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Varieties of Consequence

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I

Contemporary – metamathematical – logic operates with two kinds of consequence. In both cases the consequence in question is a relation among (sets) of well-formed formulae (wffs) in a certain formal language $\mathcal{L}$. In order to keep my exposition maximally simple I shall first consider a language for the propositional calculus, using only the connectives $\supset$ ('implication') and $\perp$ ('absurdity') as primitive, and with

$$p_0, p_1, p_2, \ldots, p_k, \ldots,$$

as propositional letters.

The (well-formed formulae of the) formal language $\mathcal{L}$ are given by a standard inductive definition:

(0) $\perp$ is an (atomic) wff in $\mathcal{L}$.
(1) p_k is an (atomic) wff in $\mathcal{L}$, for every $k \in N$.
(2) When A and B are wffs $\mathcal{L}$, then so is $(A \supset B)$.
(3) There are no other wffs in $\mathcal{L}$ than those one obtains through finitely repeated applications of (0)–(2).

The clauses (0) and (1) are the *basic* clauses for the inductive definition, whereas the clause (2) constitutes the *inductive* clause. Jointly they tell us what to put into the inductively defined class. The clause (3), finally, is the *extremal* clause, that tells us what to exclude from the class in question. (In languages with such a sparse collection of primitive notions, the other standard connectives are defined in the usual way from $\supset$ and $\neg$ ('negation'), where the stipulatory definition

$$\neg A =_{\mathrm{def}} (A \supset \perp)$$

takes care of the negation.)

Both kinds of consequence are inductively defined with respect to the build-up of the well-formed formulae of the language in question. The first notion, which is the later one from a chronological point of view, is *semantical* in that it makes use of interpretations, or models, for the formal language $\mathcal{L}$.

We consider the two Boolean truth-values **T**(rue) and **F**(alse). A *valuation* v is a function from N to {**T**, **F**}. This valuation v is then extended to a valuation v* for all of $\mathcal{L}$ via the following inductive definition:

(0) $v^*(\perp) = \mathbf{F}$, that is, from a contentual point of view, absurdity is false (under any valuation);
(1) $v^*(p_k) = v(p_k)$ (which value $\in \{\mathbf{T}, \mathbf{F}\}$);
(2) $v^*(A \supset B) = \mathbf{T}$ when $v^*(A) = \mathbf{T}$ implies that $v^*(B) = \mathbf{T}$, and $= \mathbf{F}$ otherwise.

A valuation v such $v^*(\varphi) = T$ is a *model* of the wff φ. When Σ is a set of wffs in $\mathcal{L}$, we extend v* also to the set Σ

$v^*(\Sigma) = \mathbf{T}$ when $V^*(\psi) = \mathbf{T}$, for all wffs $\psi \in \Sigma$.

Finally we are ready to define the notion of *(logical) consequence*

the *consequent* φ is a (logical) consequence of the set of *antecedents* Σ (in symbols $\Sigma = \varphi$),
iff $v^*(\varphi) = \mathbf{T}$ for any valuation v* such that $v^*(\Sigma) = \mathbf{T}$.

We write

$\psi = \varphi$

for '$\{\psi\} = \varphi$.' (The sign '=' is known as a *turnstile*.)

Accordingly, φ is a (logical) consequence of ψ when every model of a model of ψ is also a model of φ.

On this construal, then, (logical) consequence is a universal notion, defined by means of universal quantification over functions (or sets), since one considers *all* models satisfying a certain condition. (Thus, consequence is refuted by a countermodel, that is, a valuation that makes the antecedent true and the consequent false.) This universality of consequence is a typical feature which is retained also for more complex languages: for instance the above pattern is kept also for the predicate calculus, albeit that the notion of valuation is considerably more intricate in that case.

II

The other notion of consequence for the language $\mathcal{L}$ is *syntactical*, rather than semantical, in character. It is defined, not in terms of truth under all valuations, but in terms of the existence of a 'derivation' from certain 'axioms.'

