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Modality of Deductively Valid Inference

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1 Validity and Necessity

An inference is deductively valid if and only if it is logically necessary that if its assumptions are true, then its conclusions are also true; or, alternatively, if and only if it is logically impossible for its assumptions to be true and its conclusions false.

Some type of modality evidently governs the truth conditions of assumptions and conclusions in deductive inference. There are many different systems of alethic modal logic, however, and the question of which modal system is appropriate for understanding the modality of deductive validity has not been rigorously investigated. In what exact sense is it logically necessary for the conclusions of a deductively valid argument to be true if its assumptions are true? In what exact sense it is logically possible for the conclusions of a deductively invalid argument to be false when its assumptions are true? Does deductive inference presuppose the modality of, say, modal system $S1$, or T, $S2$, $S3$, $S4$, $S5$, the Brouwersche system, or yet another modal logic?

I argue in what follows that the failure of the validity or Pseudo-Scotus paradox in normal modal logics weaker than $S5$, and its provability in $S5$ and conservative extensions of $S5$, suggests that the modality of deductively valid inference must be weaker than $S5$. The sense in which it is logically necessary for the conclusions of a deductively valid inference to be true if its assumptions are true, or logically impossible for its assumptions to be true and its conclusions false, in that case must be defined in terms of a modal logic weaker than $S5$.

2 The Validity Paradox

The validity paradox, also known as the Pseudo-Scotus, is most easily understood in an impredicative formulation. Consider the following inference:

(V) 1. Argument (V) is deductively valid.

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2. Argument (V) is deductively invalid.

The paradox proceeds by projecting argument (V) into a dilemma. We assume that argument (V) is either deductively valid or deductively invalid. If (V) is deductively valid, then it is also sound, since the assumption in (1) declares that the argument is deductively valid. Sound arguments by definition have true conclusions. So, if (V) is deductively valid, then, as its conclusion states, it is deductively invalid. The second horn of the dilemma is more difficult. If (V) is deductively invalid, then, according to the definition of deductive validity, it is logically possible for the assumption of (V) to be true and the conclusion false. The assumption of the second dilemma horn thus implies only that it is logically possible, not categorically true, that argument (V) is deductively valid. It does not follow simply that if argument (V) is deductively invalid, then it is deductively valid, but at most only that if (V) is deductively invalid, then it is logically possible that (V) is deductively valid. To go beyond this, trying to deduce that (V) is deductively valid if and only if it is deductively invalid, is to commit the inelegant modal fallacy of inferring that a proposition is true from the mere logical possibility that it is true (see Jacquette 1996).

3 Gödel Arithmetizing the Validity Paradox

It might be thought that the validity paradox is improper because of its impredicative form, violating the vicious circle principle. The impredicative expression of the validity paradox as presented is nevertheless inessential. Impredication can be avoided by Gödelizing the syntax of the inference.

The validity paradox (V) is Gödelized as (GV) for $g\ulcorner V[sub_g(n)] \vdash \overline{V}[sub_g(n)]\urcorner = n \wedge sub_g(n) = \ulcorner V[sub_g(n)] \vdash \overline{V}[sub_g(n)]\urcorner$, in order to prove that $V[sub_g(n)] \leftrightarrow \overline{V}[sub_g(n)]$. The Gödel number of the argument is determined by assigning natural numbers to syntax items in the expression to be arithmetized. Each such number is made the exponent of a corresponding prime number base taken in sequence in the same order of increasing magnitude as the syntax (standardly left-to-right) in the expression to be coded. The Gödel number of the expression is the product of these primes raised to the powers of the corresponding syntax item code numbers. A Gödel substitution function, sub_g, substitutes for any whole number to which it is applied the unique syntax string, if any, which the Gödel number encodes.

$$\begin{array}{ccccccccccccccc}
V & [& sub_g & (& _ &) &] & \vdash & \overline{V} & [& sub_g & (& _ &) &] \\
| & | & | & | & | & | & | & | & | & | & | & | & | & | & | \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 6 & 7
\end{array}$$

The Gödel number of the validity paradox on this assignment of Gödel numbers to syntax items in the formula is: $2^1 \times 3^2 \times 5^3 \times 7^4 \times 11^5 \times 13^6 \times 17^7 \times 19^8 \times 23^9 \times 29^2 \times 31^3 \times 37^4 \times 41^5 \times 43^6 \times 47^7 = n$. This number is substituted for blank spaces (alternatively, free variables) to which the number 5 is here assigned in the open sentence above to complete the Gödel arithmetization in $g\ulcorner V[sub_g(n)] \vdash \overline{V}[sub_g(n)]\urcorner = n$, where by stipulation, $sub_g(n) = \ulcorner V[sub_g(n)] \vdash \overline{V}[sub_g(n)]\urcorner$.

