37

Why Fuzzy Logic?

PETR HÁJEK

It is generally understood that fuzzy logic deals with vague, imprecise notions and propositions. In spite of several successful applications, the logician may (and should) ask: is this really a logic? Does it have foundations, mathematical and/or philosophical? I shall try to give a positive answer to this question, at least as mathematical foundations are concerned, leaving philosophical foundations to professional philosophers. Due to space limitation, I can offer only a survey; but the interested reader will find enough references to detailed works.

1 Origin

Lotfi Zadeh is the author of the notion of a fuzzy set; his 1965 paper is a landmark (Zadeh 1965). A fuzzy subset X of a set A is given by its characteristic function μ_X assigning to each element $a \in A$ the degree $\mu_X(a)$ in which a belongs to X; $\mu_X(a)$ is a real number from the unit interval [0, 1]. Natural language offers plenty of examples: think, for example, of a set of people and its fuzzy subset of tall people (some are more tall, some less). Naturally, one can similarly speak on fuzzy propositions, some being more true and some less ('John is tall'). Apparently the term 'fuzzy logic' first occurs in (Goguen 1968–9) with a elucidating title "The logic of inexact concepts." The beginning of numerous applications of such fuzzy logic is Mamdani (1974), where the author describes a controller based on "fuzzy IF-THEN rules." Such rules are nowadays very popular and may look for example as follows: 'If the pressure is high and the increase of pressure is high then turn the wheel far to the left.' You see various fuzzy notions; for example the meaning of high pressure is to be understood as a fuzzy subset of the domain of pressures: each pressure is high is some degree. Observe the use of natural language (Zadeh likes to speak on "computing with worlds"). There is also some rudimentary logic ('and', 'if-then') but not much.

2 Many-Valued Logic

Clearly, the above resembles some many-valued logic; but for a long time, there were nearly no contacts between what was called fuzzy logic and the many-valued logic entertained by logicians. Early examples are Giles (1976) and Pavelka (1979). Recall that 20th-century many-valued logic started in the 1920s and 1930s in the work of Jan Łukasiewicz (1930; Łukasiewicz and Tarski 1930); later there were works on many-valued logic related to intuitionistic logic (A. Heyting, K. Gödel (1932)). Their work was continued by several authors (Dummett, Chang, Moisil, McNaughton, Scarpelini and others), Gottwald in his 1988 German book on many-valued logic has a short chapter relating many-valued logic to fuzzy logic. Note that Gottwald's book is to appear soon in a revised English version (Gottwald forthcoming). In the meantime, plenty of papers appeared claiming to deal with fuzzy logic but being logically uninteresting. Mutual contacts developed rather slowly.

3 Fuzzy Logic in a Broad and Narrow Sense

It turned out that one has to distinguish two notions of fuzzy logic. It was again Zadeh who coined the terms "fuzzy logic in broad (or wide) and narrow sense." In a broad sense, the term 'fuzzy logic' has been used as anonymous with 'fuzzy set theory and its applications'; for good monographs on this logic see Zimmermann (1991) and Klir and Yuan (1995); in the emerging narrow sense, fuzzy logic is understood as a theory of approximate reasoning based on many-valued logic. Zadeh (1994) stresses that the questions of fuzzy logic in the narrow sense differ from usual questions of many-valued logic and concern more questions of approximate inferences than those of completeness, etc.; nevertheless, with full admiration to Zadeh's pioneering and extensive work (see Klir and Yuan 1996) a logician will *first* study classical logical questions on completeness, decidability, complexity, etc. of the symbolic calculi in question and *then* try to reduce the question of Zadeh's agenda to questions of *deduction* as far as possible. This is the approach in my monograph (Hájek 1998), which I sketch below.