Any wff of $\mathcal{L}$ that is an instance of one of the following schemata is an *axiom*:

(0) $(A \supset (B \supset A))$;
(1) $((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$;
(2) $(((A \supset \perp) \supset \perp) \supset A)$;

The *theorems* ('derivable' formulae) are then defined via (yet again!) an inductive definition:

(0) Any axiom is derivable (is a theorem).
(1) If $(\varphi \supset \psi)$ and φ are derivable (theorems), then so is ψ.
(2) There are no other theorems than those obtained from repeated applications of (0) and (1).

When the wff φ is derivable we use a single turnstile, rather than the double semantical turnstile '$\models$', and write '$\vdash \varphi$.'

By the above definition every theorem is a theorem in virtue of a *derivation*. Such derivations are in tree form and have axioms at their topmost leaves: there is no other way to commence a derivation save by an axiom. Deeper down the tree is regulated by the rule of *modus ponens*:

$$\frac{\vdash A \supset B \quad \vdash A}{\vdash B.}$$

Properties of *all* theorems can then be established by 'induction over the (length of the) derivation.'

In order to obtain the syntactic notion of consequence we must extend the notion of derivability to 'derivability from assumptions in the set Σ.' We proceed (yet again) via an inductive definition:

(0) φ is derivable from assumptions Σ whenever φ is an axiom;
(1) φ is derivable from assumptions in Σ whenever $\varphi \in \Sigma$;
(2) If $(\varphi \supset \psi)$ and φ are derivable from assumptions in Σ, then so is ψ.
(3) No wff is derivable from assumptions in Σ save by a finite number of applications of (0)–(1).

The syntactic turnstile is then extended to cover also derivability from assumptions: we write '$\Sigma \vdash \varphi$' when φ is derivable from assumptions in Σ. Also theorems from assumptions in Σ have derivations (from assumptions in Σ); such derivations from assumptions in Σ allow as top-formulae, not only axioms, but also wffs from the set Σ. We then see that derivability from assumptions, that is syntactic consequence, does not share the universal form of semantic consequence. On the contrary, syntactic consequence holds in virtue of the *existence* of a suitable derivation. This generation of the syntactic notion of consequence via axioms, rules of inference, and added assumptions is not the only way of proceeding. In the early 1930s, Gentzen and Jaskowski took derivability from assumptions as the basic notion in their systems of natural deduction, using no axioms, but inference rules only, where outright derivability can be defined as derivability from no assumptions.

III

The *soundness* and *completeness* theorems for a formal system relate the semantic and syntactic notions of consequence. Soundness states that every syntactic consequence is also a semantic consequence, while the opposite direction is taken care of by completeness.

The above pattern of semantic and syntactic consequence relations is omnipresent in current metalogic. Predicate logic, second- and higher-order systems, extensions to infinitary languages, modal logics; all and sundry confirm to the basic pattern. In the early days of mathematical logic the syntactic consequence-relation was the primary one. A formal system was given showing how its theorems were generated from axioms via rules of inference. However, as more experience was gained of matters semantical, through the work of Alfred Tarski and his pupils, notably Dana Scott, the semantical perspective gained prominence. Today it is fair to say that the semantical way of proceeding is the more fundamental one, partly also because some logics (systems), such as full second-order logic or the logic of the so-called Henkin-quantifier, do not allow for complete axiomatization.

The extension of the above (excessively simple) notion of valuation to the language of first order logic proved non-trivial. In the case of a first-order language $\mathcal{L}$, containing only the two-place predicate symbol R, the individual constant c, and for simplicity, no further function symbols, we interpret with respect to a relational structure

$$A = \langle A, R^A, c^A \rangle,$$

where the set $A \neq \emptyset$. The problem here is that, in general, the domain of discourse, that is, the set A, contains more elements than can be named by constants of $\mathcal{L}$. This problem – technical, rather than conceptual – was solved by Tarski using *assignments*. An assignment is a function $s \in N \to A$, and the terms of the language $\mathcal{L}$ are evaluated relative to this assignment:

(0) $s^*(c) = c^A$;
(1) $s^*(x_k) = s(k)$.

