25

Property-Theoretic Foundations of Mathematics

MICHAEL JUBIEN

1 Introduction

The main goal of this essay is to show how a certain comparatively weak theory of properties is adequate to provide a 'foundation' for classical mathematics. Theories of properties have of course been enlisted in foundational efforts in the past. The most prominent example is surely Whitehead and Russell's (1910–13) theory of 'propositional functions' (where these entities are taken as properties and relations in intension). George Bealer (1982) has provided a more contemporary example. These theories are very far-reaching and intricate, and they are also very different. Whitehead and Russell's is a 'ramified type theory' in which each propositional function appears at a certain level in a complex infinite hierarchy, but Bealer's properties aren't 'stratified' at all.

In the middle decades of the twentieth century 'intensional' entities such as *properties* and *propositions* were generally either regarded with great suspicion or else rejected outright. This tendency often reflected the powerful influence of W. V. Quine, who argued that 'extensional' entities – notably *sets* – are more respectable philosophically and are capable of doing any work that might have seemed to require intensional entities (see Quine 1960). During this same period set theory flourished in its own right as an autonomous branch of mathematics and came to be regarded very generally as the ultimate foundation of classical mathematics. In this atmosphere the idea of property-theoretic foundations was so far from most people's minds that even Whitehead and Russell's effort was typically thought of as a foundation within a system of *set theory*, with the apparent dependency of sets upon intensional entities ignored or forgotten (see Parsons 1967).

But things have changed. In the past few decades intensional entities have come to enjoy a great deal of attention along with greatly revived respectability. To a large extent this traces to a surge of interest in modal logic, its semantics, and the philosophical discussion of alethic modality, especially the question of 'essentialism.' (Prominent contributors to these developments include Ruth Barcan Marcus and Saul Kripke.) The resuscitation of such a seemingly paradigmatic intensional notion as *modality* brought in its wake a renewed interest in all matters intensional, including of course intensional *entities*. This in turn prompted a reexamination of Quine's criticism of these entities, and many have concluded that his criticism fails (e.g. Jubien 1996). Further, in a stunning reversal, philosophers have begun to argue that *sets* – so recently celebrated as paragons of clarity among *abstracta* – are in fact deeply obscure and mysterious (e.g. Bealer 1982; Jubien 1989). David Lewis (1991) also argues that these entities are fundamentally mysterious, but then throws up his hands and accepts impure nonempty sets (which he prefers to call *classes*) because he thinks they provide mathematics with a much-needed foundation.

At the present moment we, therefore, find some philosophers accepting properties and rejecting sets because, ironically, they think properties are clear while sets are obscure. And we even find philosophers who accept sets doing so with real reluctance because they concede the obscurity of the notion. As a very general comment, the stock in properties has been on a decades-long winning streak, while the stock in sets, though perhaps solid, has sustained some long-term decline and has some jittery holders.

So now is a good moment to revisit property-theoretic foundations of mathematics. The foundation I will offer here may have an appeal that others lack. For it's based on a very 'sparse,' epistemically conservative theory of properties, one that postulates no properties that aren't *intrinsic* to their instances. In fact it doesn't depend on postulating many properties or sorts of properties at all. It gets much of its foundational power from a dose of mereology that will be discussed later.

2 On Foundations

What do we really mean by 'foundations of mathematics'? Does mathematics even *need* a foundation? (Putnam (1967) argues that it does not.) If it does, must it have only one or is there room for many? The work that has historically been classified as 'foundations of mathematics' is remarkably diverse, ranging from the purely logical to the purely philosophical, and as a whole it provides no clear answers to these questions. (Parsons (1967) provides a very meticulous survey of the topic.) I won't make any effort at a complete discussion, but I will try to distinguish two very broad senses in which one might speak of foundations. These two notions are easily confused because the concept of the reducibility of one formal theory to another plays a central role in each. But in one of these senses there could only be one foundation of mathematics, while in the other there are potentially many different, equally acceptable foundations.

