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Many-Valued, Free, and Intuitionistic Logics

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Standard logic is a package with two parts – a formal deductive apparatus and a conception of interpretation for the language. The deductive apparatus and the semantics are mutually reinforcing and in this chapter we examine primarily the semantic assumptions that formally justify the deductive machinery. The second part of the package, the semantics of 'standard' logic, includes the assumptions that:

- there are two and only two truth-values, True and False,
- every sentence of the language has a determinate truth-value in each interpretation,
- the truth-value of any sentence of the language in an interpretation is determined by the reference or extension of the parts of the sentence in that interpretation (together with the universe of discourse.)

This chapter concerns three historically important forms of non-standard logics:

1. *Many-valued logics* reject the assumption that there are only two truth-values – it explores the possiblities that some sentences may be neither true nor false. Among the reasons for rejecting the assumption are the belief that statements about the future, statements involving vague predicates or statements about quantum mechanical properties are always either true or false. Most many-valued logics begin by rejecting the law of excluded middle, though there are exceptions. The number of values ranges from 3 to various infinite sets. The nature of the further values varies widely from author to author as do the motivations for introducing the additonal values.
2. *Free logics* reject the assumption that truth-values depend only on the referents and extensions of the parts of the sentence. The primary motivation in this case is to give a treatment of names that have no referent.
3. *Intuitionist logic* and other constructivist logics reject the basic assumption, shared by classical logic and the alternatives listed above, that logic should be founded on truth values, and instead proposes to base logic for mathematics on the concept of a mathematical construction. The founder of intuitionistic logic, L. E. J. Brouwer, proposed this logic only for reasoning about mathematics, but various authors

have subsequently argued at length that standard logic should be replaced by intuitionistic logic in other domains.

Two other forms of nonstandard logic are discussed in Part XII: "Relevance and Paraconsistent Logics."

1 Two- and Three-Valued Logics

Frege, one of the originators of modern logic, argued that sentences designate their truth-values and assumed that there are just the two values True and False. Russell, the greatest developer and promoter of modern logic, thought of sentences as denoting propositions in his early work, including the monumental collaboration with Whitehead in *Principia Mathematica*. However, Post (1921) proved that the axiomatization for sentential logic given by Russell and Whitehead is complete with respect to a two-valued interpretation.

POST'S THEOREM Any sentence which cannot be derived from the standard axiomatization of sentential logic is false in a two-valued interpretation. The interpretation can be explicitly constructed given the sentence.

Russell and Whitehead cited Post's result approvingly in the preface to their second edition, and the two-valued interpretation of logic became standard. The introduction of the truth tables as a method of teaching and understanding the sentential connectives in place of the complicated derivations from the axioms of *Principia Mathematica* represented an enormous pedagogical gain, as well as a theoretical advance.

In the same article in which he proved the completeness of the axioms with respect to two-valued interpretations, Post explored generalizations of the truth functions to more values, and he is counted as one of the two founders of many-valued logic. Post's interests were entirely mathematical; he was interested in what happens when you generalize the two-valued interpretations to more values. His systems have been studied extensively, especially in recent decades as they provide a theoretical structure for the analysis of multi-valued switching circuits. However, they have not gathered much attention from philosophers.

The other major founder of many-valued logic is Łukasiewicz. He sketched the idea of a many-valued logic in 1920 and published a systematic account in 1930 (both are reprinted in Borkowski (1970)). Unlike Post, Łukasiewicz introduced three-valued logic for philosophical reasons, to provide a more appropriate representation for the indeterminacy of the future. He apparently was led to this concern both by a historical concern, studying Aristotle's discussion of necessity, particularly his sea battle example, and by a very contemporary concern about how to accommodate the indeterminism of modern physics within logic.

Aristotle's sea battle argument is:

1. If there will be a sea battle tomorrow, then necessarily there will be a sea battle tomorrow.

2. If there will not be a sea battle tomorrow, then necessarily there will not be a sea battle tomorrow.
3. Either there will or there will not be a sea battle tomorrow.
4. Therefore, either there will necessarily be a sea battle tomorrow or there will necessarily not be a sea battle tomorrow.

