Part X

INTUITIONISTIC, FREE, AND MANY-VALUED LOGICS

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Intuitionism

DIRK VAN DALEN AND MARK VAN ATTEN

We will view intuitionism as a philosophical program in mathematics. It was founded as such by the Dutch mathematician and philosopher L. E. J. Brouwer (1881–1966) (van Dalen 1999a). The main reference for the technical results discussed here is (Troelstra and van Dalen 1988); the original texts by Brouwer can be found in (Brouwer 1975); additional translations, and texts of other authors mentioned below, are in (van Heijenoort 1967), (Mancosu 1998), and (Gödel 1990–95).

1   Logic: The Proof Interpretation

Various arguments for intuitionistic logic have been propounded, for example by Brouwer (from the nature of mathematics), Heyting (from a concern for ontological neutrality), Dummett (from considerations on meaning) and Martin-Löf (from type theory). These are all different arguments but they lead to the same logic. We focus on Brouwer's motivation.

Brouwer thinks of mathematics first of all as an activity rather than a theory. One constructs objects in one's mind. Mathematical truth, therefore, does not consist in correspondence to an independent reality, but in the fact that a construction has been (or could be) carried out. An intuitionist accounts for the truth of $2 + 2 = 4$ by saying that if one constructs 2, constructs 2 again, and compares the overall result to a construction of 4, one sees they are the same. This construction not only establishes the truth of the proposition $2 + 2 = 4$, but is all there is to its truth.

In Brouwer's view, logic depends on mathematics and not vice versa. Logic notes and studies regularities that one observes in the mathematical construction process. For example, the logical notion of negation derives from seeing that some mathematical constructions do not go through. One constructs $2 + 3$ and sees that the outcome does not match with a construction of 4; hence $\neg(2 + 3 = 4)$.

This suggests that the construction criterion for mathematical truth also yields an interpretation of the logical connectives. We will now elaborate on this. Let us write '$a : A$' for 'a is a construction that establishes A' and call this a a *proof* of A.

A proof of $\neg A$ has to tell us that A cannot have a proof; hence we read $p : \neg A$ as 'Each proof a of A can be converted by the construction p into a proof of an absurdity (say, $0 = 1$; abbreviated $\perp$)'.

To extend this *proof interpretation* to the other connectives, it is convenient to have the following notation. (a, b) denotes the pairing of constructions, and $(c)_0$, $(c)_1$ are the first and second projections of c.

A proof of a conjunction $A \wedge B$ is a pair (a, b) of proofs such that $a : A$ and $b : B$.

Interpreting the connectives in terms of proofs means that, unlike classical logic, the disjunction has to be *effective*, one must specify for which of the disjuncts one has a proof. A proof of a disjunction $A \vee B$ is a pair (p, q) such that p carries the information which disjunct is shown correct by this proof, and q is the proof of that disjunct. We stipulate that $p \in \{0, 1\}$. So if we have $(p, q) : A \vee B$ then either $p = 0$ and $q : A$, or $p = 1$ and $q : B$.

The most interesting propositional connective is the implication. Classically, $A \rightarrow B$ is true if A is false or B is true, but this cannot be used now as it involves the classical disjunction. Moreover, it assumes that the truth values of A and B are known before one can settle the status of $A \rightarrow B$.

Heyting showed that this is asking too much. Consider A = 'there occur twenty 7's in the decimal expansion of π,' and B = 'there occur nineteen 7's in the decimal expansion of π.' $\neg A \vee B$ does not hold constructively, but in the proof interpretation, $A \rightarrow B$ is obviously correct.

It is because, if we could show the correctness of A, then a simple construction would allow us to show the correctness of B as well. Implication, then, is interpreted in terms of possible proofs: $p : A \rightarrow B$ if p transforms each possible proof $q : A$ into a proof $p(q) : B$.

The meaning of the quantifiers is specified along the same lines. Let us assume that we are dealing with a domain D of mathematical objects. A proof p of $\forall x A(x)$ is a construction which yields for every object $d \in D$ a proof $p(d) : A(d)$. A proof p of $\exists x A(x)$ is a pair (p_0, p_1) such that $p_1 : A(p_0)$. Again, note the demand of effectiviness: the proof of an existential statement requires an instance plus a proof of this instance.

The interpretation of the connectives in terms of proofs was made explicit by Heyting (1934). Around the same time, Kolmogorov gave an interpretation in terms of *problems* and *solutions*. The two are essentially the same. Note that in its dependence on the abstract concept of proof, Heyting's interpretation goes well beyond finitism (see 'The Dialectica interpretation,' below).

Here are some examples of the proof interpretation.

1. $(A \vee B) \rightarrow (B \vee A)$ Let $p : A \vee B$, then $(p)_0 = 0$ and $(p)_1 : A$, or $(p)_0 = 1$ and $(p)_1 : B$. By interchanging A and B we get, looking for $q : B \vee A$, $(q)_0 = 1$ and $(q)_1 : B$, or $(q)_0 = 0$ and $(q)_1 : A$. This comes to $\overline{\mathrm{sg}}((p)_0) = (q)_0$ and $(p)_1 : B$, or $\overline{\mathrm{sg}}((p)_0) = (q)_0$ and $(q)_1 : A$, that is, $(\overline{\mathrm{sg}}((p)_0), (p)_1) : B \vee A$. And so $\lambda p.(\overline{\mathrm{sg}}((p)_0), (p)_1) : A \vee B \rightarrow B \vee A$.

2. $A \vee \neg A$ $p : A \vee \neg A \Leftrightarrow (p)_0 = 0$ and $(p)_1 : A$ or $(p)_0 = 1$ and $(p)_1 : \neg A$.
   However, for an arbitrary proposition A we do not know whether A or $\neg A$ has a proof, and hence $(p)_0$ cannot be computed. So, in general there is no proof of $A \vee \neg A$.