Angle quotes, $\ulcorner, \urcorner$, are used conventionally to indicate that the Gödel-numbering context is intensional, since a Gödel numbering context does not support intersubstitution of logically equivalent expressions that differ syntactically in any way. A distinct Gödel number obtains for every distinct syntax combination, including logical equivalents, like $\varphi \vee \psi$ and $\neg\varphi \rightarrow \psi$, where $g\ulcorner\varphi \vee \psi\urcorner \neq g\ulcorner\neg\varphi \rightarrow \psi\urcorner$, even though $[\varphi \vee \psi] \leftrightarrow [\neg\phi \rightarrow \psi]$.

The Fundamental Theorem of Arithmetic guarantees that every number can be decomposed into a unique factorization of prime number bases raised to certain powers. When number n is factored in this way and the factors arranged in ascending order (again, from left to right) according to the increasing magnitude of prime number bases, the expression mapped into Gödel-numbered space can be read directly from the exponents of each prime, and translated back into the original logical syntax by the glossary of natural number assignments.

The Gödelized validity paradox is not impredicative, because the Gödelized paradox argument is not defined in terms of propositions that explicitly mention the argument's label or name, (V). Self-reference is instead achieved only indirectly by the stipulation that the Gödel number of the inference $V[sub_g(n)] \vdash \overline{V}[sub_g(n)]$ is n, and the definition of the Gödel substitution function sub_g, by which the Gödel coded inference is recovered in its exact syntax-item-by-syntax-item formulation. Gödelization avoids impredication in the same way that it circumvents Russell's simple type theory restriction on syntactical self-predications. The Gödel sentence predicates a semantic property only of an object, a substituend identical to the sentence obtained by applying the Gödel substitution function to a Gödel number, and not to another property represented by a predicate of the same type. Gödelization thereby also avoids the need for explicit mention of the name or label of a sentence or argument, achieving self-reference indirectly in the inference by predicating a semantic property, validity or invalidity, of the substituend represented by a Gödel code number defined as the Gödel code number of the inference itself.

4 The Validity Paradox in S5

A proof that the second dilemma horn fails in modal systems weaker than $S5$, but succeeds in modal $S5$ and its conservative extensions, can be formalized in this way for Gödelized validity paradox (GV).

The role of the iterated modalities, and their implications for the second validity paradox dilemma horn, are seen in the following derivation. Here it is obvious that the inference from the assumption that (GV) is invalid to the conclusion that (GV) is valid holds only in some but not all systems of modal logic, according to the world- or model-accessibility relations by which each distinct modal logic is defined.

To symbolize the paradox requires a metalinguistic vocabulary to formally represent specific logical and semantic properties of propositions and inferences. We stipulate as primitive metalogical predicates that A is the property of being an assumption, C the property of being a conclusion, effectively, of an argument. We assume *Truth*, T, as a primitive bivalent relation of positive correspondence between a proposition and an existent state of affairs that the proposition describes or otherwise linguistically represents. If the state of affairs the proposition represents does not exist, then the proposition is false. A state of affairs is the possession of a property by or involvement in a relation of the objects in a well-defined semantic domain; a state of affairs Fa exists when an object *a* possesses a property or is involved in a relation *F*, and fails to exist when *a* does not possess or is not involved in relation *F*. *Ramsey* reduction then states that for any proposition φ, φ is true if and only if φ, $\forall\varphi[T\varphi \leftrightarrow \varphi]$. The principle effects what is sometimes known also as the redundancy theory of truth, where to say that φ is true is just to say that φ, and to say that φ is to say that φ is true. The principle allows us to move freely back and forth from true propositions to true metalinguistic propositions that state that the propositions are true.