Aristotle suggested that premise (3), the principle of excluded middle, $\mathbf{A}\vee\sim\mathbf{A}$, should be rejected when A is a statement about a future contingency. Thus the motivation, if not the details, of many-valued logic are as ancient as the study of logic itself. Łukasiewicz developed this idea into a systematic logic.

In his original paper Łukasiewicz used 1 for truth and larger integers for other truth-values, but he later switched to using 1 for truth, 0 for falsity and intermediate values for other truth-values. Most, but not all other writers use this convention. Of course it is one thing to decide that 1/2 is your third truth-value and another to give a philosophical explanation of it. For Łukasiewicz the intermediate value is 'indeterminate.' Given this understanding, the most natural three-valued generalization of the two-valued truth tables are the following, in which negation reverses the value,

A	**~A**
1	0
1/2	1/2
0	1

conjunction takes the minimum value of the conjuncts.

A & B	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

and disjunction the maximum

A ∨ B	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

For example, the conjunction of a true sentence and an indeterminate one would seem to be indeterminate. It could become true if the indeterminacy was resolved in favor of truth, or false if it were resolved in favor of falsity.

Note that when all the components of a sentence formed from these connectives are all assigned value 1/2, then the entire sentence has value 1/2. If we introduce the conditional as $\sim\mathbf{A}\vee\mathbf{B}$, as is often done in two-valued logic, then conditionals would also have this property and there would be no sentences which are logical truths. More concretely, since that identification of the conditional with $\sim\mathbf{A}\vee\mathbf{B}$ makes $\mathbf{A} \rightarrow \mathbf{A}$ equivalent to *excluded middle* $\mathbf{A} \rightarrow \mathbf{A}$ would not be a logical truth.

Instead of using that traditional, if often questioned, equivalence, Łukasiewicz defined the conditional thus:

A → B	1	1/2	0
1	1	1/2	0
1/2	1	1	1/2
0	1	1	1

One way of describing this table is that the conditional is false only in the case of True → False, and is Indeterminate only in the two cases: True → Indeterminate and Indeterminate → False. A rationale for these choices is that if **A** is true and **B** indeterminate, then the conditional **A** → **B** could be true if **B** were to be true, and false if it were false. The choice of the value 1 when both components have value 1/2 is required if **A** → **A** is to be logically true. Further, setting the value of (**A** → **B**), which we will represent as V(**A** → **B**), equal to 1/2 when the components are both assigned 1/2 would result in every sentence having value 1/2 when all its components do, and thus there would be no logical truths.

Equivalence can be defined as usual as **A** ↔ **B iff (A → B) & (B → A)**. In all of the systems we will be considering equivalence is so treated and we will not make explicit mention of equivalence again. (In Łukasiewicz' presentation of his system, he used only negation and the conditional, having noted that **A**∨**B** can be defined as **(A → B) → B**, and then **A&B** can be defined by using the usual DeMorgan's principle.)

In two-valued logic, we define a sentence to be logically true iff it is true in all interpretations. When we have more than two truth-values, then we must indicate which subset of the values are the designated values, those which are truth-like. Our definition now becomes

A IS A LOGICAL TRUTH iff it has a designated value in all interpretations.

Since Łukasiewicz' motivation was to deny excluded middle, he chose only 1 as a designated value. This achieves the purpose of rendering excluded middle not a logical truth. It has one somewhat counterintuitive consequence though, which is that under an interpretation in which both components are assigned value 1/2, A&~A has the same truth value as A∨~A. This will be a consequence in any system of truth tables generalized along the principles above that has an odd number of truth-values, but not of those with an even number. This suggests that many-valued logics with an even number of truth-values might be preferable. Issues of the indeterminacy of the future are now generally studied within the framework of tense logic discussed in chapter 31, "Deontic, Epistemic, and Temporal Logics." Aristotle's argument is generally regarded as fallacious, but Łukasiewicz's innovations have opened the possibilities for a variety of other systems and ideas.