3. $\neg \exists x A(x) \rightarrow \forall x \neg A(x)$
   $p : \neg \exists x A(x) \Leftrightarrow p(a) : \bot$ for a proof $a : \exists x A(x)$
   We have to find a q such that $q : \forall x \neg A(x)$, i.e. $q(d) : A(d) \rightarrow \bot$ for any $d \in D$. So pick an element d and let $r : A(d)$, then $(d, r) : \exists x A(x)$ and so $p((d, r)) : \bot$. Therefore we put $q(d)(r) = p((d, r))$, so $q = \lambda r.\lambda d.p((d, r))$ and hence
   $\lambda p.\lambda r.\lambda d.p((d, r)) : \neg \exists x A(x) \rightarrow \forall x \neg A(x)$.

Brouwer employed a characteristic technique now known as 'Brouwerian (weak) coun-
terexamples' to show that certain classical statements are constructively untenable by
reducing them to unproven statements. To illustrate, here is a Brouwerian counterex-
ample to the classical trichotomy law $\forall x \in \mathbb{R}(x < 0 \vee x = 0 \vee x > 0)$.

We compute simultaneously the decimal expansion of π and a Cauchy sequence to
be specified. We use $N(k)$ as an abbreviation for 'the decimals $p_{k-89}, \ldots, p_k$ of π are all 9.'
Now we define

$$a_n = \begin{cases} (-2)^{-n} & \text{if } \forall k \leq n \neg N(k) \\ (-2)^{-k} & \text{if } k \leq n \text{ and } N(k) \end{cases}$$

a_n starts as an oscillating sequence of negative powers of -2. Should we hit
upon a sequence of 90 nines in the expansion of π, a_n becomes constant from
there on:

$$1, -1/2, 1/4, -1/8, \ldots, (-2)^{-k}, (-2)^{-k}, (-2)^{-k}, \ldots$$

The sequence a_n satisfies the Cauchy condition and in that sense determines a
real number a. The sequence is well defined, and, in principle, for each n we can check
$N(n)$.

But of this a we cannot say whether it is positive, negative, or zero:

$a > 0 \Leftrightarrow N(k)$ holds the first time for an even number
$a < 0 \Leftrightarrow N(k)$ holds the first time for an odd number
$a = 0 \Leftrightarrow N(k)$ holds for no k.

Since we as yet have no construction that determines whether $N(k)$s occur, we cannot
affirm $a < 0 \vee a = 0 \vee a > 0$ and hence the trichotomy law cannot be said to have a
proof.

Moreover, the number a cannot be irrational, for then $N(k)$ would never apply, and
hence $a = 0$, contradiction. This shows that $\neg\neg(a$ is rational. On the other hand, there
is no proof that a is rational, so $\neg\neg A \rightarrow A$ fails. Similarly, $a = 0 \vee a \neq 0$ has no proof.

This type of counterexample is called *weak* because they show that some proposition
has no proof yet, but it does not at all exclude that such a proof will be found later. (A
sequence that Brouwer employed in his own writings is 01234567890 in the expan-
sion of π; but its occurence has now been proved.)

Strong counterexamples cannot always be expected. There are, for example,
instances of the Principle of the Excluded Middle (PEM) that have no proof (in any case,
not yet), but the negation of PEM cannot be proved! $\neg(A \vee \neg A)$ is equivalent to $\neg A \wedge$
$\neg\neg A$, which is a contradiction. However, strong counterexamples to some other clas-
sical principles do exist, and some will be shown in next section.

Although Brouwer had little interest in developing logic for its own sake, some of the
finer distinctions that are common today were introduced by him. In his 1907 thesis
one can already find the explicit and fully understood notions of language, logic, meta-
language, metalogic, etc. Also, Brouwer was the first to prove a non-trivial result in
intuitionistic logic, $\neg A \leftrightarrow \neg\neg\neg A$ (1923). He discussed logic in an informal manner;

Kolmogorov (1925) and Glivenko (1929) then presented formalizations of parts of intuitionistic logic. A full system was given by Heyting (1930). As such it has become a part of mathematical logic in its own right, independent of philosophical motivations. Also, semantics other than the proof interpretation were developed that allow for sharper technical results (see 'Further semantics', below).

Gödel (1933) defined a translation ° given by

$A° = \neg\neg A$ for atomic A
$(A \land B)° = A° \land B°$
$(A \lor B)° = A° \lor B°$
$(A \to B)° = A° \to B°$
$(\forall x A(x))° = \forall x A° (x)$
$(\exists x A(x))° = \neg \forall x \neg A° (x)$

and proved that in predicate logic we have

$$\Gamma \vdash_c A \Leftrightarrow \Gamma° \vdash_i A°$$

where $\Gamma° = \{B \,|\, B \in \Gamma\}$, and $\vdash_c$ and $\vdash_i$ denote classical and intuitionistic derivation relations, respectively.

Classically, a sentence A and its translation translation $A°$ are equivalent, $\vdash_c A \leftrightarrow A°$; from an intuitionistic point of view, however, disjunctions and existential statements will be weakened by the translation. Still, Gödel's result shows that, formally, classical predicate logic can be embedded into intuitionistic predicate logic.

Taking A = ⊥ and noting that ⊥° = ⊥, it follows that classical predicate logic is consistent if and only if intuitionistic predicate logic is; so the philosophical advantages of intuitionistic over classical predicate logic must lie in its interpretation and not in its trustworthiness.

In fact, Gödel proved something stronger. Classical arithmetic (**PA**, i.e. Peano's axioms with classical logic as the underlying logic) can be embedded into intuitionistic arithmetic (**HA**, i.e. Peano's axioms with Heyting's formalized intuitionistic logic as the underlying logic):

$$\textbf{PA} \vdash_c A \Leftrightarrow \textbf{HA} \vdash_i A°$$

In particular,

$$\textbf{PA} \vdash_c 0 = 1 \Leftrightarrow \textbf{HA} \vdash_i \neg\neg 0 = 1$$
$$\Leftrightarrow \textbf{HA} \vdash_i 0 = 1$$

So **PA** is consistent if and only if **HA** is.

However, it is not always possible to embed classical systems into their intuitionistic counterparts. In particular, it turns out that intuitionistic analysis (second-order arithmetic with function variables) contradicts classical analysis. This will be elaborated on in the next section.