The formulae are then evaluated in the obvious way mimicking the inductive steps for the propositional calculus in the definition of the three-place metamathematical relation $A =_s \varphi$ – 'the assignment s satisfies the wff φ in the structure A':

(0) $A =_s R(t_1, t_2)$ iff $< s^*(t_1), s^*(t_2) > \in R^A$;
(1) not: $A =_s \bot$:
(2) $A =_s (\varphi \supset \psi)$ iff $\mathbf{A} =_s \varphi$ implies $A =_s \psi$
(3) $A =_s (\forall x_k \varphi)$ iff for all $a \in A$, $A =_{s[a/k]} \varphi$,

where the function $s[a/k] \in N \to A$ is defined by

$$\begin{aligned} s[a/k]\,(m) &=_{def} s(m) \text{ if } m \neq k; \\ &=_{def} a \text{ if } m = k. \end{aligned}$$

One should here note that traditionally, and unfortunately, the double turnstile is used for *two different notions*, namely satisfaction – a three-place relation between a structure A, a wff φ and an assignment s,

and

(logical) consequence – a two-place relations between (sets of) wffs.

The above definitions, with the relativization to varying domains of discourse, are essentially due to Tarski, and were, perhaps, first published in final form only as late as 1957. (Tarski's earlier (1936) work on the definition of logical consequence had left this relativization out of account.) Once this definition of satisfaction is given, the definition of logical consequence also for this extended language of first-order predicate logic is readily forthcoming, namely as the preservation of satisfaction by the same assignment from antecedents to consequent.

IV

The above orgy of inductive definitions, which commenced in his famous work on the definition of truth, was not Tarski's only contribution to the theory of consequence (-relations). Already in 1930 he considered an abstract theory of consequence that was obtained by generalization from the syntactic consequence relation above. We consider a set *S* of 'sentences' and a consequence operator *Cn* defined on sets of sentences. Tarski then uses axioms such as:

(0) $S \neq \emptyset$ and $\mathrm{card}(S) \leq \aleph_0$;
(1) If $X \subseteq S$, $X \subseteq Cn(X) \subseteq S$;
(2) If $X \subseteq S$, $Cn(Cn(X)) = Cn(X)$;
(3) If $X \subseteq S$, $Cn(X) = \cup \{Cn(Y): Y \subseteq X \text{ and } \mathrm{card}(Y) < \aleph_0\}$;
(4) For some $x \in S$, $Cn(\{x\}) = S$.

These axioms are clearly satisfied by the above notion of syntactic consequence: axiom (2) says that using derivable consequences as extra assumptions does not add anything, and axiom (3) expresses that a derivation makes use only of finitely many assumptions, while the absurdity $\perp$ serves as the omniconsequential sentence demanded by axiom (4).

Around the same time, Gerhard Gentzen, building on earlier work by Paul Hertz, gave a formulation of elementary logic in term of *sequents*. A sequent is an array of wffs

$\varphi_1, \ldots, \varphi_k \Rightarrow \psi$.

(In some systems Gentzen allows more then one 'succedent-formula' after the arrow.)

Then, the derivable objects of his sequent calculi are sequents, rather than wffs. Derivations begin with *axioms* of the schematic form

$A \Rightarrow A$,

that is, the wff A is a consequence of, is derivable from, the assumption A. Depending on which kind of calculus one chooses, the derivation then proceeds by adding complex formulae using either (left and right) *introduction*-rules only, in which case we have a *sequent calculus*, or introduction- and elimination-rules, which operate solely to the right of the arrow, in which case we have a *sequential formulation of natural deduction*. For instance, in the sequent calculus as well as in the sequential natural deduction calculus, the (right) introduction rule for conjunction & (where the language has been extended in the usual fashion) takes the form

$$\frac{\Gamma \Rightarrow A \quad \Sigma \Rightarrow B}{\Gamma, \Sigma \Rightarrow A \,\&\, B}$$

where Γ and Σ are lists (or sets, or 'multisets') of wffs, depending on what representation has been chosen for sequents. The left introduction rule has the form

$$\frac{A,\ B,\ \Gamma \Rightarrow C}{A \,\&\, B,\ \Gamma \Rightarrow C}$$

and is justified by (and describes) the natural deduction elimination-rules

$$\frac{\Gamma \Rightarrow A \,\&\, B}{\Gamma \Rightarrow A} \quad \text{and} \quad \frac{\Gamma \Rightarrow A \,\&\, B}{\Gamma \Rightarrow B}$$

If C can be obtained from assumptions A, B, then C can be obtained from an assumption A&B, since, by the elimination rules, from A&B one gets both A and B.

