem to X_{J_i}, $dQ^{\xi}_{J_i}/dP_{J_i}$ is given by

$$\xi_{it} = \exp\left\{-\int_0^t a_{J_i s}^\top dw_s - \frac{1}{2}\int_0^t |a_{J_i s}|^2 ds\right\}, \tag{A3}$$

where

$$a_{J_i t}^\top = -v_{J_i t}^\top [\sigma_{X_{J_i}} \sigma_{X_{J_i}}^\top]^{-1} \sigma_{J_i X}. \tag{A4}$$

Here, $v_{J_i t} = (v_{j_1 t}, \ldots, v_{j_{n_i}} t)$, $[\sigma_{X_{J_i}} \sigma_{X_{J_i}}^\top]$ is the instantaneous variance–covariance matrix of X_{J_i}, and $\sigma_{J_i X}$ is the matrix whose rows are those of σ_X that correspond to X_{J_i}. Furthermore, by Girsanov's Theorem,

$$dw_t^{\xi} = dw_t + a_t dt \tag{A5}$$

is a Brownian motion under Q^{ξ}, and

$$dX_t = [\mu_X(X_t, t) + v_t]\, dt + \sigma_X(X_t, t) dw_t^{\xi}. \tag{A6}$$

Let

$$V_t = \inf_{\xi}\left\{u(c_t)\Delta + e^{-\rho\Delta}\left[\sum_{i}^{K} \psi(E_t^{\varepsilon}[V_{t+\Delta}])\phi_i L(\xi_{it+\Delta}) + E_t^{\xi}[V_{t+\Delta}]\right]\right\}. \tag{A7}$$

Then

$$[V_t - u(c_t)\Delta]e^{\rho\Delta} = \inf_{\xi}\left\{\psi(E_t^{\varepsilon}[V_{t+\Delta}]) \sum_{i=1}^{K} \phi_i L(\xi_{it+\Delta}) + E_t^{\xi}[V_{t+\Delta}]\right\}, \tag{A8}$$

and thus,

$$0 = \inf_{\xi}\left\{\psi(E_t^{\varepsilon}[V_{t+\Delta}]) \sum_{i=1}^{K} \phi_i L(\xi_{it+\Delta}) + E_t^{\xi}[V_{t+\Delta}] - V_t + V_t - [V_t - u(c_t)\Delta]e^{\rho\Delta}\right\}. \tag{A9}$$

Dividing by Δ, and letting $\Delta \to 0$,

$$\frac{[E_t^{\xi}[V_{t+\Delta}] - V_t]}{\Delta} \to \mathcal{A}[V_t] + v_t^{\top} V_X, \tag{A10}$$

and

$$-\frac{[[V_t - u(c_t)\Delta]e^{\rho\Delta} - V_t]}{\Delta} \to [u(c_t) - \rho V_t]. \tag{A11}$$

Since $d\xi_{it} = -\xi_{it} a_{J_i t}^{\top} dw_t$, we have,

$$L(\xi_{it+\Delta}) = \frac{E_t[\xi_{it+\Delta} \ln \xi_{it+\Delta}] - E[\xi_{it+\Delta}] \ln E_t[\xi_{it+\Delta}]}{E_t[\xi_{it+\Delta}]} \tag{A12}$$

$$= \frac{E_t[\xi_{it+\Delta} \ln \xi_{it+\Delta}] - \xi_{it} \ln \xi_{it}}{\xi_{it}}. \tag{A13}$$

After a straightforward calculation using Itô's Lemma, we have

$$L(\xi_{it+\Delta})/\Delta \to \frac{1}{2} a_{J_i t}^{\top} a_{I_1 t} = \frac{1}{2} v_{J_i t}^{\top} [\sigma_{X_{J_i}} \sigma_{X_{J_i}}^{\top}]^{-1} v_{J_i t}. \tag{A14}$$

Substituting (A10)–(A14) into (A9) yields the desired result. Q.E.D.

Proof of Theorem 2

The first-order condition for the minimization problem in (20) is

$$0 = \begin{bmatrix} V_W W \pi_t + V_R \\ V_Y \end{bmatrix}^{\top} + \psi(V_t) \begin{bmatrix} v_R \\ v_Y \end{bmatrix}^{\top} \Phi. \tag{A15}$$

The first-order conditions for the maximization problem are

$$0 = u'(c) - V_W, \tag{A16}$$

$$0 = V_W[(\mu_R - r_t) + v_R] + \sigma_R \sigma_Y^{\top} V_{WY} + \sigma_R \sigma_R^{\top} V_{WR} + W_t V_{WW} \sigma_R \sigma_R^{\top} \pi_t. \tag{A17}$$

The first part of the theorem follows directly from these two first-order conditions. For the second part, rewrite the first part of the theorem as

$$\pi = -\frac{1}{W V_{WW}} [\sigma_R \sigma_R^{\top}]^{-1} \left(V_W(\mu_R - r) + \sigma_R \sigma_Y^{\top} V_{WY} + \sigma_R \sigma_R^{\top} V_{WR} + V_W \begin{bmatrix} I_R & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_R^* \\ v_Y^* \end{bmatrix} \right) \tag{A18}$$

and

$$\begin{bmatrix} v_R^* \\ v_Y^* \end{bmatrix} = -\frac{V_W W}{\psi(V)} \Phi^{-1} \begin{bmatrix} I_R \\ 0 \end{bmatrix} \pi - \frac{1}{\psi(V)} \Phi^{-1} \begin{bmatrix} V_R \\ V_Y \end{bmatrix}. \quad \text{(A19)}$$

Solving for π yields the closed-form solution. Q.E.D.