2 Analysis: Choice Sequences

A *choice sequence* is a potentially infinite sequence of mathematical objects $\alpha = \alpha(0)$, $\alpha(1)$, $\alpha(2)$, . . . chosen, one after the other, from a fixed collection of mathematical objects by the individual mathematician (from 1948 on, Brouwer explicitly speaks of *the creating subject* although he must have had the notion already in 1927). Here we will limit our discussion to choice sequences of natural numbers and rationals. A choice sequence is an incomplete object, for it is never finished.

Choice sequences come in many varieties, depending on how much freedom one allows oneself in making the successive choices. The two extreme cases are the lawless sequences, where there is no restriction whatsoever on future choices, and the lawlike sequences, where one simply takes the numbers generated by a law or algorithm. One may (but need not) identify 'lawlike' with 'recursive'. (A lawlike sequence need not be thought of as an incomplete object, provided one is willing to make the additional abtraction from the temporal unfolding of the sequence.)

There are various reasons why this variety is relevant. First, a type need not be closed under a given operation. Consider the sum of two lawless sequences $\gamma = \alpha + \beta$, i.e.

$$\gamma = \alpha(0) + \beta(0), \alpha(1) + \beta(1), \alpha(2) + \beta(2), \ldots$$

This γ is itself neither lawless (because it depends on α and β), nor lawlike (because α and β are lawless). Second, lawlike sequences are needed to instantiate specific existence claims. Third, lawless sequences are important for metamathematical purposes.

Brouwer probably came to accept choice sequences as objects of intuitionistic mathematics in 1914, but theory development began in 1916–17. He showed how, using choice sequences, one can formulate a theory of the continuum that does not let it dissolve into separate points. Thus, Brouwer was the first to show how to incorporate into mathematics a point already made by Aristotle and others: a set of discrete elements cannot represent the geometrical or intuitive continuum. Discreteness and continuity are inseparable, complementary notions, that cannot be reduced to one another. Neither Cantorian set theory nor earlier constructivist analyses of the continuum (e.g. Poincaré, Borel, Brouwer in his dissertation of 1907, Weyl in 1918) had been able to accommodate this insight.

How does this work? Brouwer *identifies* a 'point' with a choice sequence of numbers that represent, through some coding, rational intervals on the continuum; these intervals should satisfy the Cauchy condition. A point, then, is 'becoming' and often to some extent undetermined. Brouwer then notices that, in general, extensional identity of choice sequences is undecidable. This models the non-discrete nature of the continuum.

The undecidability of extensional identity follows from the incompleteness of choice sequences: at any particular time, all there is of a choice sequence is a finite initial segment with an open end. Even if the initial segments of two sequences are the same, still nothing can be said about whether they will always have the same values. (In the case of two lawlike sequences, one may be able to show extensional identity by proving equivalence of the laws governing them.)

Choice sequences are generated freely, and at any time we have no more than a finite initial segment of them, perhaps together with some self-imposed restrictions. But then a sequence cannot, at any stage, have (or lack) a certain property if that could not be demonstrated from the information available at that stage. It follows that bivalence, and hence PEM, does not hold generally for statements about choice sequences. For example, consider a lawless sequence α of which we have so far generated the intial segment 8, 1, 3, and the statement P = 'The number 2 occurs in α.' Then we cannot say that $P \vee \neg P$ holds. Note how this argument against the validity of PEM depends on both the freedom of generation and the potential infinity of the sequences. We see that acceptance of choice sequences as mathematical objects forces a revision of logic along the lines of the proof interpretation given above. According to Placek (1999), this is the strongest argument in favor of intuitionistic logic currently available. (The philosophical thesis that logic may vary according to the ontological region one is speaking about, has been elaborated by Tragesser (1977), taking his cue from Husserl; in category theory, the phenomenon is familiar from *topoi*.)

Just as in classical mathematics elements are collected into a set, choice sequences are held together in a *spread* ('Menge,' in Brouwer's original, somewhat confusing terminology). A *spread law*, which should be decidable, either admits an initial segment or inhibits it; a further condition on the spread law is that of each admitted segment, at least one immediate extension should be admitted as well. The admitted segments form a growing tree, hence they are also known as nodes. Because of the second conditon, there will be no finite maximal paths in the tree. Choice sequences correspond to the infinite paths, and are called the *elements* of the spread.

A special case is the universal spread, which admits all choice sequences. The spread of all choice sequences satisfying the Cauchy condition is one way to represent the continuum.

For a few particular classes of choice sequences, there are translation theorems. For simplicity, we look at the case of lawless sequences, but the arguments that follow are general. Troelstra, developing earlier work by Kreisel, presented a formal system LS describing lawless sequences, together with a mapping τ into a subsystem without variables for lawless sequences IDB_1, such that

1. $\tau(A) \equiv A$ for A a formula of IDB_1
2. $LS \vdash A \leftrightarrow \tau(A)$
3. $(LS \vdash A) \Leftrightarrow (IDB_1 \vdash \tau(A))$

Such translation theorems show the coherency of the translated notion as a mathematical notion, and are important for metamathematical purposes. However, it cannot be concluded right away that translations explain lawless sequences away. These translations take the form of equivalences. An interest in ontological reduction would demand that we regard them as contextual definitions of quantification over lawless sequences. However, such a demand would have to be supported by arguments against such sequences that are independent of the axiomatization, for as the translation is symmetric, it could just as well be taken to mean that in some cases, quantification over lawlike sequences is best explained as quantification over lawless sequences.

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More generally, such translations depend on specific axiomatizations of choice sequences. (In fact, lawless sequences have been axiomatized in different ways (Kreisel, Myhill, Troelstra), that are not always equivalent.) But an axiomatization is a way to present mathematical content; it is not identical with it. Hence the need for independent arguments. Brouwer certainly thought of choice sequences of any type as genuine objects of mathematics, constructed by the creating subject. (A phenomenological justification of this conviction can be found in van Atten 1999.)