4 The Basic Fuzzy Propositional Calculus

The calculus we are going describe is a result of the following 'design choices' (they are not obligatory but are apparently rather reasonable:

1. The real unit interval [0, 1] is taken to be the *standard set of truth values*, 1 meaning absolute truth, 0 absolute falsity. The usual ordering $\leq$ of reals serves as a comparison of truth-values; we build the logic as a logic with a *comparative notion of truth*. Other structures of truth-values, possibly only partially ordered, are not excluded.
2. The logic is *truth-functional*, that is connectives are interpreted via their truth functions; then for example the truth-value of a conjunction $\varphi\&\psi$ is uniquely determined by the truth-value of φ, of ψ and by the chosen truth function of &.
3. *Continuous t-norms* are taken as possible truth functions of *conjunction*. These operations are broadly used by a fuzzy community; a binary operation $*$ on [0, 1] is a

t-norm if it is commutative ($x*y = y*x$), associative ($x*(y*z) = (x*y)*z$), non-decreasing in each argument (if $x \le x'$ then $x*y \le x'*y$ and dually) and 1 is a unit element ($1*x = x$). The t-norm * is a continuous t-norm if it is continuous as a real function. The three most important continuous t-norms are:

$$\begin{array}{ll} x*y = \max(0, x+y-1) & \text{(Łukasiewicz } t\text{-norm)},\\ x*y = \min(x, y) & \text{(Gödel } t\text{-norm)},\\ x*y = x \cdot y & \text{(product } t\text{-norm)}. \end{array}$$

(For the names see Historical remarks in Hájek (1998).) Note in passing that each continuous t-norm is built from these three in a certain way.

4. The truth function of *implication* is the *residuum* of the corresponding t-norm. If * is your continuous t-norm then its residuum is the operation $\Rightarrow$ defined as follows:

$$x \Rightarrow y = \max\{z | x*z \le y\}.$$

Note that $x \Rightarrow y = 1$ iff $x \le y$; for $x > y$ the residua of the above t-norms are

$$\begin{array}{ll} x \Rightarrow y = 1 - x + y & \text{(Łukasiewicz)},\\ x \Rightarrow y = y & \text{(Gödel)},\\ x \Rightarrow y = y/x & \text{(product)}. \end{array}$$

(One calls these implications R-implications, R for residuum.)

5. The truth function of negation is $(-)x = x \Rightarrow 0$ (x implies falsity).

The resulting logic is called BL – the basic fuzzy propositional logic. We sketch its main properties.

Work with propositional variables $p_1, p_2 \ldots$ and connectives &, $\to$ (strong conjunction, implication) and truth constant $\overline{0}$ (falsity). Formulas are defined in obvious way; $\neg\varphi$ stands for $\varphi \to 0$. Given a continuous t-norm * (and thus its residuum $\Rightarrow$), each evaluation e of propositional variables by truth degrees from [0, 1] extends to an evaluation e_* of all formulas; thus $e_*(\overline{0}) = 0$, $e_*(\varphi\&\psi) = e_*(\varphi) * e_*(\psi)$, $e_*(\varphi \to \psi) = e_*(\varphi) \Rightarrow e_*(\psi)$. Call φ a **-tautology* if $e_*(\varphi) = 1$ for each evaluation e; call φ a *t-tautology* if it is a *-tautology for each * (i.e. however you interpret your propositional variables and connectives, φ is true).

The following t-tautologies are taken to be *axioms* of BL:

$$\begin{array}{ll} \text{(A1)} & (\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))\\ \text{(A2)} & (\varphi\&\psi) \to \varphi\\ \text{(A3)} & (\varphi\&\psi) \to (\psi\&\varphi)\\ \text{(A4)} & (\varphi\&(\varphi \to \psi)) \to (\psi\&(\psi \to \varphi))\\ \text{(A5a)} & (\varphi \to (\psi \to \chi)) \to ((\varphi\&\psi) \to \chi)\\ \text{(A5b)} & ((\varphi\&\psi) \to \chi) \to (\varphi \to (\varphi \to \chi))\\ \text{(A6)} & ((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)\\ \text{(A7)} & \overline{0} \to \varphi \end{array}$$

The deduction rule is *modus ponens* (from φ and $\varphi \to \psi$ infer ψ), proofs and provability are defined in the obvious way.

COMPLETENESS: For each formula φ, BL proves φ iff φ is a t-tautology.

(For a proof see Cignoli et al. (submitted); Hájek (1998) presents another completeness for BL, relating provability in BL to tautologicity with respect to so-called *BL-algebras*. Each continuous t-norm defines a BL-algebra but not conversely.)