V

The above pattern with two metamathematical consequence relations, one syntactic and one semantic, is present throughout the whole gamut of (metamathematical) logic; it has been carried out for classical logic (and its intuitionistic rival). Among so called 'philosophical logics' not only familiar modal logic(s) and the logic of counterfactual conditionals have been so treated, but also more exotic members of the wide logical family such as doxastic and erotetic logic, relevance logic, paraconsistent logic, and so on, have been brought within the fold. You name your favorite logical system and the chance is very high, indeed, that it has a syntax and semantics, with ensuing soundness and completeness theorems. When the entire pattern cannot be upheld, the semantic definition is generally given pride of place. Soundness of the syntactic consequence relative to the semantic one is a *sine qua non*, whereas completeness of the syntactic rule-system with respect to the semantic consequence is a strong desideratum, naturally enough, but cannot always be guaranteed. As already noted, full second order logic, with quantification over really *all* subsets of the universe cannot be effectively axiomatized with decidable axioms and rules of inference. As is well-known (from the work of Richard Dedekind), using full second-order quantification, it is possible to characterize the natural numbers up to isomorphism. Thus, in view of Tarski's theorem concerning the arithmetical undefinability of arithmetical truth, theoremhood in second-order logic cannot be arithmetical, much less recursively enumerable. So, therefore, there are no appropriate syntactic characterizations of this prior semantic notion of second-order logical truth and consequence. This failure – unexpected, unavoidable, and unwanted – of completeness in full second-order logic holds with respect to a prior more or less 'natural' semantics. Sometimes though, especially in the case of various (artificial) modal and tense logics, the opposite direction poses the challenging task of actually designing syntactic rule-systems that (provably) have no complete semantics of a given kind. Such constructions, though, are of limited philosophical interest in themselves. To my mind, they can be compared to the construction of pathological counter-examples in real analysis, for example of a non-differentiable contiunuous function: *that* there is such a function is interesting, but the function itself is not very interesting.

VI

The wffs are considered solely as metamathematical objects and also their 'interpretation' was metamathematical rather than semantic, that is, no proper meaning has been assigned to the formulae. When considering natural-language interpretations of the formal calculi, I shall use the following terminology. An assertion is commonly made through the utterance of a declarative that expresses a statement. (This is not to say that every utterance of a declarative is an assertion; it is, however, a convention concerning the use of language that an utterance of a declarative, in the absence of appropriate counter-indications, is held to be an assertion.) The content of the statement expressed by a declarative is a proposition. Propositions can be indicated by means of nominalized declaratives, that is, by *that clauses*. Thus, for instance,

that snow is white,

is a proposition. However, one cannot make an assertion by means of a proposition only; for this we need to add

. . . is true,

to the that-clause, in order to get a statement in declarative form, by means of which an assertion can be effected. Thus

that snow is white is true

is the explicit form of the statement expressed by the declarative

snow is white.

The content of the statement in question is the proposition that snow is white. Thus the declarative *snow is white* expresses the statement that snow is white is true

which has the proposition *that snow is white* as its content.

When the wffs are interpreted as propositions, they may be thought of as that-clauses, that is, nominalizations of declarative natural language sentences, such as

that snow is white and *that grass is green*,

an *implication* wff $(\varphi \supset \psi)$ is interpreted as, for instance, the proposition

that *that snow is white* implies *that grass is green*,

which is the same proposition as

the implication of *that snow is white* and *that grass is green*.