The incompleteness of choice sequences guarantees properties that are desirable to model the continuum, but may at the same time seem to make them unworkable in practice. For if mathematics is to be based on constructions, what place is there for objects that at no stage have been completely constructed? Fortunately, there is a continuity principle. Essentially, this says that all one has to know to make a predication of some choice sequence is an initial segment. Unlike the sequence itself, its initial segments are given in a finite construction.

$$(\textbf{WC-N}) \ \forall\alpha\exists x \, A(\alpha, x) \Rightarrow \forall\alpha\exists m\exists x\forall\beta[\overline{\beta}m = \overline{\alpha}m \rightarrow A(\beta, x)]$$

where α and β range over choice sequences of natural numbers, m and x over natural numbers, and $\overline{\alpha}m$ stands for $\langle\alpha(0), \alpha(1), \ldots, \alpha(m-1)\rangle$, the initial segment of α of length m. 'WC-N' stands for 'Weak Continuity for Numbers': weak, as it only says something about each α individually (local continuity).

From WC-N, two theorems follow that show that intuitionistic analysis is not just an amputation of classical mathematics, but contains new results that are classically not acceptable. (It is true that there is no contradiction between the classical and intuitionistic systems of analysis as such, as key terms ('point,' 'function') are defined differently; but contradiction arises when one realizes that both systems try to capture the *same*, pre-formal notions of 'continuum' and so on.)

Veldman (1982) has shown that from WC-N one can derive

THE CONTINUITY THEOREM A real function whose domain of definition is the closed segment [0, 1] is continuous on [0, 1]:

$\forall\varepsilon\forall x_1\forall\delta\forall x_2(|x_1 - x_2| < \delta \rightarrow |f(x_1) \rightarrow f(x_2)| < \varepsilon)$
for positive δ, ε and $x_1, x_2 \in [0,1]$.

THE UNSPLITTABILITY OF THE CONTINUUM The continuum cannot be split into two nontrivial subsets: if $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$, then $A = \mathbb{R}$ or $B = \mathbb{R}$.

Weyl announced the continuity theorem in 1921, but this is not really the same strong result as Brouwer's. Weyl defined real functions in such a way that they are continuous by definition, that is via mappings of the intervals of the choice sequence determining the argument to intervals of the image sequence. This way, the function type is reduced from $\mathbb{R} \rightarrow \mathbb{R}$ to $\mathbb{N} \rightarrow \mathbb{N}$ (initial segments to initial segments). Brouwer, on the other hand, established the continuity of functions from choice sequences to choice sequences, by showing how this followed from intuitionistic principles and the functional character (the $\forall\exists!$-combination).

Brouwer did not explicitly state the continuity theorem, instead he proved the stronger

UNIFORM CONTINUITY THEOREM   A real function whose domain of definition is the closed segment [0, 1] is uniformly continuous on [0, 1]

$$\forall\varepsilon\exists\delta\forall x_1\forall x_2(\,|\,x_1 - x_2\,| < \delta \to |\,f(x_1) \to f(x_2)\,| < \varepsilon)$$

for positive δ, ε and x_1, $x_2 \in [0, 1]$.

Brouwer used the bar theorem (see below) to prove the uniform continuity theorem and seems to have believed that the continuity theorem can only be obtained as a corollary from it. Likewise, Brouwer in his proof of the unsplittability of the continuum appealed to the fan theorem (see below), where the simpler WC-N suffices, for unsplittability is a direct consequence of the continuity theorem: suppose $\mathbb{R} = A \cup B$ and $A \cap B = \emptyset$, then f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in A \\ 0 & \text{if } x \in B \end{cases}$$

is total and therefore, by the continuity theorem, continuous. But then f must be constant, so either $\mathbb{R} = A$ or $\mathbb{R} = B$. An instance of unsplittability is that it is not true that every real number is either rational or irrational. For if it were, we could obtain a non-trivial splitting of the continuum by assigning 0 to rational, and 1 to irrational real numbers.

Also note that WC-N by itself already suffices to refute PEM: consider $\forall\alpha[\forall x(\alpha x = 0) \vee \neg\,\forall x(\alpha x = 0)$, 'Every choice sequence is either the constant zero sequence, or not.' This is equivalent to $\forall\alpha\exists z[(z = 0 \to \forall x(\alpha x = 0)) \wedge (z \neq 0 \to \neg\forall x(\alpha x = 0))]$. Applying WC-N to this gives:

$$\forall\alpha\exists z\exists m\forall\beta(\overline{\beta}m = \overline{\alpha}m \to [(z = 0 \to \forall x(\beta x = 0)) \wedge (z \neq 0 \to \neg\forall x(\beta x = 0))])$$

Now take $\alpha = \lambda u \cdot 0$ and determine the z and m that WC-N correlates to this α. Then the above says that each β with an initial segment of m zeros will consist of zeros throughout, which is of course not the case.

Also refuted by WC-N is Church's Thesis in the form

CT $\forall a\exists x\forall y\exists z[T(x, y, z) \wedge U(z) = a(y)]$

that says, for every sequence a there exists a Turing machine with index x that calculates, for a natural number y, the $y + 1$th member of the sequence (z represents the computation process, and $U(z)$ its result). CT fails if a ranges over the whole universe and WC-N is true. For in that case, the index x would always have to be determined from just an initial segment of a, which is impossible.

An application of WC-N to a predicate $A(\alpha, x)$ determines a set of initial segments (nodes) that suffice to calculate an x such that $A(\alpha, x)$ holds. Such a set is called a *bar*.

To see if one can arrive at stronger results in analysis, one would have to know whether bars have structural properties. Brouwer managed to find such a property. For convenience, we consider a *thin* bar B, that is one with the property that if $\vec{n} \in B$ and $\vec{m} < \vec{n}$ (in the ordering of the tree), then $\vec{m} \notin B$ (i.e. a thin bar contains no initial segments that are longer than strictly necessary). Brouwer's *bar theorem* shows that the collection of thin bars, call it *ID*, is inductively defined (they are well-ordered). The clauses are:

1. Every singleton tree is in *ID*;
2. If $T_1, T_2, T_3, \ldots$ are in *ID*, then so is the tree obtained by adding a top to the direct sum of $T_1, T_2, T_3, \ldots$

This is a powerful insight, for it allows one to use induction in reasoning about thin bars. One sees from the order of the quantifiers why uniform continuity is stronger than ordinary continuity: for a given ε, uniform continuity demands that the same δ work for the whole interval, whereas for ordinary continuity, δ may vary with each x_1. This observation makes it plausible that uniform continuity should require knowledge of the structure of the bar whereas ordinary continuity does not.