The three important t-norms defined above (Ł – Łukasiewicz, G – Gödel, Π – product) give us three important and well-known logics stronger than BL:

Łukasiewicz logic can be axiomatized by adding the schema of double negation $\varphi \equiv \neg\neg\varphi$ to BL. Formulas provable in this logic (developed also by Ł) are exactly all Ł-tautologies. (See Cignoli et al. (2000) for extensive analysis and deep theory of Łukasiewicz logic.)

Gödel logic G (related to Godel (1932)) is BL plus the schema $\varphi \equiv (\varphi \& \varphi)$ of idempotence of conjunction. Formulas provable in *G* are exactly all *G*-tautologies.

Product logic Π is BL plus two additional axioms $(\varphi \to \neg\varphi) \to \neg\varphi$ and $\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$. (The latter axiom expresses cancellation by a non-zero element.) Π proves exactly all Π-tautologies.

It should be mentioned that *G* contains the *intuitionistic logic* (so *G* is an intermediate logic between intuitionistic and classical logic). Neither Ł nor Π contain intuitionistic logic since they have a non-idempotent conjunction.

In BL we may define derived connectives: min-conjunction $(\varphi \wedge \psi) \equiv \varphi \& (\varphi \to \psi)$ whose truth function is a minimum, and max-disjunction $(\varphi \vee \psi) \equiv (((\varphi \to \psi) \to \psi)\ \psi\ ((\psi \to \varphi) \to \vee)$ (maximum).

The truth function $(-)$ of negation in Ł is $(-)x = 1 - x$; but the negation of *G* is Gödel negation: $(-)0 = 1$, $(-)x = 1$ for $x > 0$. Also Π has Gödel negation.

This means that in general the strong conjunction has in BL no dual disjunction; only in Ł, whose negation is involutive, that is $(-)(-)x = x$, the strong disjunction $(\varphi \oplus \psi) \equiv \neg(\neg\varphi \& \neg\psi)$ behaves well. But you may extend both *G* and Π by Łukasiewicz negation (if you want to work with so-called t-conorms; see Esteva et al. (2000)) for a reasonable axiomatization.

Another important extension results when we add to Łukasiewicz logic truth constant $\bar{r}$ for each rational $r \in [0, 1]$ (Pavelka logic), postulating $e_Ł(\bar{r}) = r$. Then evidently $e_Ł(\bar{r} \to \varphi) = 1$ iff $e_Ł(\varphi) \geq r$, which gives us the possibility of expressing estimates of the truth degree of a formula. This extension of Ł has very pleasing properties; an analogous extension of *G* or Π is more complicated. We note in passing that for example. Novák et al. (2000) considers Pavelka logic to be *the* fuzzy logic; I do not share this opinion.

Summarizing this section, continuous t-norm propositional logics are well understood, have pleasant properties and are presently the subject of intensive study.

5 The Basic Fuzzy Predicate Calculus

Extending the developed propositional calculus to a predicate calculus is very natural and a generalization of Tarskian truth definition is immediate. Take some predicates P_1, . . . , each having its arity (unary, binary, . . .), object variables x, y, . . . , connectives &,

$\rightarrow$, truth constant $\overline{0}$, quantifiers $\forall$, $\exists$. (We disregard object constants and function symbols for simplicity.) Formulas are defined in the usual way. An *interpretation* (of $P_1, \ldots, P_n$) is a structure $\mathbf{M} = (M,(r_P)_{P \text{ predicate}})$ where M is a nonempty set (domain) and for each predicate P of arity n, r_P is an n-ary fuzzy relation on M, that is a mapping associating with each n-tuple $(a_1, \ldots, a_n)$ of elements of M a truth degree $r_P(a_1, \ldots, a_n) \in [0, 1]$. The *truth-value* of a formula φ in $\mathbf{M}$ depends (besides $\mathbf{M}$) on a given evaluation e of object variables by elements of M (an M-evaluation, actual meaning of variables) and on the chosen semantics of connectives, that is on the t-norm *. We write $\|\varphi\|^*_{M,e}$ for this. It is defined inductively as follows:

$\|P(x_1, \ldots, x_n)\|^*_{\mathrm{M},e} = r_P(e(x), \ldots e(x_n))$;
$\|\varphi\&\psi\|^*_{\mathrm{M},e} = \|\varphi\|^*_{\mathrm{M},e} * \|\varphi\|^*_{\mathrm{M},e}$
$\|\varphi \rightarrow \psi\|^*_{\mathrm{M},e} = \|\varphi\|^*_{\mathrm{M},e} \Rightarrow \|\psi\|^*_{\mathrm{M},e}$
$\|(\forall x)\varphi\|^*_{\mathrm{M},e} = \inf\ e_x\|\varphi\|^*_{\mathrm{M},e}$
$\|(\exists x)\varphi\|^*_{\mathrm{M},e} = \sup\ e_x\|\varphi\|^*_{\mathrm{M},e}$

where e_x runs over all evaluations differing from e at most in the value for the argument x. The atomic case can be paraphrased thus: the formula saying that $(x_1, \ldots, x_n)$ are P has the truth-value equal to the degree in which the objects $e(x_1) \ldots e(x_n)$ (being the meanings of $x_1, \ldots x_n$) are in the relation r_P (which is the meaning of P). The definitions for $\forall$, $\exists$ naturally generalize the two-valued case. Now the reader expects the following definitions:

A formula φ is a *-tautology (of the predicate calculus) if $\|\varphi\|^*_{\mathrm{M},e} = 1$ for each interpretation $\mathbf{M}$ and M-evaluation e. φ is a t-tautology if it is a *-tautology for each *.

We may call φ **-true* in $\mathbf{M}$ of $\|\varphi\|^*_{M,e} = 1$ for each e. Thus φ is a *-tautology if φ is *-true in each interpretation.

Note that this may be generalized from t-norms to BL-algebras; then r_P is a mapping into the domain of the algebra. But for quantified formulas $\|\varphi\|^*_{\mathrm{M},e}$ ($\mathbf{L}$ being a BL-algebra) may be defined if the corresponding infimum/supremum does not exist in $\mathbf{L}$. One defines an $\mathbf{L}$-*safe* interpretation to be an $\mathbf{L}$-interpretation in which $\|\varphi\|^{\mathbf{L}}_{\mathrm{M},e}$ is total; φ is an $\mathbf{L}$-*tautology* if it is $\mathbf{L}$-true in each $\mathbf{L}$-safe interpretation.

The basic fuzzy predicate logic BL$\forall$ has the above axioms for BL and the following axioms for quantifiers:

($\forall$1) $(\forall x)\varphi(x) \rightarrow \varphi(y)$
($\exists$1) $\varphi(y) \rightarrow (\forall x)\varphi(x)$
($\forall$2) $(\forall x)(\chi \rightarrow \psi) \rightarrow (\chi \rightarrow (\forall x)\psi)$
($\exists$2) $(\forall x)(\varphi \rightarrow \chi) \rightarrow ((\exists x)\varphi \rightarrow \chi)$
($\forall$3) $(\forall x)(\varphi \vee \chi) \rightarrow ((\forall x)\varphi \vee \chi)$

These formulas are well-known from classical logic; they are all predicate t-tautologies (and even BL-tautologies – are $\mathbf{L}$-true in each safe $\mathbf{L}$-interpretation).

Deduction rules are modus ponens and generalization (from φ infer $(\forall x)\varphi$) – as in classical logic.

EXERCISE. Just for refreshment, take the trivial example 5.1.2 from Hájek (1998): $M = \{1, 2, 3\}$, binary predicate *likes*. r_{likes} given by the table

	1	2	3
1	1	0.3	0.7
2	0.9	0.9	0
3	0.9	0.1	0.2

Compute the truth value of $(\forall x, y)(\text{likes}(x, y) \to \text{likes}(x, x))$ (saying 'everybody likes himself/herself most') for Ł, G, Π. (Hint: for Ł it is 0.9.)