The *apparent* subject matter of classical mathematics includes such seemingly diverse entities as points, lines, natural numbers, real and imaginary numbers, ordered n-tuples, infinite sequences, functions, vector spaces, topological spaces, groups, rings, and so forth. Working mathematicians have typically proceeded as if these are independently existing 'Platonic' entities, and they haven't worried too much about whether this presupposition is ontologically defensible. But philosophers have often worried about it, and they have explored many positions, most of which may be seen as versions either of realism (Platonism), conceptualism, or nominalism, or as hybrids involving the reduction of entities of some kinds to others. We may think of such philosophical positions as *ontological* foundations, for at bottom they are claims about what mathematical entities really exist, along with claims about their ultimate natures.

For example, noting the reducibility of first-order versions of the various branches of classical mathematics to set theory, it might be held that it is really only *sets* that comprise the true subject matter of mathematics. Sets would be seen as abstract entities of their own special sort, and the idea that there are *also* such varieties of entities as numbers, vectors, and functions (etc.) would be abandoned. In this way sets (and set theory) would be seen as the ontological foundation of mathematics. A proponent of this position would owe a principled reason for preferring sets over any other sort of entities (such as properties) to which the apparent mathematical objects might also happen to be reducible. One ingredient of the reason could be the recent ascendency of sets themselves as apparent mathematical entities, but on its own this wouldn't be very convincing. For if the other apparent entities are up for grabs, why should sets be any different? The important point here is that since any proposed ontological foundation incorporates a claim about the real subject matter of mathematics, it is incompatible with each of its ontological alternatives.

Although I cannot try to establish it here, I think it is far from clear that the original motivation for seeking foundations required them to be *ontological*. Very roughly, I think the motivation was the worry that certain concepts of analysis involving the infinite were not well understood and threatened paradox. What is needed to address this kind of concern is a way to arrive at an adequate grasp of the concepts, one that increases our confidence that they aren't inherently incoherent or paradoxical. There is no good reason to suppose that this couldn't be done without making controversial ontological commitments. Nor is there any good reason to suppose it could only be done one way. Because the primary goal of foundations in this sense is improved understanding, we may think of such foundations as *epistemological.*

One way to arrive at a better understanding of a given concept is to 'model' it with other concepts. When specific theories are in hand, we may do this *syntactically* by carrying out a formal reduction of one theory to the other. If we initially understand the structural features of the reducing concepts better than those of the given concept, we automatically improve our grasp of that concept simply by doing the reduction. In the case of analysis and set theory, very intricate notions are readily 'modeled' in a theory whose sole primitive – the binary membership relation – is remarkably simple and, at least structurally, very accessible.

The reduction of one theory to another is also, in effect, a relative consistency proof: if the reducing theory is consistent, then so is the reduced theory. Of course this means the reducing theory is, logically speaking, at least as strong as the reduced theory. Despite this, it is possible to gain confidence in the coherence of the reduced theory as a result of the reduction. For our initial, intuitive conviction that the reducing theory is coherent may be substantially greater than our initial confidence in the coherence of the reduced theory, perhaps as a result of a greater accessibility of its primitive concepts. So the reduction can have the effect of boosting our confidence in the reduced theory rather than undermining our confidence in the reducing theory. (Of course we know from Gödel's work that the consistency of any first-order theory strong enough to reduce classical mathematics is in effect a matter of faith rather than proof.)

The reduction of analysis and the rest of classical mathematics to set theory should be seen as a great conceptual advance whether one accepts the existence of sets or not. For it shows that the various mathematical concepts are structurally related to the set concept in certain ways regardless of whether that concept actually has instances. Since, under the epistemological conception of foundations, there would be no claim that sets comprise the ultimate subject matter of mathematics, it may not even matter whether they exist or not. We may be able to attain sufficiently improved understanding and confidence simply as a result of the formal reduction.