Another reason that has led philosophers and logicians to explore many-valued logics is to attempt to avoid paradoxes such as the Liar. The Liar sentence L is:

L. Sentence L is false.

This produces a paradox: if the sentence is false, what it asserts is correct and it is true; if the sentence is true, then what is asserts is correct and it is false. Introducing a third truth-value 'paradoxical' gives a way out of the paradox. Bochvar was the first to suggest a three-valued logic as treatment for the paradoxes. His system differed from Łukasiewicz' since Bochvar's third value was 'paradoxical,' in contrast to Łukasiewicz' 'indeterminate.' Bochvar had a double set of connectives, but we will only mention the first set here. Since a paradoxical component, according to Bochvar infected an entire sentence, his truth table for conjunction was:

A & B	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	1/2
0	1	1/2	1

In this system, every sentence of the language has value 1/2 when all of its components are assigned 1/2, and thus there are no logical truths. There is, however, a related notion, that of a sentence which is never false. This set coincides with the classical two-valued logical truths.

However, the relief from paradox is at most temporary because the revised Liar L′:

L′: L′ is false or indeterminate.

produces a new but closely related paradox.

The other relatively well-known system of three-valued logic is due to Kleene. His motivation was to deal with statements or equations involving partially defined functions and consequently his third truth-value was 'undefined.' Since a conjunction of a false sentence and an 'undefined' could not turn out to be anything but false, his truth tables for conjunction, disjunction, and negation were the same as Łukasiewicz. However, for the conditional, Kleene regarded a conditional with both antecedent and consequent 'undefined' to have the value undefined. Thus his conditional was characterized as:

A → B	1	1/2	0
1	1	1/2	0
1/2	1	1/2	1/2
0	1	1	1

As in the Bochvar system no sentence receives value 1 on all interpretations. The most significant and plausible application of Kleene's system in philosophy was given by Korner (1966) in relation to the concept of an inexact class. Various linguists have also made use of the Kleene connectives in application to natural languages.

It is also worth mentioning that Reichenbach introduced a three-valued logic as part of an attempt to provide a better logical framework in which to understand quantum mechanics. This was a complex system with three negations and three conditionals. This approach was superseded by quantum logic; it is controversial whether quantum logic is to be considered a many-valued logic. For further discussion, we refer the reader to Part XI, "Inductive, Fuzzy, and Quantum Logics for Probability."

2 Finite Valued Systems with more than Three Values

The Łukasiewicz three-valued generalization can be systematically carried further. The n-valued generalization consists of taking the values $i/n - 1$ for $0 \leq i \leq n - 1$. Conjunction will take the minimum value of the conjuncts, and disjunction the maximum value; the value of a negation is 1 minus the value of the negated sentence. For the conditional $\mathbf{A} \rightarrow \mathbf{B}$ we have two clauses:

$V(\mathbf{A} \rightarrow \mathbf{B}) = 1$ if $V(\mathbf{A})$ is less than or equal to $V(\mathbf{B})$, and

$V(\mathbf{A} \rightarrow \mathbf{B}) = [1 - V(\mathbf{A})] + V(\mathbf{B})$ otherwise.

In all of the Łukasiewicz systems the only designated value is 1. Excluded middle will not be logically true in any of these systems, though in the even valued systems excluded middle is always truer than the contradiction $\mathbf{A}\&{\sim}\mathbf{A}$. Systems with more than 1 designated value were mentioned by Post and this variation on Łukasiewicz systems was studied by Slupecki and others.

Four-valued logic was proposed for modal logic, the values being 'necessarily true,' 'contingently true,' 'contingently false,' and 'necessarily false.' The Łukasiewicz definitions of the usual connectives can be used and a modal operator added. The necessity operator will map 'necessarily true' onto itself and all other values onto 'necessarily false.' While these truth tables have some uses, they have been superseded by the possible worlds approach to modal logic discussed in chapter 29, "Alethic Modal Logics and Semantics."