The sequent

$\varphi \Rightarrow \psi$,

on the other hand, is then interpreted as the consequence-statement

that grass is green is true under the assumption (on condition, provided) that *that snow is white* is true.

However, the statement

that snow is white is true

is the same as the statement

snow is white,

that is, the same assertion would be effected by uttering either.

Accordingly, the above consequence-statement is the same statement as

grass is green under the assumption (on condition, provided) that snow is white,

or, indeed, in conditional form,

if snow is white, then grass is green.

(Thus, I take it, these example show that 'implication' is different from the conditional 'if . . . , then ____'; an implication takes propositions (that is, what that-clauses stand for) and yields a statement, whereas the conditional takes statements and yields a statement. Finally, in order to saturate the expression 'the implication of . . . and ____,' two that-clauses are needed, and we then get a term that stands for an implicational proposition.)

VII

Both approaches to consequence – semantic and syntactic – have counterparts in a long-standing logical tradition. In order to understand this it is necessary to memorize one of the decisive steps in the development of logic that was taken by Bolzano in his monumental *Wissenschaftslehre*, Theory of Science, from 1837. There he discarded the traditional, two-term form of judgement [S is P] and replaced it with the unary form

the proposition A is true.

Bolzano's propositions are *Sätze an sich* and serve as contents of judgments. Frege, indeed, used "judgable content" for the very same notion. They are independent of any *Setzung* whether by mind or language and do not belong in the physical or mental realm, but belong to a platonic third realm, having no spatial, temporal, or causal features. Thus they exhibit the same pattern as Frege's *Gedanken* ("Thoughts"), that is, the judgable contents in a later guise. Bertrand Russell, in what is surely the worst mistranslation in the history of logic, rendered Frege's "Gedanke" as "proposition" in *The Principles of Mathematics*, and he and G. E. Moore, who had inspired Russell's use, bear the responsibility for the resulting confusion. Throughout the earlier logical tradition the term *proposition* was invariably used for speaking about judgments and not about their contents. This (unacknowledged and maybe even unwitting) change in the use of the term has had dire consequences for our understanding of the theory of inference.

In the late middle ages (±1300) a novel genre was added to the logical repertoire. Around that time tracts "on consequences" (*De Consequentiis*) begin to appear, in which the theory of inference was treated differently from what had been common up till then. The old theories had been squarely syllogistic in nature, studying (what amounts essentially to) Aristotelian term-logic, whereas now one begins to find treatments of a more propositional kind. Today, introductory courses in logic commonly teach students to look for 'inference indicators' when analyzing informal arguments. Typical such indicator-words are

therefore, *thus*, *whence*, *hence*, *because*, and, sometimes even, *if . . . , then*.

The medieval *consequentia* were at least of four kinds and knew the indicator words:

(i) *si* (if): If snow is white, then grass is green;
(ii) *sequitur* (follows from): That grass is green follows from that snow is white;
(iii) *igitur* (therefore): Snow is white. Therefore: grass is green;
(iv) *quia* (because): Grass is green.

Note that these words did not serve to indicate different notions: on the contrary, all four point to one and the same notion. Thus the laws for *consequentia* should hold under all four readings. Today, it must be stressed, it would seem more apposite to distinguish four different notions, rather than to have the four versions of the medieval notion:

(i′) *Conditional*, which forms a statement out of statements;
(ii′) *Consequence*, which forms a statement out of (modern) propositions;
(iii′) *Inference*, which is a passage from known statement(s) to a statement;
(iv′) *Causal grounding*, which is a relation between state of affairs or events.

The medievals applied a single correctness-notion *tenere* (holds) to *consequentia*. Each of the four current notions, however, matches its own correctness-notion. The appropriate notions of correctness are, respectively:

When correct
(i″) conditionals are *true*;
(ii″) consequences *hold;*
(iii″) inferences are *valid*;
(iv″) causal groundings *obtain*.

Unless we wish to follow the medieval pattern, we shall have to inquire into the conceptual order of priority, if any, among the various kinds of *consequentia* and their matching correctness notions. It will then also prove convenient to add a fifth notion, namely

(v) Implication, which takes two propositions and yields a proposition, namely: the implication of that snow is white and that grass is green
[= the proposition that *that snow is white* implies *that grass is green*].