But it is also possible that the foundational gain would be even greater if we were convinced that the reducing entities really did exist. For example, someone who already believes the real ontology of mathematics is just its apparent Platonic ontology is likely to find it more satisfying to think that there actually exist *non-mathematical* entities that display the structural complexity of the putative mathematical objects. For then the foundation would not only enhance clarity and the conviction of coherence, it might also make Platonic mathematical objects seem more plausible by providing an independent precedent for Platonic entities of that level of complexity.

I will stop short of endorsing an ultimate ontology for mathematics. Instead I'll just mention what I think are the two most plausible candidates. They reflect a pair of ontological convictions. One is that the philosophical difficulties of sets are so overwhelming that sets should be rejected. The other is that the case for Platonic properties is very strong and doesn't rest on considerations about mathematics. Given these convictions, one candidate, inspired by Whitehead and Russell and by Frege, is that mathematics does have a genuinely ontological foundation in properties (and property theory). On this view, properties are the ultimate (and exclusive) subject matter of mathematics. Despite its historical moorings, this view would surely be seen as revisionary. The other candidate is more of a 'face-value' position. It's the view that mathematics really has no ontological foundation, that its ultimate subject matter is its original apparent ontology (of course, not including sets), and that an epistemological foundation in properties supports this ontology (as suggested above) while achieving the original foundational goals of clarity and coherence. I believe there is a good deal that can be said in favor of either of these positions. Of course they both require property-theoretic foundations, even if for one they are ontological and for the other they are not.

3 Properties, Sums, Plurality, and Reality

The theory to be offered here is basically the outgrowth of three simple ideas, two metaphysical and one purely logical. I believe both metaphysical notions are extremely compelling intuitively, and that the logical notion has a solid grounding in ordinary language. The first idea is a Platonic principle about properties: that properties 'constitute' things as being how they are. For a thing to be green just *is* for it to instantiate the property of *being green*, nothing more, nothing less. A corollary of this principle of constitution is that for any entity at all, since it is a *specific* entity, there is a property of *being that entity*. Intuitively, any such property must be *intrinsic* (and also *essential*) to its unique instance, but it must not be a *part* of that instance. The theory will postulate properties of this kind and also the property of *being self-identical*, which certainly is also intrinsic to each of its instances. The intrinsicalness of the postulated properties (along with the fact that they are all instantiated) ensures that the theory is compatible with what is commonly called a 'sparse' conception of properties. And because it postulates only these very basic sparse properties, the theory is conservative epistemically.

The second idea is that any entities have a mereological sum, regardless of their individual natures. So not only are there sums of physical objects, there are also sums of abstract entities, and sums of entities some of which are concrete and others of which are abstract. The sum of any entities has each of them as a part, and has no part that isn't a part of some of those entities. (For a defense of the idea that mereology extends to the realm of the abstract, see Lewis (1991).)

The logical notion is 'plural' quantification. As it happens, plural quantifiers are quite common in ordinary English (but the phenomenon is still relatively unexplored in logic). Plural quantifiers range over the various pluralities of things, but without presupposing that the pluralities are individual entities like sets (or 'totalities' of any other kind). These quantifiers are not reducible to ordinary (singular) objectual quantifiers. A classic example of an ordinary English sentence with a plural quantifier is 'Some critics admire only each other.' It is clear what this sentence means, but that meaning cannot be captured with ordinary first-order quantifiers. Another example is the first sentence of the previous paragraph. (For more on plural quantification, see Boolos (1984).)

Unfortunately, these three very appealing ideas cannot be adopted in full generality, for then there would be a sum of *everything*, say S, and the property of *being S* would have to be a part of S. This is harsh reality. To avoid the problem we have to restrict at least one of the three ideas, and it is a further reality that no way of doing this can be quite as elegant and simple as the unadulterated combination of ideas. But this kind of reality has a familiar precedent, for the simplest and most elegant theory of *sets* – that based on unrestricted comprehension – proved to be self-contradictory. Our approach will be to limit the formation of sums and leave the other two notions alone. Other approaches are also possible, but we will not debate the relative merits of the various possibilities here. Our overall goal is merely to show that a great deal can be done even with an epistemically conservative conception of properties.