Brouwer's proof of the bar theorem strongly depends on the intuitionistic notion that truth of a proposition consists in having a construction for it, and on reflection on the available means to construct proofs concerning bars.

A bar may well be an infinite tree (not in depth, but in width). A *fan* is a finitely branching tree. A corollary of the bar theorem is the *fan theorem*: if B is a thin bar for a fan, then there is an upper bound to the length of the nodes in B. Briefly put, a thin bar for a fan is finite. (The contrapositive of the fan theorem is better known, but was proven later (1927): König's infinity lemma, which says that a fan with infinitely many nodes contains an infinite path. It is not constructively valid, for there is no effective method to pick out a path that is infinite.)

The unit continuum [0, 1] can be represented by a fan, for example by demanding that for every n, the nth interval is of the form

$$\left[\frac{a}{2^{n+1}}, \frac{a+2}{2^{n+1}} \right]$$

where $2 \leq a + 2 \leq 2^{n+1}$, as then the number of alternatives at choice n is finite. Thus it is that Brouwer could prove theorems about the continuum (such as the uniform continuity theorem) from a theorem on the constructively more tractable finitary trees. This shows the power of the fan theorem.

The weak counterexamples, of which we saw an example in the section on logic, require no more than lawlike sequences and intuitionistic logic. By exploiting the presence of sequences that are not lawlike but involve genuine choice, Brouwer in 1949 found a systematic and explicit way to construct *strong* counterexamples, which show that, if one accepts non-lawlike sequences, certain classical principles are not only without proof so far but could never be proven at all, as they are contradictory. These strong counterexamples are based on the theory of the creating subject; we adopt Kreisel's terminology here.

Let $\square_n A$ stand for 'the creating subject experiences A (has full evidence for A) at time n'. The following principles (Kripke, Kreisel) are evident:

1. $\forall n \forall m\, (\square_n A \rightarrow \square_{n+m} A)$
   that is evidence never gets lost;
2. $\forall n (\square_n A \vee \neg \square_n A)$
   that is at every moment the creating subject can decide whether it has full evidence for A or not;
3. $A \leftrightarrow \exists n \square_n A$
   A holds exactly if the creating subject has full evidence for it at some moment. (Kreisel dubbed this the 'Principle of Christian Charity,' or, alternatively, the 'Principle of Infinite Vanity': if something is true, the creating subject will sooner or later experience this.)

These principles more or less define the intuitionistic conception of truth.

On the basis of 1−3, one can associate with each proposition A a choice sequence α that 'witnesses' A:

$$a(n) = \begin{cases} 0 & \text{if } \neg \square_n A, \\ 1 & \text{else} \end{cases}$$

The statement that such an α exists is known as 'Kripke's Schema':

(KS) $\exists \alpha (A \leftrightarrow \exists x \cdot \alpha(x) = 1)$

Brouwer used the principles 1−3, and implicitly Kripke's Schema, to establish strong counterexamples.

For example, in 1949 he showed

$$\neg \forall x \in \mathbb{R}(\neg\neg x > 0 \rightarrow x > 0)$$

and, by an argument of the same type,

$$\neg \forall x \in \mathbb{R}(x \neq 0 \rightarrow x \,\#\, 0)$$

(# denotes *apartness* of two real numbers: $a \,\#\, b \equiv \exists n(|a - b| > 2^{-n})$. In the proof interpretation, this is stronger than $\neg(a = b)$.)

In the proofs of these counterexamples choice sequences are employed that depend on the creating subject's having experienced either the truth or the absurdity of a particular mathematical assertion; these sequences are not lawlike. These methods are very powerful, for example one can prove that already the irrationals are unsplittable (van Dalen 1999b).

Besides analysis and counterexamples, other uses of choice sequences have been found. They are used in certain completeness proofs for intuitionistic predicate logic, and, together with KS, allow the definition of the (intuitionistic) powerset of $\mathbb{N}$ as a spread.

3   Further Semantics

As remarked, it is not easy to get model-theoretic results out of the proof interpretation, as the notion of 'construction' as employed there is still informal and not very specific. Therefore, various alternative semantics for intuitionistic logic have been developed (topological models, realizability, Kripke models, Beth trees, Martin-Löf's type theory, the Dialectica interpretation, sheaf semantics, topos models). The investment into various codifications of formal proof-notions should be rewarded by perspicuous effectiveness: the first prize being the 'existence property' or 'effective definability property': if $\exists x P(x)$ is proved constructively, the interpretation should supply us with an effective procedure to compute (or define) an object a and a proof of $P(a)$.

We will present four: realizability, Kripke semantics, the Dialectica interpretation, and Martin-Löf's type theory.

Realizability

Starting considerations from the finitary standpoint of Hilbert-Bernays, Kleene suggested that provability in HA of a statement of the form $\forall x \exists y \phi(x, y)$ should be taken to mean that there exists a recursive (choice) function f such that $\forall x \phi(x, f(x))$. Thus, the original statement is only an 'incomplete communication' (a notion introduced by Weyl), a full statement gives the choice function as well. Similarly, $\exists x \phi(x)$ is an incomplete communication of a full statement that specifies an object a such that $\phi(a)$. The idea behind Kleene's *recursive realizability* (or 1945-realizability) is to code all the information necessary to prove a particular statement ϕ into a natural number n. The notation is $n \mathbf{r} \phi$, 'n *realizes* ϕ'.