REFERENCES

Anderson, E., L. Hansen, and T. Sargent, 1999, Robustness detection and the price of risk, Working paper, University of Chicago.

Chen, Z., and L. Epstein, 2002, Ambiguity, risk and asset returns in continuous time, *Econometrica* 70, 1403–1443.

Cooper, I., and E. Kaplanis, 1994, Home bias in equity portfolios, inflation hedging and international capital market equilibrium, *Review of Financial Studies* 7, 45–60.

Coval, J., and T. Moskovitz, 1999, Home bias at home: Local equity preference in domestic portfolios, *Journal of Finance* 54, 2045–2073.

Duffie, D., and L. Epstein, 1992, Stochastic Differential Utility, *Econometrica* 60, 353–394.

Ellsberg, D., 1961, Risk, ambiguity and the Savage axioms, *Quarterly Journal of Economics* 75, 643–669.

Epstein, L., and J. Miao, 2003, A two-person dynamic equilibrium under ambiguity, *Journal of Economic Dynamics and Control* 27, 1253–1288.

Epstein, L., and M. Schneider, 2002, Recursive multiple-priors, *Journal of Economic Dynamics and Control*, forthcoming.

Epstein, L., and T. Wang, 1994, Intertemporal asset pricing under Knightian uncertainty, *Econometrica* 62, 283–322.

Epstein, L., and S. Zin, 1989, Substitution, risk aversion and the temporal behavior of consumption and asset returns: A theoretical framework, *Econometrica* 57, 937–969.

French, K., and J. Poterba, 1990, Japanese and U.S. cross-border common stock investments, *Journal of the Japanese and International Economics* 4, 476–493.

French, K., and J. Poterba, 1991, Investor diversification and international equity markets, *American Economic Review* 81, 222–226.

Gorman, L., and B. Jorgensen, 1999, Domestic versus international portfolio selection: A statistical examination of the home bias, Working paper, Harvard University.

Grinblatt, M., and M. Keloharju, 2001, Distance bias, language bias, and investor sophistication: Results from Finland, *The Journal of Finance* 56, 1053–1073.

Hansen, L., and T. Sargent, 2001a, Acknowledging misspecification in macroeconomic theory, *Review of Economic Dynamics* 4, 519–535.

Hansen, L., and T. Sargent, 2001b, Robust control and model uncertainty, *American Economic Review* 91, 60–66.

Hansen, L., T. Sargent, and T. Tallarini, 1999, Robust permanent income and pricing, *Review of Economic Studies* 66, 873–907.

Hansen, L., T. Sargent, G. Turmuhambetova, and N. Williams, 2002, Robustness and uncertainty aversion, Working paper, University of Chicago.

Hansen, L., T. Sargent, and N. Wang, 2002, Robust permanent income and pricing with filtering, *Macroeconomic Dynamics* 6, 40–84.

Huberman, G., 2001, Familiarity breeds investment, *Review of Financial Studies* 14, 659–680.

Lewis, K., 1999, Trying to explain home bias in equities and consumption, *Journal of Economic Literature* 37, 571–608.

Lucas, R., 1978, Asset prices in an exchange economy, *Econometrica* 46, 1429–1445.

Maenhout, P., 1999, Robust portfolio rules and asset pricing, Working paper, Harvard University.

Merton, R., 1971, Optimum consumption and portfolio rules in a continuous-time model, *Journal of Economic Theory* 3, 373–413.

Merton, R., 1973, An intertemporal asset pricing model, *Econometrica* 41, 867–888.

Merton, R. C., 1980, On estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics* 8, 323–361.

Schultz, E., 1996, Workers put too much in their employer's stock, *Wall Street Journal*, September 13.

Skiadas, C., 2003, Robust control and recursive utility, *Finance and Stochastics* 7, 475–489.

Stulz, R., 1995, International portfolio choice and asset pricing: An integrative survey, in R. Jarrow, V. Maksimovic, W. Ziemba, eds: *Handbook in Operations Research and Management Science, Finance* vol. 9 (North-Holland, Amsterdam).

Tesar, L., and I. Werner, 1995, Home bias and high turnover, *Journal of International Money and Finance* 4, 467–492.

Tversky, A., and C. Heath, 1991, Preferences and beliefs: Ambiguity and competence in choice under uncertainty, *Journal of Risk and Uncertainty* 4, 5–28.