4 Mereological Property Theory

One of the central concepts in this theory (MPT) is the binary relation of *intrinsicalness*, which we will express with the symbol 'μ.' Thus 'xμy' means that y is intrinsic to x. Another key concept is expressed by the unary function symbol 'β,' which maps any entity x to the 'individuality' property of *being x* ('βx'). A third basic notion is the part–whole relation of mereology, which we will express with the binary predicate letter 'ρ,' so that 'xρy' means x is part of y. The theory will also employ identity and both singular and plural (objectual) quantification. Plural quantifiers will be enclosed within square brackets (e.g. '[∀x]' and '[∃x]') and singular quantifiers are left unenclosed ('∀x' and '∃x'). An occurrence of '[x]y' (etc.) within the scope of a plural quantifier on x is understood to mean that y is an x (that is, y is one of the x's). MPT has four nonlogical axioms and an axiom scheme. (Here we view the axioms for identity and the functionality of β as logical axioms.)

1. *Self-Identity*: $\exists! x \forall y((y \mu x \leftrightarrow y = y) \& (y \rho x \rightarrow y = x))$.

This simply says that there is a unique property of which everything is an instance and which has no proper parts. We will call this property 'a' in the metalanguage. A consequence is that a instantiates itself. Of course there may be many distinct properties of which everything is an instance, but in practice we only need one, and we can think of it as the property of being self-identical. (An 'impure' version of MPT could be obtained by replacing Axiom 1 with an axiom postulating the existence of a concrete entity.)

2. *Individuality*: $\forall x \exists! y(y = \beta x \& \neg y \rho x \& \forall z((z \mu y \leftrightarrow z = x) \& (z \rho y \rightarrow z = y)))$.

For any entity x there is a unique property of *being x*, which is never a part of (and so never identical with) x itself, which has x as its only instance, and which has no proper parts. We will call any property that is a βx for some x a 'β-property.' Notice that the 'atomicity' of the β-properties ensures that any sum of given β-properties can have no other β-properties as parts. The β-properties are therefore in a certain sense independent of each other. We now adopt an axiom to ensure that the part–whole relation of MPT conforms to that of standard mereology.

3. *Mereology*: $\forall x \forall y \forall z(x \rho y \ \& \ y \rho z \rightarrow x \rho z) \ \&$
$\forall x \forall y \exists z[x \rho z \ \& \ y \rho z \ \& \ \forall w(w \rho z \rightarrow \exists v(v \rho w \ \& \ (v \rho x \vee v \rho y)))]$

The first conjunct ensures the transitivity of the part–whole relation and the second entails that any finite number of entities have a unique mereological sum. Hereafter we will not hesitate (in the metalanguage) to write '$x + y$' for the sum of x and y. (Obviously + is commutative and associative.) The next axiom provides us with an entity that has infinitely many non-overlapping parts.

4. *Infinity*: $\exists x[\exists w(\forall y(y \mu w \leftrightarrow y = y) \ \& \ \beta w \rho x) \ \&$
$\forall y(y \rho x \ \& \ \neg(y = x) \ \& \ \forall z(\beta z \rho y \rightarrow (z \rho y \vee z = w))) \rightarrow \beta y \rho x)]$.

The entity x therefore has βa as a part and, for any part y other than x itself, it has βy as a part provided that whenever a property βz is part of y, either z = a or z is also part of y. We may think of sums of β-properties that meet this last condition as 'β-transitive.' Since βa is β-transitive, $\beta\beta$a must be part of x. But since $\beta\beta$a isn't β-transitive, the axiom doesn't require $\beta\beta\beta$a to be part of x. On the other hand, βa + $\beta\beta$a *is* β-transitive, so its β-property is part of x. And since βa + $\beta\beta$a + β(βa + $\beta\beta$a) is *also* β-transitive, *its* β-property is part of x, and so on. Next we introduce an axiom scheme that, in combination with Axioms 1 and 4, 'generates' a vast universe of mereological sums.