The defining clauses of $\mathbf{r}$ mirror those of the proof interpretation. We use some notation from recursion theory: $\{x\}y$ for application, and $\downarrow$ for convergence.

$$
\begin{aligned}
x \mathbf{r}\, \phi &:= \quad \text{for atomic } \phi \\
x \mathbf{r}\, (\phi \wedge \psi) &:= \quad (x)_0 \mathbf{r}\, \phi \wedge (x)_1 \mathbf{r}\, \psi \\
x \mathbf{r}\, (\phi \vee \psi) &:= \quad ((x)_0 = 0 \rightarrow (x)_1 \mathbf{r}\, \phi) \wedge ((x) \neq 0 \rightarrow (x)_1 \mathbf{r}\, \psi) \\
x \mathbf{r}\, (\phi \rightarrow \psi) &:= \quad \forall y (y \mathbf{r}\, \phi \rightarrow \{x\}y \downarrow \wedge \{x\}y \mathbf{r}\, \psi) \\
x \mathbf{r}\, \exists y \phi(y) &:= \quad (x)_1 \mathbf{r}\, \phi((x)_0) \\
x \mathbf{r}\, \forall y \phi(u) &:= \quad \forall y (\{x\}y \downarrow \wedge \{x\}y \mathbf{r}\, \phi(y))
\end{aligned}
$$

According to the first clause, any number realizes an atomic sentence; no number, however, realizes a false atomic sentence. The second clause is obvious. The third clause shows the effective nature of the disjunction: as we can effectively test whether $(x)_0 = 0$ or $(x)_0 \neq 0$, the 'realizer' of a disjunction gives us all the information needed to indicate the desired disjunct. Similarly, the fifth clause says that a realizer of $\exists y \phi(y)$ codes the required instance and the information that realizes it. The fourth and sixth clauses are like the proof interpretation: the realizer of an implication transforms any realizer of ϕ into a realizer of ψ; the realizer of a universal statement is a partial recursive function that yields a realizer for any instance.

Note that $n \mathbf{r} \phi$ is itself a formula of HA, so realizability can be viewed as an interpretation of HA in itself. Therefore, it makes sense to ask for the truth of an instance of $n \mathbf{r} \phi$, or whether it is derivable in HA.

Since the introduction of realizability by Kleene, many variations on the original notion have been developed. In particular we mention 'truth realizability' **rt**, which is defined like **r** but with an extra condition in the clause for implication:

$$x \; \mathbf{rt} \; (\phi \to \psi) \quad := \quad \forall y(y \; \mathbf{rt} \; \phi \to \{x\}y \downarrow \land \{x\}(y) \; \mathbf{rt} \; \psi) \land (\phi \to \psi)$$

Truth realizability is particularly useful in showing how realizability renders the relation between existential statements and instantiations explicit. One can prove that

HA* ⊢ $t \; \mathbf{rt} \; \psi \to$ and
HA* ⊢ $\psi \Rightarrow$ HA* ⊢ $t \; \mathbf{rt} \; \psi$ for a suitable term t,

where HA* is a suitable extension of HA in which partial terms are allowed, and which allow for a formalization of the basis of recursion theory. This fact is used to obtain an effective version of the existence property

HA* ⊢ $\exists x P(x) \Rightarrow$ HA* ⊢ $P(\bar{n})$ for suitable $\bar{n}$

Moreover, HA is closed under Church's rule:

HA ⊢ $\forall x \exists y P(x, y) \Rightarrow$ HA ⊢ $P(x, \{e\}y)$ for a suitable e.

Since the index of the recursive (choice) function can be effectively determined, realizability provides the (admittedly not very practical) machinery needed to extract programs from proofs.

Kripke's semantics

In Kripke's semantics, the activity of the creating subject is modeled; it strongly resembles the theory of the creating subject mentioned above. At each point in time, the subject has constructed a collection of objects and has experienced a number of truths. The subject is free to take its activity of construction to a next stage; at each moment there is a number of possible next stages (or *possible worlds*). Thus, the stages for the individual form a partially ordered set (even a tree) $\langle K, \leq \rangle$; $k \leq \ell$ is taken to mean 'k is before, or coincides with, ℓ.' We write '$k \models A$' for 'A holds at stage k;' the standard terminology is 'k forces A.' With every $k \in K$ we associate its local domain of objects created so far, denoted by $D(k)$. A reasonable assumption is that objects, once created, are not destroyed later: $k \leq \ell \Rightarrow D(k) \subseteq D(\ell)$.

The interpretation of the logical connectives now consists in spelling out the clauses of the proof interpretation in this possible-world model of the subject's activity. Then the inductive definition of the forcing relation is obvious:

For atomic A, $k \models A$ is given; $\perp$ is never forced.

$$
\begin{array}{lll}
k \Vdash A \land B & \Leftrightarrow & k \Vdash A \text{ and } K \Vdash B \\
k \Vdash A \lor B & \Leftrightarrow & k \Vdash A \text{ or } k \Vdash B \\
k \Vdash A \to B & \Leftrightarrow & \forall \ell \geq k(\ell \Vdash A \Rightarrow \ell \Vdash B)
\end{array}
$$

$$k \Vdash A \quad \Leftrightarrow \quad k \Vdash A \to \bot$$
$$\Leftrightarrow \quad \forall \ell \geq k(\ell \Vdash A \Rightarrow \ell \Vdash \bot)$$
$$\Leftrightarrow \quad \forall \ell \geq k(\ell \nVdash A)$$
$$k \Vdash \exists x A(x) \quad \Leftrightarrow \quad \exists \alpha \in D(k)k \Vdash A(a)$$
$$k \Vdash \forall x A(x) \quad \Leftrightarrow \quad \forall \ell \geq k \forall a \in D(\ell) \Vdash A(a)$$

Note that the cases of $\wedge$, $\vee$ and $\exists$ are determined on the spot, whereas $\to$, $\neg$ and $\forall$ essentially refer to the future.

A *Kripke model* $\mathcal{K}$ is concrete partially ordered set with an assignment of domains and relations. A is true in a Kripke model $\mathcal{K}$ if for all $k \in K$, $k \Vdash A$. A is true, simpliciter, if A is true in all Kripke models. Semantical consequence is defined as follows: $\Gamma \Vdash A$ iff for all Kripke models $\mathcal{K}$ and all $k \in K$ $k \Vdash C$ for all $C \in \Gamma \to k \Vdash A$.