3 Infinite Valued Systems

The Łukasiewicz n-valued generalization can be systematically carried further – Łukasiewicz also studied the cases where the set of truth-values consists of all rational numbers in the interval [0,1] and where the values consist of all real numbers in the same interval. As before, conjunction will take the minimum value of the conjuncts, and disjunction the maximum value; the value of a negation is 1 minus the value of the negated sentence. For the conditional $\mathbf{A} \rightarrow \mathbf{B}$ we again have the two clauses:

$V(\mathbf{A} \rightarrow \mathbf{B}) = 1$ if $V(\mathbf{A})$ is less than or equal to $V(\mathbf{B})$, and

$V(\mathbf{A} \rightarrow \mathbf{B}) = [1 - V(\mathbf{A})] + V(\mathbf{B})$ otherwise.

Some applications and extensions of these systems will be discussed in later sections. An important breaking point with respect to axiomatizability occurs in this region. All of the finite Łukasiewicz logics are axiomatizable in both their sentential and quantificational forms. In the finite-valued logics the quantifiers are straightforward generalizations of the principles for conjunction and disjunction. A universally quantified expression has as its value the minimum of the values of the Fx. However, in the infinite case, the set of values of Fx may be a set whose minimum, greatest lower bound, is not a member of the set. For this reason, the rationals [0.1] are a satisfactory logic for sentential logic, but the full continuum [0,1] is required for quantificational Łukasiewicz systems. It has been shown that the infinite-valued quantified Łukasiewicz logic is not recursively axiomatizable.

4 Vagueness, Many-valued and Fuzzy Logics

Another philosophical perplexity for which many-valued logics have been prescribed as remedy concerns vagueness. A natural first step in dealing with borderline cases would be to introduce a third truth-value. However, this seems unsatisfactory for it merely replaces the unrealistically sharp boundary between true and false with two unrealistically sharp boundaries, one between true and indefinite, and the other between indefinite and false. More finite values seem only to make the problem worse, and even moving to the infinite case seems to render in appropriate results inasmuch as seems counterintuitive to suppose that a vague statement has a precise real number as its truth-value. However, an important proposal for analyzing vagueness has been based on the continuum valued Łukasiewicz logic.

Zadeh (1975) first introduced the conception of a *fuzzy set* – a set for which membership is not a dichotomous matter but where the membership can take on any of the continuum of values in [0,1] He then replaced the idea of a precise truth mathematical truth-value with fuzzy linguistic truth-values. His truth-values are the countably infinite set: {true, very true, very very true, rather true, not true, false, very false, not very true and not very false, . . . } each of which is a fuzzy subset of the continuum [0,1]. Zadeh's ideas were further developed by Goguen (1968–9) who related them to inexact concepts.

Fuzzy logic and set theory have been enormously successful as tools in engineering and artificial intelligence, and many intelligent control systems from elevators to washing machines have been designed using fuzzy logic. However, as an approach to vagueness it has not been widely accepted in the philosophical community. Part of the resistance may be due to the fact that without the 'fuzzy linguistic values' the approach imputes too much precision to vague contexts, and on the other hand the 'fuzzy linguistic values' seem too unclear and undeveloped to be philosophically respectable. It is also possible that philosophers lack the mathematical sophistication to fully appreciate the approach.

5 Boolean Valued Systems

Another family of interpretations with a different flavor are the interpretations in which the truth-values are the elements of a Boolean algebra. A Boolean algebra is a generalization of principles that are common to elementary set theory and sentential logic. A Boolean algebra consists of a set of elements $\boldsymbol{B}$ with two distinguished elements, $\mathbf{0}$ and $\mathbf{1}$, a one place operation – and two two place operations $\cup$ and $\cap$, which satisfy a set of equations to be enumerated in a moment. We are using the familiar symbols in **bold** for the Boolean notions for heuristic reasons, but it is important to distinguish the Boolean symbol $\cup$ from the set theoretic symbol $\cup$. We will see that the set theoretic operations are one instance of the Boolean operations.