5. *The Scheme of Generation*: All formulas
$\forall x\exists! y\mathbf{A}(x,y). \rightarrow .\forall z[\forall w][\forall u([w]u \rightarrow u\rho z) \rightarrow$
$\exists v[\forall x\forall y(([w]x \;\&\; \mathbf{A}(x,y)) \rightarrow y\rho v) \;\&$
$\forall s(s\rho v \rightarrow \exists x\exists y\exists t([w]x \;\&\; \mathbf{A}(x,y) \;\&\; t\rho y \;\&\; t\rho s))]]$,

where x and y are free in the formula $\mathbf{A}(x,y)$. Although it looks rather complex, this scheme merely says, for each 'functional' formula $\mathbf{A}(x,y)$ (with parameters as required), that for any object z, and any w's, if the w's are parts of z, then there is a unique sum of the **A**-images of those w's. As a simple application, let z be any infinite sum conforming to Axiom 4. Then βa, ββa, β(βa + ββa), and so on are all parts of z. Let 'the w's' be the parts of z that are β-properties, and let $\mathbf{A}(x,y)$ be the formula '$\exists s(x = \beta s \;\&\; \exists t(trs \;\&\; \neg(t = s) \;\&\; y = x) \vee y = u)$', where u is parametrized to β(βa + ββa). $\mathbf{A}(x,y)$ is easily seen to be functional, for it maps each w = βs to itself if s has a proper part, and maps every other w to β(βa + ββa). So the axiom delivers a sum of which β(βa + ββa) is a part but βa and ββa are not.

For another example, let $\mathbf{A}(x,y)$ be the formula 'y = βx' (which is obviously functional). Again take z to be any sum conforming to Axiom 4, and let 'the w's' be all parts of z. Then the resulting v is the sum of all the β-properties of its parts. So, for example, β(βa + β(βa + ββa)) is part of v and so is βz.

Although it generates a boundless universe, MPT is very 'minimalistic' from a property-theoretic perspective. In a nutshell, it says that there is a property (*self-identity*) that everything instantiates, that for anything at all there is the (mereologically atomic) 'individuality' property of *being that thing*, that there is a sum having infinitely many β-properties among its parts, and that the (intra-theoretically describable) functional *relata* of any parts of any entity has a sum. So, from the perspective of the theory, the only properties that need exist are *self-identity* and various β-properties, and the only nontrivial sums are those given by Axiom 4 and generated by Scheme 5. But even this 'minimal model' would constitute a remarkably lavish universe of entities – easily enough to provide an interpretation of ZF.

5 Foundations of Mathematics

Let's begin by making the (uncontroversial) assumption that classical mathematics is formally reducible to ZF. Then it suffices for our purposes if ZF, in turn, is formally reducible to MPT. In other words, MPT is an adequate foundation provided that '∈' is definable in the language of MPT in such a way that the translations of the ZF axioms under the definition are theorems of MPT. A careful demonstration that this is indeed the case would be lengthy and tedious. So in this section I will address the question semantically rather than axiomatically, and will do so in an informal and rather incomplete way. The goal is just to provide the construction that delineates a 'domain' of a 'model' of ZF within an arbitrary 'model' of MPT, along with an 'interpretation' of '∈'. Verification that the result actually is a 'model' will be omitted.

For this project to make sense we must employ a conception of *model* (and *interpretation* generally) that departs ontologically from the usual set-theoretic one. (Hence the quotation marks in the previous paragraph.) The main reason, as I see it, for exploring property-theoretic foundations is the conviction that there really are no sets (whether pure or impure) and it is an immediate consequence that the sort of set-theoretic constructions that are usually thought to be the objects of model theory simply do not exist. If there are no sets, then model theory, as it is typically understood, has no subject matter. So we need to reinterpret it, and various approaches are possible.