There is an extensive model theory for Kripke semantics. It is strongly complete for intuitionistic logic, that is $\Gamma \vdash_i A \Leftrightarrow \Gamma \Vdash A$, and in particular $\vdash_i A \Leftrightarrow A$ is true. Predicate logic is complete for Kripke models over trees, and for propositional logic we even have the *finite model property*: $\nvdash A \Rightarrow A$ is false in a Kripke model over a finite tree.

From the completeness over tree models, one proves the *disjunction property*:

(DP) $\vdash_i A \vee B \Rightarrow \vdash_i A$ or $\vdash_i B$

A straightforward proof of DP is as follows. Suppose $\nvdash_i A$ and $\nvdash_i B$, then there is a tree model $\mathcal{K}_1$ that does not force A, and a tree model $\mathcal{K}_2$ that does not force B. Now $\mathcal{K}_1$ and $\mathcal{K}_2$ are glued together: put the two models side by side and place a new node k below both. In k no proposition is forced. The result is a correct Kripke model, and since $\vdash_i A \vee B$ (given), $k \Vdash A \vee B$, and hence $k \Vdash A$ or $k \Vdash B$. But that contradicts the fact that A and B are not forced in $\mathcal{K}_1$ and $\mathcal{K}_2$; therefore, $\vdash_i A$ or $\vdash_i B$.

Similarly, there is the *existence property*

(EP) $\vdash_i \exists x A(x) \Rightarrow \vdash_i A(t)$ for a closed term t

The theorems support the intuitionistic intended meaning of 'existence,' but the straightforward proofs use *reductio ad absurdum* and are therefore not constructive. Here proof theoretical devices have come to the rescue (the normal form theorem):

1. If $\vee$ does not occur positively in any formula in $\Gamma \vdash_i A \vee B$, then $\Gamma \vdash_i A$ or $\Gamma \vdash_i B$
2. If $\exists$ and $\vee$ do not occur positively in any formulas in Γ and $\Gamma \vdash_i \exists x A(x)$, then $\Gamma \vdash_i A(t)$ for some closed term t.

The Dialectica *interpretation*

Gödel's *Dialectica* interpretation (1958) is an interpretation of HA where, as the primitive notion, 'construction' of the proof interpretation is traded in for 'computable function of finite type over the natural numbers,' axiomatized in his system T. The latter notion is both more specific and less abstract (i.e. closer to Hilbert's 'concrete' finitary methods). The main result can be stated

If $HA \vdash A$, then $T \vdash \exists x \forall y A_D(x, y)$ (A_D is quantifier-free; see below)

This explains Gödel's philosophical motivation to devise the interpretation; for combined with a willingness to grant that the principles of T are evident, the main result yields a consistency proof of HA (and, in combination with Gödel's embedding of PA into HA mentioned above, a consistency proof of PA). In other words, Gödel aimed to show that if one wants to go beyond Hilbert's finitary arithmetic (and to prove its consistency, one has to), the required non-finitary elements need not be as abstract as the intuitionistic notion of proof.

The interpretation A^D of a formula A is defined by induction on the number of logical operators in A ($s \ldots z$ and $V \ldots Z$ stand for finite (possibly empty) sequences of, respectively, arbitrary type or higher type; in particular, x and u denote the sequences of free variables in A and B):

$$A^D := A \text{ for atomic } A$$

For the induction step, suppose $A^D = \exists y \forall z A_D(y, z, x)$ and $B^D = \exists v \forall w B_D(v, w, u)$; then

$(A \wedge B)^D = \exists yv \forall zw(A_D(y, z, x) \wedge B_D(v, w, u))$
$(A \vee B)^D = \exists yvt \forall zw(t = 0 \wedge A_D(y, z, x) \vee t = 1 \wedge B_D(v, w, u))$
$(A \rightarrow B)^D = \exists V\, Z \forall yw(A_D(y, Z(yw), x) \rightarrow B_D(V(y), w, u))$
$(\exists s A)^D = \exists sy \forall z A_D(y, z, x)$
$(\forall s A)^D = \exists Y \forall sz A_D(Y(s), z, x)$
(negation is defined by $\neg A := A \rightarrow 0 = 1$)

The interpretation reduces the logical complexity of sentences at the cost of increasing the type of the objects. The interplay between, on the one hand, the connectives and, on the other, the quantifiers as constructively construed, introduces the higher-order functions and thereby removes quantifiers from the connected statements. As statements without quantifiers are decidable, the connectives between them become simple computable (truth) functions.

For example, $\exists x A(x) \rightarrow \exists u B(u)$ (for atomic A and B) is translated as $\exists U \forall x(A(x) \rightarrow B(U(x)))$. This renders exactly the constructive reading of the original formula: 'Given an object with property A, one can construct an object with property B', that is, there is a construction that takes an object with property A as input and yields an object with property B as output. Such constructions are the values for U in the translated formula.

It cannot be excluded that an intuitionistic proof of a statement invokes proofs of more complex statements; this exhibits a form of impredicativity in the proof interpretation. The *Dialectica* interpretation does not fare better here, as functionals of a higher type could be used to define functionals of a lower type. Also, unless one is willing to take the notion of 'computable functional' as primitive, logic will be needed again in the precise defintion of the intended class of functionals. For these reasons, it is not easy to assess the exact epistemological advantage of the *Dialectica* interpretation.

Martin-Löf's type theory

Per Martin-Löf was the first logician to see the full importance of the connection between intuitionistic logic and type theory. Indeed, in his approach the two are so closely interwoven, that they actually merge into one master system. His type systems are no mere technical innovations, but they intend to capture the foundational mean-

ing of intuitionistic logic and the corresponding mathematical universe (Martin-Löf 1975, 1984).

Martin-Löf points out that we not only consider propositions (statements) but also make judgments about them. That is, we may hold propositions true. The basic judgments we have to consider are:

1. A is a type
2. A and B are equal types
3. a is an element of the type A
4. a and b are equal terms of the type A.

We have the following correspondence between propositions and proofs on the one hand, and types and elements on the other hand:

A is a type	a is an element of the type A	A is inhabited
A is a proposition	a is a proof of the proposition A	A is true

The type formation corresponds exactly to the formation of propositions, as used in logic. It is a basic idea of Martin-Löf's type theory, that elements and types have canonical forms. This explains, for example, equality judgments. Why is $2 + 3 = 4 + 1 : \mathbb{N}$? That is to say, why are the terms $2 + 3$ and $4 + 1$ equal in the type $\mathbb{N}$? The answer is that $2 + 3$ and $4 + 1$ have the same canonical form 5 (i.e. $(1 + (1 + (1 + (1 + 1))))$). The rules for equality have to be understood in this way, for example

$$\frac{a = b : A}{b = a : A}$$

A particular feature of Martin-Löf's type theory that is the system does not take anything for granted, but always makes explicit all required assumptions. Thus, when making up a type from parts, all those parts have to satisfy the necessary requirements.