Here the appropriate correctness notion is *truth* (for propositions), naturally enough.

VIII

Aristotle, in the *Posterior Analytics*, imposed three conditions on the principles that govern demonstrative science: ultimately, a proof, or demonstration, has to begin with principles that are (1) general, (2) *per se*, and (3) universal. The generality in question means that first principles should be in a completely general form: they speak about all things of a certain kind. Particular knowledge of particulars does not constitute the right basis for logic. To some extent the demand for universality is related to this: it comprises a demand for topic-neutrality. The general principles must be applicable across the board; not only within geometry or arithmetic or biology, but in any discipline. These demands for generality and universality on the basic principles of demonstrative science have a counterpart in one of the ways in which the medieval treated of the validity of inference, namely the *Incompatibility* theory. It goes back to Aristotle's *Prior Analytics* and was perhaps first clearly enunciated in the Stoic propositional approach to logic. It was firmly upheld by Parisian logicians in the early fifteenth century. The general inference I:

$$\frac{J_1 \ldots J_k}{J}$$

is held to be valid if the *truth of the premises* $J_1, \ldots, J_k$ is *incompatible* with *the falsity of the conclusion* J. Thus, by trivial computation in Boolean and modal logic, we get

[A is true. *Therefore*: B is true] is valid
iff
[A true **and** B false] are incompatible
iff
$\neg\Diamond$[A true **and** B false]
iff
$\Box\neg$[A true **and** B false]
iff
$\Box$ [**if** A true, **then not**-(B false)]
iff
$\Box$ [**if** A true, **then** B true].

The question is now how the modal box '$\Box$', that is, the necessity in question, should be interpreted. One natural way of proceeding here is to take necessity in the sense of 'holds in all alternatives.' This was done by an influential school of medieval logicians, who read the universality and topic neutrality as 'holds *in omnis terminis*' and so the logically valid is that which holds in all terms. The above chain of equivalences the continues:

for any variation $'$ with respect to sub-propositional parts
if A' true, **then** B' true.

This is how Bernard Bolzano defined his notion of *Ableitbarkeit* (consequence) in 1837; note that this is a *three*-place relation between antecedent(s), consequent(s) and a collection of ideas-in-themselves (that is, the relevant sub-propositional parts, with respect to which the variation takes place). *Logische Ableitbarkeit* – *logical* consequence – then involves variation of with respect to all non-logical sub-propositional parts. Similarly, Bolzano held that a *Satz an sich*, that is, a proposition, was 'analytic in the logical sense' if the proposition remained true with respect to arbitrary variation at all nonlogical sub-propositional parts. One century later, essentially the same characterization of logical truth was offered by Ajdukiewicz and Quine (for *sentences* though, rather than Bolzano's propositions).

To some extent this notion of truth under variation is captured by the modern model-theoretic notion. The parallel is not exact, though. In the Bolzano–(Ajdukiewicz–Quine) conception variation takes place with respect to the proposition (or sentence), whereas in the semantic, model-theoretic notion what is varied is *not* the metamathematical counterpart to the proposition (sentence), that is, the well-formed formula. On the contrary, the variation takes place with respect to the relational structure A. Thus, if anything, it is the *world*, rather than the *description* thereof, that is varied. Thus, the notion of a *tautology*, that is, a proposition of logic, from Wittgenstein's *Tractatus* is a better contentual counterpart to the model theoretic notion of logically true wff. A tautology is a proposition which is true, come what may, independently of what is the case in the world (irrespective of how the world is or of what states of affairs obtain in the world), and similarly for the notion of consequence. This onto-logical conception of logical truth and validity seems to me to capture best the intuitions that are formalized in the model-theoretic notion of semantic consequence.