The approach we will adopt avoids thinking of models as *individual entities* at all. Instead, the fundamental notion will be that of (plurally, now!) *some entities* modeling a theory by virtue of relations they bear to each other. As an example, suppose we have a first-order theory with a single binary relation symbol. In a typical model-theoretic approach, an *interpretation* of the theory is an ordered pair consisting of a nonempty set (the domain) and a set of ordered pairs of members of the domain (or some close variation on this idea). Such an interpretation is a *model* if the axioms are true when the quantifiers range over the domain and the relation symbol is interpreted as having the set of ordered pairs as its extension. So one model might be the ordered pair of the set of people who live in Detroit and the set of ordered pairs $\langle x,y \rangle$ of Detroiters x and y such that x is a parent of y. But, convenient though they may be, we don't need the sets and ordered pairs at all, because we understand perfectly well what it takes for the axioms of the theory to be true of the people in Detroit when the relation symbol is understood as expressing the parent-of relation. (Similarly, if there really are natural numbers, then *they* – with their characteristic relations – obey the axioms of number theory whether or not there exist any sets or ordered n-tuples of them from which to concoct standard model-theoretic interpretations.)

Now, using plural quantifiers in the metalanguage, we can say what it means to *specify an interpretation* without having to think of interpretations as specific *entities* of any sort whatever. We succeed in 'specifying an interpretation' whenever we give a clear specification of *some entities* (the intuitive *domain*) together with an appropriate association of *properties* and *relations* with the nonlogical symbols of the theory. If the theory's axioms happen to hold for those entities under those associations, then we have 'specified a *model*.' That there is no actual entity available to call 'the model' is at worst a mild inconvenience (and indeed is one that could be avoided in a full-blown *property-theoretic* model theory). The other side of this coin is that interpreting formal theories *via* set-theoretic objects is just a minor convenience, and is in no way a special source of mathematical or semantical rigor. In a successful case, the *specification* of the various sets that comprise a set-theoretic interpretation of a given theory would have exactly the same informal-though-precise status as the specifications of 'interpretations' in the present sense.

So now let's suppose that we have an arbitrary model of MPT. Then it contains entities we may view as ordinal numbers. In fact Axiom 4 very conveniently delivers an infinitude of such entities. We will define the ordinals in two stages, the first of which replicates the intuitive notion of β-transitivity mentioned above.

DEFINITION 1 A sum T of β-properties is β-*transitive* if for all x, if βx is part of T, then either x is part of T or x = a.

DEFINITION 2 A β-transitive sum T of β-properties is an *ordinal* if for any distinct x and y, if βx and βy are parts of T, then either βx is part of y or βy is part of x.

As an example, consider βa + ββa + βββa. It is β-transitive, but it isn't an ordinal. For notice that if x = a and y = ββa, then both βx and βy are parts of the sum, but βx is not part of y nor is βy part of x. On the other hand, βa + ββa + β(βa + ββa) is easily seen to be an ordinal.

Because we have chosen not to view a as an ordinal, all ordinals are sums of β-properties. The ordinals begin like this:

βa; βa + ββa; βa + ββa +β(βa + ββa); . . .

It is easy to see that the ordinals (up to any point) are well-ordered by the part–whole relation. We may define a *limit* ordinal as any ordinal other than βa that has no part βx such that every other β-part is part of x, and a *successor* ordinal as one that is neither βa nor a limit. (The existence of a limit ordinal follows from Axiom 4 and Scheme 5. To see this, note that the definitions of 'ordinal' and 'limit ordinal' may be captured by formulas of MPT, say '**O**(x)' and '**L**(x).' Now let **A**(x,y) be the formula '(**O**(x) & ∀y((yρx → ¬**L**(y)) & y = x) ∨ y = u),' with u parametrized to βa. Then **A**(x,y) is functional. Now let the z of Scheme 5 be any sum that conforms to Axiom 4, and let 'the w's' be all parts of v. The object v delivered by the resulting instance of the scheme is then the sum of all 'finite' ordinals and is easily seen to be a limit ordinal itself.) We may also define surrogates for the natural numbers by setting 0 = βa and the successor of n = the sum of n and βn.