Validity, *V*, is defined as a relation among the truth conditions of the assumptions and conclusions of an inference, such that it is logically impossible for the assumptions to be true and the conclusions false. The truth of logically necessary propositions is invoked in step (7) as $\Box\varphi \to \varphi$. The formalism reflects the intuitive reasoning that if (*GV*) is valid, then it is also sound, since its assumption says that it is valid. But, as we have seen, since sound arguments necessarily have true conclusions, it follows in that case that (*GV*), as its conclusion states, is invalid.

PROOF 1 *Validity Horn of the Validity Paradox*

(1)	$\forall x[Vx \leftrightarrow \Box[\forall y[Ayx \wedge Ty] \to \forall y[Cyx \to Ty]]]$	*Validity*
(2)	$V[GV]$	*Assumption*
(3)	$\Box[[\forall y[\mathrm{A}y[GV]] \wedge Ty] \to \forall y[Cy[GV] \to Ty]]$	(1,2)
(4)	$\forall y[TAy[GV]] \leftrightarrow V[GV]$	(*GV*)
(5)	$\forall y[TCy[GV]] \leftrightarrow \overline{V}[GV]$	(*GV*)
(6)	$\Box[TV[GV] \to T\overline{V}[GV]]$	(3,4,5)
(7)	$TV[GV] \to T\overline{V}[GV]$	(6, $\Box\varphi \to \varphi$)
(8)	$V[GV] \to \overline{V}[GV]$	(7, *Ramsey*)

The second dilemma horn is more difficult. It is blocked by modal fallacy, except where the accessibility relations defining a strong system of modality like *S*5 or its conservative extensions make it possible to infer necessity from possible necessity. To demonstrate the difference in strengths of modalities in deriving the inference that $\overline{V}[GV] \to V[GV]$, we first show that the inference fails in weak modal systems, and then offer a formal proof of the second paradox dilemma horn invoking the characteristic axiom of modal *S*5. This is how the proof is blocked in weak systems of modality:

PROOF 2 *Failure of Invalidity Horn of Validity Paradox in Modal Systems Weaker than S*5

(1)	$\forall x[Vx \leftrightarrow \Box[\forall y[Ayx] \wedge Ty \to [\forall y[Cyx \to Ty]]]]$	*Validity*
(2)	$\forall x[\overline{V}x \leftrightarrow \Diamond[\forall y[Ayx] \wedge Ty \wedge \exists y[Cyx \wedge \overline{T}y]]]$	(1)
(3)	$\overline{V}[GV]$	*Assumption*
(4)	$\Diamond[\forall y[Ay[GV]] \wedge Ty \wedge \exists y[Cy[GV] \wedge \overline{T}y]]$	(2,3)
(5)	$\forall y[TAy[GV]] \leftrightarrow V[GV]$	(*GV*)
(6)	$\forall y[TCy[GV]] \leftrightarrow \overline{V}[GV]$	(*GV*)
(7)	$\Diamond[TV[GV] \wedge \overline{T}\,\overline{V}[GV]]$	(4,5,6)
(8)	$\Diamond V[GV]$	(7, *Ramsey*)
(9)	$\overline{V}[GV] \to \Diamond V[GV]$	(3–8)

The conclusion falls short of the second horn of the validity paradox in the categorical form, $\overline{V}[GV] \to V[GV]$, and thereby of the entire validity paradox, $\overline{V}[GV] \leftrightarrow V[GV]$. The mere logical possibility of the deductive validity of (*GV*) is all that is validly derivable from the assumption that (GV) is deductively invalid, if the modality of deductive inference is weaker than $S5$.

By contrast, we now see how the proof goes through in modal system $S5$ and its conservative extensions. The proof depends on the principle that for any inference φ, $\Box(\varphi \rightarrow \Box\varphi)$, invoked at step (9), according to which it is logically necessary that if an argument is deductively valid, then it is logically necessarily valid, or valid in every logically possible world. The intuitive justification is that the same abstract set of propositions, true or false, for states of affairs that are realized or unrealized in any logically possible world, is ideally available for combination into all the same arguments, and the same logical laws of valid deductive inference standardly prevail, in every logically possible world. The first unproblematic half of the paradox, that $V[GV] \rightarrow \overline{V}[GV]$, is recalled without further ado as the conclusion of *Proof 1*, in step (20). We also appeal to weak standard principles of *Necessitation*, $\Box[\varphi \rightarrow \psi] \rightarrow [\Box\varphi \rightarrow \Box\psi]$, in step (10), and *Duality*, $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$, in step (13). The proof hinges essentially on the characteristic axiom of modal $S5$, $\Diamond\Box\varphi \rightarrow \Box\varphi$, introduced in step (16).