Many alternative sets of axioms are available for Boolean algebras; a simple one that is not too redundant, where **x**,**y** and **z** are any elements of ***B***

B1	$\mathbf{-0 = 1}$	$\mathbf{-1 = 0}$
B2	$\mathbf{x \cap 1 = x}$	$\mathbf{x \cup 0 = x}$
B3	$\mathbf{x \cap - x = 0}$	$\mathbf{x \cup - x = 1}$
B4	$\mathbf{x \cap y = y \cap x}$	$\mathbf{x \cup y = y \cup x}$
B5	$\mathbf{(x \cap y) \cup z = (x \cap z) \cup (y \cap z)}$	$\mathbf{(x \cup y) \cap z = (x \cup z) \cap (y \cup z)}$

One example of a Boolean algebra is to take ***B*** as the pair of truth-values {T,F}, with T as **1**, F as **0**, and negation, conjunction, and disjunction as the operations. Another family of examples of Boolean algebras is obtained by taking any nonempty set S, and letting ***B*** be the power set, the set of all subsets of S, with S as **1**, the empty set as **0**, set complement, union, and intersection as the operations.

What is of interest for our purposes it that if we take the elements of any Boolean algebra as truth-values, and then let our valuation function be defined for negation, disjunction, and conjunction by the Boolean operations, we find that we have a many-valued logic which validates exactly the same set of sentences as the standard two-valued. Post's theorem is sometimes taken as establishing that standard logic is two-valued, but in fact the correct statement is that standard logic is Boolean valued, and the two-valued interpretation is just the simplest Boolean algebra.

The Boolean valued systems are importantly different from the Łukasiewicz and the many-valued approaches discussed above because the values are not linearly ordered. For example, if we take a two element set S = {a,b) we generate a Boolean algebra with the four elements {a,b}, {a}, {b} and { }. If we now consider a disjunction $A \vee B$ and give an interpretation in which V(**A**) = {a} and V(**B**) = {b}, then the disjunction will have the union of these as its value, that is {a,b}. Thus in Boolean valuations a disjunction receives the least value which is greater than or equal to the values of the two disjuncts. Unlike the other many-valued logics above, a disjunction can be truer than either disjunct.

6 Supervaluations are Boolean Valued Logics

Supervaluations are an approach that was first suggested by Mehlberg in connection with vagueness, but were first developed formally by van Fraassen in the context of free logic (to be discussed in the next section). If we consider a vague predicate such as 'bald,' there is a natural intuition that there are some clear positive applications some clear negative applications and some borderline cases. One approach to vagueness is to use one of the Łukasiewicz systems and deny that excluded middle holds if we are considering a borderline case.

The supervaluation approach is to consider the set of all precisifications of the concept bald, that is all of the ways that the concept could be turned into a precise one by adjudicating among the borderline cases while preserving the positive and negative. We then call a statement Supertrue if it is true in all precisifications. Given our remarks above about Boolean algebras, it is evident that supervaluations are essentially a many-valued approach in which the values are members of a Boolean set algebra – the relevant set being that of the precisifications.

One of the advantages of this many-valued approach to vagueness is that we can make distinctions among the borderline cases. If Fred and Paul are both among the borderline cases of bald, but Fred has more hair than Paul, then in a supervaluation approach it will be true in fewer precisifications that 'Fred is bald,' and thus that sentence will receive a lower truth value than 'Paul is bald.' The main philosophical weakness of the approach is that the fundamental assumptions about precisifications and the specification of positive and negative cases have not yet been made sufficiently clear.

7 Free Logic

Arisototelian syllogistic logic assumed that the general terms involved in reasoning were nonempty. That is, in treating sentences of the form 'All Gs are Hs' it was assumed in evaluating the validity of arguments that there is at least one G and at least one H. Thus 'All unicorns are white' would not fall within the scope of syllogistic in spite of its form, since there are no unicorns. Modern logic does not make this assumption and sentences of the form $\forall x(Gx \rightarrow Hx)$ are permitted even when G or H are assigned the empty set.