We now employ transfinite induction to give the construction that will provide the basis for the domain of our model of ZF. The rough idea is to mimic the power set operation at successor stages and to mimic unions at limit stages. The result is a hierarchy of objects, each one a β-property, and hence each one a property with exactly one instance. From this hierarchy we may 'filter out' the domain of the model, and then define the surrogate of the membership relation. The first two stages of the construction are given explicitly by:

S(0) = 0 (i.e. βa) and S(1) = β0 (i.e. ββa).

Next, for any ordinal α greater than 0, let

$S(\alpha + 1) = \beta(x + \sum\beta y{:}y\rho x)$, where $S(\alpha) = \beta x$,

where 'Σβy : yρx' denotes the sum of the β's of the parts of x, guaranteed to exist by Scheme 5. (Note that, on the left side, '+1' means ordinal successor, not mereological sum.) So, at a given stage α, S(α) is some β-property, say βx. Then S(α + 1) is the result of applying β to the result of summing x with the sum of all objects βy, where y is a part of x. Finally, if λ is a limit ordinal, let

$S(\lambda) = \beta\sum S(Y){:}Y\rho\lambda$.

S(λ) is therefore obtained by applying β to the sum of all earlier stages (which sum must exist by an application of Scheme 5 to the ordinal parts of λ *via* a functional formula reflecting the method of construction). Now, given the entire hierarchy of S(δ)s, we are able to describe the domain of the model: an object is in the range of the ZF quantifiers iff it is a β-property that is a part of an instance of an S(δ). To illustrate, consider the first four stages:

S(0) = βa;
S(1) = ββa;
S(2) = β(βa + ββa); and
S(3) = β(βa + ββa + βββa + β(βa + ββa)).

So, for example, β(βa + ββa) is in the domain because it is a β-property that is a part of the instance of S(3). But, for example, because βa + ββa is *not* a β-property, it isn't in the domain even though it is a part of the instance of S(3) (and, for that matter, also a part of the instance of S(2)). That each S(α) is a β-property contributed to the domain by S(α + 1) is also illustrated here. For example, we have:

S(3) = β(S(0) + S(1) + βS(1) + S(2)).

Notice that every member of the domain is the result of applying the β-operator to a or to a sum of β-properties. Intuitively, βa will represent the null set, and every other member of the domain will represent the set whose members are precisely the sets represented by the β-properties that are parts of the sum. What makes this work is the 'independence of individuality' that is guaranteed by the atomicity of β-properties: no sum of any specific β-properties can have a β-property as a part that isn't one of those specific β-properties. It is now easy to provide the interpretation of '∈' in the given domain:

x ∈ y iff ∃z(y = βz & xρz).

Notice that there is no need for a clause stating that x is a β-property since only β-properties fall in the range of the quantifiers in the first place. Thus it is clear that βa is the unique object from the domain that has no members – it's the surrogate of the empty set. There are no deep difficulties in verifying the other axioms of (pure) ZF. Thus we have shown how to construct a model of ZF from materials available in any model of MPT. It follows that MPT has at least as much foundational punch as ZF.

References

Bealer, G. (1982) *Quality and Concept*. Oxford: Clarendon Press.

Boolos, G. (1984) To be is to be the value of a variable (or to be some values of some variables). *Journal of Philosophy*, 81, 430–49.

Jubien, M. (1989) Straight talk about sets. *Philosophical Topics*, 17, 91–107.

Jubien, M. (1996) The myth of identity conditions. *Philosophical Perspectives*, 10, 343–56.

Lewis, D. (1991) *Parts of Classes*. Oxford: Basil Blackwell.

Parsons, C. (1967) Mathematics, foundations of. In P. Edwards (ed.), *The Encyclopedia of Philosophy*. (Vol. 5, pp. 188–213). New York: Macmillan and the Free Press.

Putnam, H. (1967) Mathematics without foundations. *Journal of Philosophy*, 64, 5–22.
Quine, W. V. O. (1960) *Word and Object*. Cambridge, MA: MIT Press.
Whitehead, A. N. and Russell, B. A. W. (1910–13) *Principia Mathematica*. Cambridge: Cambridge University Press (2nd edn., 1925–7).