PROOF 3 *Invalidity Horn of the Validity Paradox in S5*

(1)	$\forall x[Vx \leftrightarrow \Box[\forall y[Ayx \wedge Ty] \rightarrow [\forall y[Cyx \rightarrow Ty]]]]$	*Validity*
(2)	$\forall x[\overline{V}x \leftrightarrow \Diamond[\forall y[Ayx \wedge Ty] \wedge \exists y[Cyx \wedge \overline{T}y]]]$	(1)
(3)	$\overline{V}[GV]$	*Assumption*
(4)	$\Diamond[\forall y[Ay[GV] \wedge Ty] \wedge \exists y[Cy[GV] \wedge \overline{T}y]]$	(2,3)
(5)	$\Diamond[\exists y[Cy[GV]] \wedge \overline{T}y]$	(4)
(6)	$\forall y[TCy[GV]] \leftrightarrow \overline{V}[GV]$	(GV)
(7)	$\Diamond\overline{T}\,\overline{V}[GV]$	(5,6)
(8)	$\Diamond V[GV]$	(7, *Ramsey*)
(9)	$\Box[V[GV] \rightarrow \Box V[GV]]$	$\Box(\varphi \rightarrow \Box\varphi)$
(10)	$\Box[\varphi \rightarrow \psi] \rightarrow [\Box\varphi \rightarrow \Box\psi]$	*Necessitation*
(11)	$\Box[\neg\Box V[GV] \rightarrow \neg V[GV]] \rightarrow [\Box\neg\Box V[GV] \rightarrow \Box\neg V[GV]]$	(10)
(12)	$\Box[V[GV] \rightarrow \Box V[GV]] \rightarrow [\neg\Box\neg V[GV] \rightarrow \neg\Box\neg\Box V[GV]]$	(11)
(13)	$\Box[V[GV] \rightarrow \Box V[GV]] \rightarrow [\Diamond V[GV] \rightarrow \Diamond\Box V[GV]]$	(12, *Duality*)
(14)	$\Diamond V[GV] \rightarrow \Diamond\Box V[GV]$	(9,13)
(15)	$\Diamond\Box V[GV]$	(8,14)
(16)	$\Diamond\Box V[GV] \rightarrow \Box V[GV]$	$(S5)$
(17)	$\Box V[GV]$	(15,16)
(18)	$V[GV]$	(17, $\Box\varphi \rightarrow \varphi$)
(19)	$\overline{V}[GV] \rightarrow V[GV]$	(3–18)
(20)	$V[GV] \rightarrow \overline{V}[GV]$	(*Proof 1*)
(21)	$V[GV] \leftrightarrow \overline{V}[GV]$	(19,20)

5 Validity, Necessity, and Deductive Inference

The validity paradox can only be avoided by disallowing formulations of the modality governing the logical necessity of deductively valid inference as strong as or stronger than $S5$. The fact that the validity paradox goes through in modal $S5$ and stronger logics, but not in weaker systems, suggests that the modality of deductive inference, on pain of contradiction in the derivation of inferences that are deductively valid if and only if they are deductively invalid, must be weaker than *S*5. Needless to say, the status of deductively valid inference in *S*5 is also thereby placed in doubt.

If *S*5 itself is redefined to embody a sufficiently nonstandard model of deductively valid inference that avoids the validity paradox, then it might be possible to interpret the modality of deductively valid inference in terms of such an appropriately nonstandard *S*5. The defender of *S*5 as the modality of deductive validity nevertheless cannot reasonably appeal to the intuition that a deductively valid inference accessible from the actual world ought to be deductively valid in every logically possible world accessible from any logically possible world. An equivalence relation for accessibility provided for the model set theoretical semantics of *S*5, involving reflexivity, symmetry and transitivity, must be adequate even for deductively valid inferences involving modal structures in which not all models contain all the same objects. It must be adequate, indeed, for deductively valid inference in any modal environment weaker than *S*5, and so, by the same reasoning, presumably, weaker than *S*4, and so on, down to the weakest modal logic. The conclusion to which the provability of the validity paradox in *S*5 ultimately points is that the modality of deductively valid inference in general cannot be stronger than that formalized by the weakest modal system interpreted only as reflexive world-accessibility.

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