What about *completeness?* Note that BL$\forall$ is complete will respect to general interpretation: BL$\forall$ proves φ iff φ is an **L**-tautology for each BL-algebra **L**. Similarly, the predicate versions Ł$\forall$, G$\forall$, $\Pi\forall$ of the corresponding propositional logics *are* complete with respect to (safe) interpretations over algebras from the corresponding subclasses of the class of BL-algebras (called MV-algebras, G-algebras and product algebras for Łukasiewicz, Gödel, and product logic respectively.

With respect to interpretations over [0, 1] the situation is more complicated: the set of all predicate t-tautologies (i.e. formulas being *-tautologies for each continuous t-norm *) is not recursively enumerable (for specialists: it is Π_2-hard). Similarly, neither the set of predicate Łukasiewicz tautologies (tautologies w.r.t. Łukasiewicz t-norm) nor the set of predicate product tautologies is recursively axiomatizable. (For BL$\forall$ not yet published; for Ł$\forall$ first proved by Scarpelini, see Hájek (1998).) But the set of predicate G-tautologies *is* completely axiomatized by BL$\forall$ plus the axiom schema $\varphi \equiv (\varphi\&\varphi)$.

To get a full picture of these logics one has to have some knowledge about formulas provable in them; this is found in Hájek (1998). (Such knowledge is necessary for *proofs* of completeness results.)

Again, various extensions of these logics have been described. Furthermore, there are results on theories over these logics; we have no room to go into details. Similarly as above, let us summarize that the basic fuzzy predicate calculus is reasonably well developed and well behaving. Concerning the results on non-axiomatizability, compare this with the situation of classical second order logic: in the intended standard semantics it is not recursively axiomatizable, but it has a recursive axiomatization which is complete with respect to a generalized (Henkin) semantics.

In the rest of this chapter we shall describe some uses of fuzzy logic that may be of interest for the philosophically minded reader.

6 Similarity

Similarity is a fuzzy equality; the notion appears to be well-known in the fuzzy community. Let $x \approx y$ stand for 'x is similar to y'; the following are axioms of similarity:

$x \approx x$ (reflexivity)
$x \approx y \to y \approx x$ (symmetry)
$(x \approx y \& y \approx z) \to x \approx z$ (transitivity).

What do models of these axioms look like? First observe a non-model: For M being the real line define x, y to be 'similar' if $|x - y| \le 1$. This is a crisp relation (yes–no) and is *not* transitive: 3 and 4 are 'similar,' 4 and 5 also, but 3 and 5 not. *Make it fuzzy*: define

$$r \approx (x, y) = \max(0, 1 - |x - y|).$$

EXERCISE: Draw the graph of the function $x \approx 4$: it is zero for $x \le 3$ and $x \ge 5$ and goes up linearly from the point (3, 0) to (4, 1) and the down linearly from (4, 1) to (5, 0).

Is this relation transitive? It depends on your logic. The axiom of transitivity does say that if $x \approx y$ and $y \approx z$ are (absolutely) true then so is $x \approx z$; but it says *much more*, namely that the truth degree of $x \approx y$ & $y \approx z$ is less than or equal to the truth degree of $x \approx z$. Take Łukasiewicz logic and compute:

$$\|x \approx y \& y \approx z\| = \max(0, \|x \approx y\| + \|y \approx z\| - 1).$$

If this is 0 nothing is to be proved; otherwise continue:

$$\|x \approx y\| + \|y \approx z\| - 1 = 1 - |\, x - y| + 1 - |y - z| - 1 = 1 - (|x - y| + |y - z|) \le 1 - |x - z|$$

by the well-known triangle of inequality; and the last term equals $\|x \approx z\|$. Thus we have verified that the truth value of the transitivity axiom is 1 (the axiom is absolutely true) for our interpretation.

Similar examples for Gödel and product logic are easy to find. Now observe that if $\approx$ satisfies the axioms of similarity and we define $x \approx^2 y$ to be $x \approx y$ & $x \approx y$ then $\approx^2$ is again a similarity; for example in our example $r{\approx}2(x, y) = \max(0, 1 - 2|x - y|)$. For more information see Hájek (1998).