IX

Given Bolzano's form of judgment, the general form of inference I is transformed into I′:

$$\frac{A_1 \text{ is true} \ldots A_k \text{ is true}}{C \text{ is true.}}$$

Bolzano reduces the validity of this inference I′ to the *logical* holding of the sequent

$$A_1, \ldots, A_k \Rightarrow C.$$

This, in turn, is equivalent to that

$$A_1 \& \ldots \& A_k \supset C \text{ is logically true.}$$

This reduction is exactly parallel to his reduction of the correctness ('truth') of the statement to the *propositional* truth of the content A. Bolzano here says that the judgment [A is true] is correct (*richtig*) when the proposition A really is true. Stronger still, the judgment

[A is true] is (a piece of) knowledge (*ist eine Erkenntnis*)

when the proposition A is true. This, however, admits of the unpalatable consequence that blind judgments are knowledge, irrespective of any epistemic grounding. (The apt term *blind judgment* was coined by Franz Brentano.) The statement

The Palace of Westminster has 1,203,496 windowpanes

is *knowledge* if by fluke, but not by telling, I have happened to choose the right number when constructing the example, that is, if the proposition

that The Palace of Westminster has 1,203,496 windowpanes

is a propositional truth (*an sich*, as Bolzano would say).

Entirely parallel considerations yield that also *blind* inference, without epistemic warrant, is valid under the Bolzano reduction. This, to me, is sufficient to vitiate the Incompatibility theory with its Bolzano reductions and thus I prefer to search for other accounts of validity that do not allow for the validity of blind inference. One such is readily forthcoming in the *Containment* theory. This also has Aristotelian roots, was perhaps first adumbrated by Peter Abailard, and was squarely defended by 'English logicians' at Padua in the fifteenth century. Here an inference is valid if the truth of the conclusion is somehow analytically contained in the truth of premises. The Bolzano-reduction reduced the correctness of a judgment to the propositional (bivalent) truth of its content. This, while pleasingly simple, leaves the vital epistemic justification completely out of the picture and it was left to Franz Brentano to suggest an evidence theory of correctness ('truth') for statements:

a statement is correct if it can be made evident.

Correctness, or truth, at the level of statements (judgments), is accordingly a modal notion.

Indeed, at this level, the equation

true = evidenceable, knowable, justifiable, warrantable,

holds. It must be stressed here that it is at the level of what is known that correctness coincides with knowability. A true, or correct, statement is knowable, but one must not export this to the propositional content of the statement in question. The object of the act of knowledge is a judgment concerning the truth of a propositional content and it is the statement which is knowable if correct. The notion of propositional truth, whether bivalent or not, is not couched in terms of knowability; propositions are not the objects of (acts of) knowledge.

X

Turning now to the validity of inference, we recall that the premise(s) and conclusion of the completely general inference-figure, inference I, are statements (judgments). Accordingly the appropriate notion of truth to be used here is that of knowability, and the inference I has to preserve knowability from premise(s) to conclusion. Thus, one has to know the conclusion under the assumption that one knows the premise(s). In other words, the conclusion must be made evident, given that the premise(s) have been made evident. This is now where the insights of the containment theory come to aid. All (true, correct, that is) knowable judgments can be made evident, and for some judgments their evidence(ness) rests ultimately upon that of other evident judgments. Certain correct judgments, though, are such that their evidenceability rests upon no other judgments than themselves: these are judgments which are *per se nota*, or analytic in the sense of Kant. The can be known *ex vi terminorum*, in virtue of the concepts out of which they have been formed. Axioms in the original (Euclidean, but not Hilbertian, hypothetico-deductive) sense are examples of this: they can be known but they neither need nor are capable of further demonstration by means of other judgments. In the same way certain inferences are 'immediately' evident upon knowledge of the constituent judgments. Note though that the immediacy is not temporal but conceptual; the inference in question neither can nor needs to be justified in terms of further inferences. The introduction and elimination rules in the natural-deduction systems of Gerhard Gentzen are examples of such immediate inferences.