However, standard logic does make existence assumptions in two forms. First, the domain of quantification must be *nonempty*. The symbolic representation of the assumption is the validity of the sentence $\forall xGx \rightarrow \exists xGx$. Second, it is assumed that all constants in the language denote some object. This is reflected in the validity of sentences of the form $Gc \rightarrow \exists xGx$.

Free logic dispenses with these assumptions. There are two main, and slightly different, motivations for this step. One is a methodological or ontological concern to make the foundations of logic as free from existential assumptions as possible. The second is an interest in applying logic to natural languages where, many believe, there are non-denoting terms such as 'Zeus' and 'Sherlock Holmes.' (It should be noted that there are opposing views on which 'Zeus' denotes a mythological god and 'Sherlock Holmes' a fictional detective.)

As with many-valued logics, there are a variety of proposals for free logic systems and a large and ongoing research program concerning them. In systems which include identity as a logical operation, the fact that a constant c denotes can already be expressed as $\exists x(x = c)$; in systems which do not include identity, a new logical expression, usually either '**E**' or '**E!**' is introduced as a one-place predicate. Exactly how one modifies the axioms and rules of inference of standard logic varies in detail depending on the particular formulation of standard logic, but the basic ideas are fairly straightforward. In place of the standard rule of existential introduction, which permits the inference from Gc to ∃xGx, we have the slightly more complicated rule which requires an additional premise, namely ∃x(x = c). Universal elimination (or instantiation) is similarly modified.

This negative pruning of the derivational system is straightforward and agreed upon, but there agreement ends. The problems arise when we consider how to evaluate the truth of Gc when 'c' is a non-denoting term. *Negative free logic* declares all atomic sentences containing non-denoting terms to be false. *Positive free logics* declare at least some atomic sentences containing non-denoting terms, for example c = c, to be true. Neutral free logics are non-committal. Negative free logic satisfies the methodological concern, but is less satisfying to those who are motivated by natural language considerations because the latter often want a theory in which sentences such as 'Zeus is Zeus,' 'Sherlock Holmes is a fictional detective' and perhaps even 'Sherlock Holmes lived in London' are true.

There is also one version of positive free logic which satisfies the methodological but not the linguistic concerns. On this theory not only is 'c = c' true for all terms, 'c = d' is also true for any pair of non-denoting terms. This makes 'Zeus is Zeus' true, but also makes true the unwanted 'Zeus is Sherlock Holmes'!

Matters become even more complex if we consider a language with a definite description operator. Following Russell we use ιxGx to stand for 'the object which is G.' However, while Russell regarded statements including the description to be paraphrasable into standard logic without descriptions, free logic takes the definite description as basic. And very unlike Russell, positive free logics treat some of the atomic occurrences of non-denoting descriptions as true. One plausible further principle is to extend the validity of self-identity to all descriptions regardless of whether they denote, that is to make ιxGx = ιxGx valid regardless of the interpretation of G.

A tempting further extension would be to declare that each definite description satisfies the condition of the description, that is to say that the winged horse is winged, and so on. However this temptation must be resisted as it leads to an inconsistent system when we take G as ~x = x, because then we obtain both ιx(~x = x,) = ιx(~x = x,) from our previous principle, and ~[ιx(~x = x,) = ιx(~x = x,)] from our new principle.

Given the disagreement over which free logic principles are correct, it is not surprising that there are a variety of semantic proposals. Many of the proposals introduce a second domain to the interpretations. The first domain is the domain over which quantifiers range, but the non-denoting terms are associated with various objects in the second domain. Technically the second domain is impeccable, but the philosophical interpretations of it are varied and controversial.

A slightly different approach to free logic stems from a concern that logical principles should be true regardless of the denotation of terms, that is excluded middle should be valid even in instances like 'Either Zeus was blue-eyed or Zeus was not blue-eyed.' A method of achieving this end while avoiding issues about the truth of atomic sentences is to use supervaluations. A supervaluation in this context is a set of interpretations which assign objects to the constants which lack denotations in the starting interpretation. Since any assignment of an object 'Zeus' will make one or the other of the disjuncts true, the disjunction true though neither disjunct is. Some authors describe supervaluations as 'non-truth functional' in this context, but the view given above seems more accurate.