7 The Liar and Dequotation

Here I assume some knowledge of Gödel's technique of self-reference in arithmetic. **N** stands for the structure of natural numbers with zero, successor, addition, and multiplication. PA is Peano arithmetic. The undefinability of truth in arithmetic means the following: Add a unary predicate *Tr* to the language of arithmetic and add the axiom schema of dequotation: $\varphi \equiv Tr(\overline{\varphi})$ to the axioms of PA (φ being an arbitrary sentence of the langauge of PA extended by the predicate *Tr*, and $\overline{\varphi}$ being the numeral naming the Gödel number of φ). Then the resulting theory (PA+Tr) is contradictory over classical logic since one can construct the liar's formula λ such that (PA+Tr) proves $\lambda \equiv \neg Tr(\overline{\lambda})$ and hence proves $\lambda \equiv \neg\lambda$, which is classically inconsistent. Over Łukasiewicz logic the last equivalence is not contradictory, it just forces the truth-value of λ to be 1/2. But we may ask more: Take, inside Łukasiewicz logic, crisp Peano arithmetic, add the predicate *Tr* (which may be fuzzy) and add the dequotation schema. Is this theory consistent (over Łukasiewicz)?

This was answered in Hájek et al. (2000) as follows: (PA+Tr) *is* consistent over the Łukasiewicz predicate logic, hence it has a model (which is crisp for arithmetic and fuzzy for *Tr*); but the standard model **N** cannot be expanded by a fuzzy predicate to a model of (PA+Tr). All models of (PA+Tr) are nonstandard (not isomorphic to **N**). To prove the last claim one constructs a formula that, over **N**, behaves as a 'modest liar formula' – saying 'I am at least a little false.' A detailed analysis shows that this leads to a contradiction.

Let us call the reader's attention to the remarkable book, Grim et al. (1992), where the authors present several self-referential formulas and analyze them by the framework of Łukasiewicz (propositional) logic.

8 Very True

When describing fuzzy logic in the narrow sense, Zadeh claims that it should go beyond the usual many-valued logic, admitting fuzzy truth-values like 'very true,' 'more-or-less true,' etc. Such truth values are understood as fuzzy subsets of the set of truth-values ('true' being just the diagonal; 'very true' being for example the fuzzy set with the characteristic function x^2 on [0, 1]). This was criticized by Haack (1996) as not well founded, unnecessary, etc. Haack herself was criticized by Dubois and Prade (1993), defending Zadeh and fuzzy logic. Here I do not want to enter this discussion but only want to show that 'very true' accommodates well in the 'standard' many-valued approach to fuzzy logic not going beyond it. The idea is to understand 'very true' as a new unary connective.

Recall that in the classical (two-valued) logic we may explicitly have, besides negation (which sends 1 to 0 and 0 to 1) a unary connective t (which sends 1 to 1 and 0 to 0). The formula tφ (evidently equivalent with φ) can then be read 'yes, φ' or 'truly, φ' or just 'φ is true' (not understood as a metatheoretical statement on φ, but just as a part of the object language). In fuzzy logic each mapping of the interval [0, 1] into itself may be taken as the truth function of a unary connective (such connectives are called *hedges*); in particular the identity ($t(x) = x$ for all x) may be taken as the truth function of the fuzzy unary connective t, tφ being read 'yes, φ' or just 'φ is true.' What about a connective vt, where vt(φ) is read 'φ is very true'? What properties shoud it have? Let us call a mapping *vt* of [0, 1] into itself a *truth-stresser* (with respect to a continuous t-norm *) if the following holds for each x, y:

$$vt(1) = 1, \quad vt(x) \leq x, \quad vt(x \Rightarrow y) \leq vt(x) \Rightarrow vt(y).$$

Let BL(vt) be the extension of our logic BL by the following axioms for vt:

(VT1) $\mathrm{vt}(\varphi) \to \varphi$
(VT2) $\mathrm{vt}(\varphi \to \psi) \to (\mathrm{vt}(\varphi) \to \mathrm{vt}(\psi))$
(VT3) $\mathrm{vt}(\varphi \vee \psi) \to \mathrm{vt}(\varphi) \vee \mathrm{vt}(\psi)$.