The validity of an inference is secured by means of the (constructive) existence (= possibility to find) of a chain of immediately evident inferences linking premise(s) and conclusion: the transmission of evidence finds place by means of immediate evident steps that are such that when one knows the premise(s) and understands the conclusion nothing further is needed in order to know the conclusion. The possession of such a chain guarantees that the conclusion can be made evident under the assumption that the premises have been made evident, that is, are known. Mere possession, though, does not suffice for drawing the inference; in order to know the conclusion I must actually have performed the immediate inferences in the chain. Thus, the modern notion of syntactic consequence, under the containment theory of inferential validity, has a counterpart in the chain of immediate inferences that constitutes the ground for the validity of an inference.

Thus, we have found a difference between inferences and consequence: a correct consequence, be it logical or not, preserves truth from antecedent propositions to consequent proposition (possibly under all suitable variations) whereas a valid inference-figure preserves knowability from premise-judgment(s) to conclusion judgment. In particular, the inference Σ:

$$\frac{A \Rightarrow B \text{ holds} \quad A \text{ is true}}{B \text{ is true}}$$

is valid; indeed, the holding (but not the *logical* holding, under all variations) of the sequent $A \Rightarrow B$ is explained in such a way that the inference from the truth of proposition A to the truth of proposition B is then immediate. Thus the holding of sequents is reduced to, or explained in terms of, the validity of inference.

An attempt, on the other hand, to reduce the validity of inference to the (possibly logical) holding of consequences to validity will engage us in an infinite regress of the kind that Lewis Carroll ran for Achilles and the Tortoise in *Mind* 1895. Then the inference

A is true. *Therefore*: B is true

is valid if the sequent $A \Rightarrow B$ holds. But the inference

$A \Rightarrow B$ holds, A is true. *Therefore*: B is true

is certainly valid by the explanation of $\Rightarrow$: $A \Rightarrow B$ holds when B is true if A is true. Thus, by the reduction of validity, the (higher-level!) sequent

$[A \Rightarrow B, A] \Rightarrow B$

must hold. But then the inference

$[A \Rightarrow B, A] \Rightarrow B$ holds, $A \Rightarrow B$ holds, A is true. *Therefore*: B is true

is valid. Thus, by the reduction of inferential validity to that of holding for consequence, the (even) higher-level sequent

$$[[A \Rightarrow B, A] \Rightarrow B, A \Rightarrow B, A] \Rightarrow B$$

must hold. But then, yet again, a certain inference is valid and so we get a tower of ever higher-level consequences that have to account for the validity of the first inference in question.

The criticisms that have been voiced against Frege's account of inference, on the present analysis, are nugatory. Frege was absolutely right in that inference proceeds from known premises and obtains new knowledge. This is also accounted for by the explanation of validity. The conclusion-judgment must be made evident, given that – under the assumption that – the premises are known. In the (logical) holding of consequence, on the other hand, there is no reference to knowledge: a sequent holds if propositional truth is transmitted from antecedent(s) to consequent. Thus, the criticisms of Frege seem to stem from a conflation of (the validity of) inference with (the holding of) consequence.

Further Reading

Bolzano, Bernard (1837) *Wissenschaftslehre*, I–IV. Sulzbach: J. Seidel. English translation edn. by Jan Berg: *Theory of Science*, Dordrecht: Reidel, 1973 (especially §§ 34, 36, 148 and 155).

Enderton, Herbert (1972) *A Mathematical Introduction to Logic*. New York: Academic Press. (Rigorous, yet accessible, treatment of the model-theoretic notion of consequence from a metamathematical perspective.)

Gentzen, Gerhard (1969) *Collected Papers*, ed. Manfred Szabo. Amsterdam: North-Holland. (Contains the original papers on the sequent calculus and natural deduction systems.)

Monk, Donald (1976) *Mathematical Logic*. Berlin: Springer. (Contains pellucid expositions of model-theoretic consequence. Possibly hard for philosophers.)

Sundholm, Göran (1998) Inference, consequence, implication. *Philosophia Mathematica*, 6, 178–94. (Spells out the present framework in greater detail.)

Tarski, Alfred (1956) *Logic, Semantics, Metamathematics*. Oxford: Clarendon. (Contains his classical papers on truth and consequence.)