All of the above discussion, however, is based on free logics which accept the two-valued assumption. That is, they reject the existence assumptions of classical logic but accept the two-valuedness assumption. More radical approaches to free logic (Jacquette 1996) also move to a many-valued set of truth-values. It is possible that the combination of these approaches will prove more philosophically compelling than the separate strands.

Further discussions of the topics of this section are to be found in Part IV: "Truth and Definite Description in Semantic Analysis" and Part VI: "Logic, Existence, and Ontology."

8 Intuitionism

Intuitionistic logic was created by L. E. J. Brouwer, a Dutch mathematician, in response to the set theoretic paradoxes, also discussed in Part VIII: "Logical Foundations of Set Theory and Mathematics," and also due to a general dissatisfaction with the understanding of the logic of mathematics as being a logic of independently existing objects, properties, and relations. In Brouwer's neo-Kantian philosophy, mathematics is a human creation and the fundamental notion is one of a mathematical construct, rather than truth and reference. For the classical logician, the statement that every natural number has a successor is true because there exist infinitely many natural numbers and the successor relation picks out a relation which holds between adjacent numbers. For Brouwer, the statement that every natural number has a successor is known because we know that there is a construction which for every natural number gives a successor natural number.

Brouwer's explanation of the logical connectives is given in terms of constructions. A construction establishes a conjunction if it consists of two parts, one of which establishes each conjunct; a construction establishes a disjunction iff it establishes one of the disjuncts and specifies which. A construction establishes a negation $\sim\mathbf{A}$ iff it is a construction which shows that if there were a construction establishing A, then we could also establish $0 = 1$. A construction establishes a conditional $\mathbf{A} \rightarrow \mathbf{B}$ iff it is a construction which, applied to any construction which establishes A, establishes that B. Note that in these last two clauses we are appealing to the application of constructions to constructions.

For the quantifiers we have, in the domain of natural numbers, a construction establishes $\forall x\mathbf{F}x$ iff it is a construction which for any natural number $\mathbf{n}$ produces a construction establishing $\mathbf{Fn}$. Analogously, in the domain of natural numbers, a construction establishes $\exists x\mathbf{F}x$ iff it is a construction which produces a natural number $\mathbf{n}$ and a construction which establishes $\mathbf{Fn}$.

Given this understanding of the connectives, instances of excluded middle such as $\exists x\mathbf{F}x \vee \sim\exists x\mathbf{F}x$, are not valid. If we take $\mathbf{F}$ to be a complex mathematical formula there is no reason to think that we can either find a specific instance of $\mathbf{F}$, or give a proof that the existence of such an $\mathbf{F}$ would imply a contradiction. Similarly, the classically valid inference from $\sim\forall x\mathbf{F}x$, which can be obtained by showing that the assumption $\forall x\mathbf{F}x$

leads to a contradiction, is insufficient to establish $\exists x \sim \mathbf{F}x$ since the proof does not typically provide a specific counterexample n.

Another classical principle which is not valid is double negation elimination: $\sim\sim\mathbf{A} \rightarrow \mathbf{A}$, although the subcase of it $\sim\sim\sim\mathbf{A} \rightarrow \sim\mathbf{A}$ is intuitionistically valid. Brouwer also opposed the then standard view that logic provided a foundation for mathematics. In Brouwer's view, mathematics required no foundation and logic was merely a reflection of mathematical practice not its basis. He also opposed the formalization of logic.

However, his student, Heyting, in an effort to generate more interest in and sympathy for intuitionism provided a formalization.