An informal example: in order to know that $a + b$ is a number, we have to know that a and b are numbers, formally stated: a and $b \in N \Rightarrow a + b \in N$. Or, considering types depending on a parameter, one has to make sure that the parameters are correctly chosen: $a \in N \Rightarrow A(a)$ is a type. Given the required rules for equality, substitution, etc., one goes on to list the various type constructions. Here are some basic rules governing judgments:

natural numbers	N *type*
product	$\dfrac{x : A \Rightarrow B(x) \, \text{type}}{\Pi x : A \cdot B(x) \, \text{type}}$
sum	$\dfrac{x : A \Rightarrow B(x) \, \text{type}}{\Sigma x : A \cdot B(x) \, \text{type}}$
disjoint sum	$\dfrac{A \, \text{type} \; B \, \text{type}}{A + B \, \text{type}}$
identity	$\dfrac{t \in A, s \in A, A \, \text{type}}{I(A, t, s) \, \text{type}}$

In the common set theoretical practice, $\Pi x : A.B(x)$ is the cartesian product, $\Sigma x : A.B(x)$ is the disjoint sum of the family $\{B(x) \mid x \in A\}$, $A + B$ is the disjoint sum of two sets. The identity type is rather unusual, it is a set which is inhabited if t and s are identical, otherwise it is empty.

Note that there is a dual reading: $\Pi x : A.B(x)$ becomes $\forall x : A.B(x)$ in the logical notation, etc.

The characteristic properties of the various types and their canonical elements are laid down by a number of rules:

Natural numbers

$$N I \quad 0:N \quad \frac{t:N}{St:N}$$

(0 is a natural number, and if t is a natural number then its successor St is also a natural number). These rules introduce numbers.

$$N E \quad \frac{t:N \;\; t_0 : A[0/x] \;\; x:N, y:A \Rightarrow t_1 : A[Sx/x] \;\; x:N \Rightarrow A \text{ type}}{R_{x,y}(t, t_0, t_1) : A[t/x]}$$

R_{xy} is the recursor operator, its nature will be explained below.

$$\Pi I \quad \frac{x:A \Rightarrow t:B \qquad x:A \Rightarrow B \text{ type}}{\lambda x \cdot t : \Pi x : A \cdot B}$$

$$\Pi E \quad \frac{t:(\Pi x:A) \cdot B \qquad t':A \qquad x:A \Rightarrow B \text{ type}}{App(t, t') : B[t'/x]}$$

The introduction rule is the common λ- abstraction. The elimination rule yields the application of the functional term t to the 'input' term t, usually written as $t(t')$, or tt'.

$$\Sigma I \quad \frac{t:A \qquad t':B[t/x] \qquad x:A \Rightarrow B \text{ type}}{p(t,t') : (\Sigma x:A) \cdot B}$$

(elements of the disjoint sum are thought of as pairs, the first item is from the 'parameter set' A, the second on from the parametrized set B_x)

$$\Sigma E \quad \frac{t:(\Sigma x:A) \cdot B \qquad A \text{ type}}{p_0(t) : A} \qquad \frac{t:(\Sigma x:A) \cdot B \qquad x:A \Rightarrow B \text{ type}}{p_0(t) : B[p_0(t)/x]}$$

For the remaining rules see for example Troelstra and van Dalen (1988: 580).

In addition one has to give rules for 'computing' terms. Here are some examples:

$$
\begin{cases}
R_{x,y}(0, t_0, t_1) & \triangleright \quad t_0 \\
R_{x,y}(St, t_0, t_1) & \triangleright \quad t_1[x, t/y, R_{xy}(t_1, t_0, t_1)]
\end{cases}
$$
$$App.(\lambda x \cdot t, t') \triangleright t[t'/x]$$
$$p_i(t_0, t_1) \triangleright t_i \; (i = 0, 1),$$
$$(p_0(t), p_1(t)) \triangleright t$$

Where ▷ stands for 'converts to.' In the formalism these conversions are also presented in the form of rules.

The system with the above types and terms is a kind of minimal system, there are a number of meaningful types to be added to make it more convenient and to strengthen it. But as it is, one can demonstrate a few characteristic features.

The properties that one establishes for types and terms can immediately be copied for propositions and proofs. If one suppresses, as is usual, the proof terms, the old natural deduction rules reappear. Example:

$$\frac{x:A \Rightarrow B\,type, x:A \Rightarrow t:B}{\lambda x.t:A \rightarrow B} \quad \text{becomes} \quad \frac{A\,true \Rightarrow B\ true}{A \rightarrow B\,true}$$

Thus we can get the intuitionistic provable proposition by operating in type theory.

Actually we get a few extras for working in a constructive setting. For example the axiom of choice becomes derivable. In ordinary language the axiom reads:

$$\forall x \in A \exists y \in B(x)C[x, y] \rightarrow \exists f \in \Pi x:A.B \forall x \in A\ C[x, fx]$$

In type theory one can indeed find a term t such that:

$$t:\Pi x:A\Sigma y:B.C[x, y] \rightarrow \Sigma z:(\Pi x:A.B)\Pi x:A.C[x, zx]$$

This confirms the intuitive argument that one would make in the proof interpretation. Note that in the proper reading of the axiom of choice, one exploits the hybrid nature of the system, terms may be elements or proofs. This is a strong practical feature of Martin-Löf's type theory.

We have barely scratched the surface of the theory, but one can see the striking similarity to the proof interpretation. To some extent, choice sequences have been incorporated in this framework as well, by admitting non-standard type theories.

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