These axioms are *-tautologies iff vt is interpreted by a *-truth stresser. (VT1) says that if φ is very true then φ (is true); (V2) says (modulo a simple transformation) that if both φ and $\varphi \to \psi$ are very true then ψ is very true. (V3) says that if a disjunction $\varphi \vee \psi$ is very true then one of the disjuncts is very true.

One can show completeness of BL(vt) with respect to a naturally defined class of BL(vt)-algebras. Several interesting examples of truth stressers (for a given t-norm) can be given. For example, one can define vt(φ) to be just φ&φ; or, independently on the t-norm take just $vt(x) = x^2$ (real square – this is the product conjunction but works as a truth stresser also for Ł and G). Proofs are found in Hájek (submitted).

The above is possibly not too surprising but hopefully the reader will agree that saying in fuzzy logic 'φ is very true' we are doing nothing mysterious or deviant. Similarly one could axiomatize other 'fuzzy truth values.'

9 Probability

We stressed that probability on formulas (of classical logic) cannot be understood as an assignment of truth-values in the sense of a (truth-functional) fuzzy logic; but still there are bridges between probability and fuzziness. We describe one of them (see Hájek (1998) originally started by Hájek et al. (1995)). Fuzzy logic speaks in a fuzzy way on some quantities (e.g. 'Temperature is high.') *Probability* is also a quantity and one may say 'The probability of . . . is high' or just '. . . is probable.' The dots stand for any formula of Boolean logic; the word 'probably' acts as a fuzzy modality. Consider a propositional language with two kinds of formulas: *non-modal* – formulas of the classical propositional calculus built from propositional variables and connectives, and *modal* formulas: atomic modal formulas have the form $P\varphi$ where φ is any non-modal formula ($P\varphi$ is read 'φ is probable') and other modal formulas are built from the atomic modal formulas using connectives of Łukasiewicz logic. A *model* of this is a (Kripke) structure $\mathbf{K} = (W, e, \mu)$ where W is a nonempty set of possible worlds, e is a Boolean evaluation assigning to each $w \in W$ and to each propositional variable p the value $e(p, w)$ (zero or one); finally μ is a probability on subsets of W (assume W finite for simplicity). Each non-modal formula has in each possible world either the value 1 or the value 0; the truth value $\|P\varphi\|_{\mathbf{K}}$ of $P\varphi$ in $\mathbf{K}$ is the probability of φ, that is $\mu\{w|\varphi \text{ true in } w\}$). Sentences built from atoms of the form $P\varphi$ are evaluated using truth functions of Łukasiewicz logic. The following formulas are then tautologies:

(FP1) $P(\neg\varphi) \equiv \neg P\varphi$
(FP2) $P(\varphi \to \psi) \to (P\varphi \to P\psi)$,
(FP3) $P(\varphi \vee \psi) \to ((P\varphi \to P(\varphi \wedge \psi)) \to P\psi)$.

EXERCISE. Denote by a, b, c, d the probability of $\varphi\wedge\psi$, $\varphi\wedge\neg\psi$, $\neg\varphi\wedge\psi$, $\neg\varphi\wedge\neg\psi$ respectively; thus for example $a + b$ is the probability of φ. Verify tautologicity of (F1) to (F3). Note that for example (F2) reads: 'If $\varphi \to \psi$ is probable then if also φ is probable then ψ is probable.'

Postulating axioms of classical logic for non-modal formulas, axioms of Łukasiewicz logic plus our (FP1) to (FP3) for modal formulas and taking as deduction rules *modus ponens* and necessitation (from φ infer $P\varphi$) you get a logic complete with respect to the above semantics.

10 Conclusion

Fuzzy logic in the narrow sense *is* a logic, a logic *with a comparative notion of truth*. It is mathematically deep, inspiring and in quick development; papers on it are appearing in respected logical journals. (Besides the monographs already mentioned, Hájek (1998), Cignoli et al. (2000), let us also mention Turunen (1999), Gottwald (submitted), Novak et al. (2000) and (slightly older) Gottwald (1993).) The bridge between fuzzy logic in the broad sense and pure symbolic logic is being built and the results are promising.

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