H1	$\mathbf{A \rightarrow (A \,\&\, A)}$
H2	$\mathbf{(A \,\&\, B) \rightarrow (B \,\&\, A)}$
H3	$\mathbf{(A \rightarrow B) \rightarrow ((A \,\&\, C) \rightarrow (B \,\&\, C))}$
H4	$\mathbf{((A \rightarrow B) \,\&\, (B \rightarrow C)) \rightarrow (A \rightarrow C)}$
H5	$\mathbf{A \rightarrow (B \rightarrow A)}$
H6	$\mathbf{(A \,\&\, (A \rightarrow B) \rightarrow B)}$
H7	$\mathbf{(A \rightarrow (A \vee B))}$
H8	$\mathbf{(A \vee B) \rightarrow (B \vee A)}$
H9	$\mathbf{((A \rightarrow B) \,\&\, (C \rightarrow B)) \rightarrow ((A \vee C) \rightarrow B)}$
H10	$\mathbf{\sim A \rightarrow (A \rightarrow B)}$
H11	$\mathbf{((A \rightarrow B) \,\&\, (A \rightarrow \sim B)) \rightarrow \sim A}$

Adding either excluded middle or double negation elimination, as H12, gives an axiomatization of the standard two-valued logic. Adding the usual axioms for the quantifier expressions to Heyting's system H1–11 provides an axiomatization of quantified intuitionistic logic.

How do we know that excluded middle does not follow in some subtle way from these axioms, showing that either Heyting's axiomatization is wrong or that intuitionism is incoherent? Heyting provided a three-valued interpretation in which all the Heyting axioms always have value 1 but excluded middle does not. Since modus ponens can be seen to preserve logical truth, excluded middle does not follow. This is an example of the use of many-valued logics in independence proofs we alluded to in discussing three-valued logics.

In Heyting's interpretation, conjunction and disjunctions behave as in the Łukasiewicz systems but negation and conditional are slightly different:

A	**~A**	**A → B**	1	1/2	0
1	0	1	1	1/2	0
1/2	0	1/2	1	1	0
0	1	0	1	1	1

Does this mean that intuitionistic logic and the rich structure of mathematical constructions can be represented by the three-value tables? No, because the Heyting interpretation gives an interpretation on which excluded middle has value 1/2 while all the axioms uniformly have value 1, but there are other schemas which receive value 1 on all interpretations but are not intuitionistic truths. Specifically, it is not intuitionistically valid to assert that for any four sentences **A**, **B**, **C**, and **D**

$$(\mathbf{A} \rightarrow \mathbf{B}) \vee (\mathbf{B} \rightarrow \mathbf{C}) \vee (\mathbf{C} \rightarrow \mathbf{D})$$

But this sentence must always receive value 1 according to the Heyting scheme.

The extension of this to n sentences is also not intuitionistically correct, but as we observed earlier an $n - 1$ valued logic in which conditionals have value 1 when the antecedent has a value less than or equal to that of the consequent, the principle will always have value 1. Thus no finite valued logic can correctly represent intuitionism. Jaskowski proposed an infinitely valued logic which does exactly match.

While Jaskowski's proposal provides an exact characterization of the sentences which are always true in Heyting's logic, it seems to be a technical fact and does not provide any connection with the underlying motivations. Other semantics for intuitionistic logic which are not many-valued but rely instead on tree structures or topological spaces seem somewhat more satisfying. Details can be found in Dummett (1977).

9 Conclusions

The nonstandard logics discussed above were each proposed to deal with a philosophical problem, and the innovator felt that moving beyond the standard framework would provide progress toward an answer. Many of the systems have proved to be enormously productive as applied to practical problems unforeseen by their inventors, and almost all of them have provided fruitful ground for mathematical development. However, none have succeeded in displacing standard two-valued logic based on truth and reference in the philosophical canon. In many cases, as noted above, the many-valued approaches proved to be first approximations to *extensions* or *enrichments* of classical systems rather than *replacements* for them. Łukasiewicz's concern for indeterminism is now addressed within tense logic; intuitionism is now seen, by most logicians, as providing a more refined analysis of concepts and proofs within classical mathematics rather than as challenging it. Of the major approaches discussed, free logic remains the area most likely to be adopted as a new standard approach, although it is possible that fuzzy logic or supervaluationism will become the standard in treatments of vagueness.

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