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Philosophical Implications of Logical Paradoxes

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Dr. Seuss' *On Beyond Zebra* opens with a young boy proudly writing on a blackboard. Conrad Cornelius o'Donald o'Dell, has demonstrated his exhaustive knowledge of the alphabet: A is for Ape, B is for Bear, . . . and Z is for Zebra. An older boy compliments Conrad. He breezily concedes to young Conrad that *most* people stop with Z. But *his* alphabet continues beyond Z. The extra letters let him spell new things. The older boy thus introduces Conrad to an otherwise inaccessible realm of exotic creatures. For instance, the Q-ish letter quan is for the vertically symmetric Quandary who lives on a shelf

> In a hole in the ocean alone by himself
> And he worries, each day, from the dawn's early light
> And he worries, just worries, far into the night.
> He just stands there and worries. He simply can't stop . . .
> Is his top-side bottom? Or bottom-side top?

A metaphysician who tags along on the tour given to Conrad will be reminded of other never-never lands.

1 Paradoxes Stimulate Theory Development

Why aren't statements that assert the existence of an entity trivially true? If 'Santa Claus exists' is false, then the sentence must be about Santa Claus. 'About' is a two-place relation, so the sentence is about Santa only if there is a Santa. But then 'Santa Claus exists' is true! Alexius Meinong challenged the transition from 'There is a Santa' to 'Santa exists.' Meinong believed there is a complex domain of nonexistent objects that have been neglected by metaphysicians much as astronomers long neglected dark matter. Meinong's metaphysics illustrates how a logical paradox can stimulate the development of a philosophical theory.

The excesses of such a theory provoke debunkers. After initial sympathy with Meinong, Bertrand Russell (1957) traced belief in nonexistent objects to an illusion about 'the.' 'The present king of France is bald' appears to refer to the present king of

France. Russell dismantled this appearance. His theory of descriptions underwrites the synonymy of 'The present king of France is bald' and 'There is exactly one king of France and whoever is king of France is bald.' The first conjunct of this sentence can be denied without referring to any particular entity.

The hypothesis that names are disguised definite descriptions subsumes 'Santa Claus does not exist' under Russell's theory. Subsequent philosophers supported Russell's linguistic thesis with increasingly sophisticated proposals as to what the disguised definite description is. Typically, deflationary accounts generate auxiliary linguistic theories and distinctions. After all, logic can be applied only with the help of assumptions about how key locutions operate. Even those who do not accept the dissolutions (whether through substantive doubts or indifference toward the motivating problem) have been impressed by some of these auxiliary theories. Linguists were especially quick to incorporate the great handmaiden of logic, H. P. Grice's theory of conversational implicature.

When the theory development takes place within logic itself, the result is a powerful constraint on all future theorizing. Any scientific result is a constraint in the sense that it constitutes grounds against any future theory that conflicts with the result. But most scientific results are domain specific and of a limited degree of necessity. For instance, it is physically impossible to make a left shoe into a right shoe by turning it over through a fourth dimension. But crystallographers, topologists and philosophers can coherently study this kind of mirror reversal. They are interested in a wider domain of possibility. Logic is at the limit of this scale of possibility. Consequently, logical impossibilities are maximally coercive. Even the skeptic is careful to keep his scenarios within the bounds of logical law.

2 An Analogy with Perceptual Illusions

Russell (1957: 47) tested his theory of descriptions by how it handled related paradoxes such as the surprising informativeness of identity statements. He advised logicians to keep a stock of logical puzzles on the grounds that these play the same role as experiments do for scientists.

Some logicians are unimpressed with Russell's analogy. They adopt the same attitude toward logical paradoxes that the perceptual psychologist J. J. Gibson took toward perceptual illusions. According to Gibson's ecological view, perception must be understood within the perceiver's natural environment, as an adaptation toward practical ends. Gibson dismissed perceptual illusions as largely irrelevant, confined to picture books and computer generated toy worlds.

After the entrenchment of Aristotle's logic, little distinction was made between sophistry and logical paradoxes. The liar paradox and the sorites paradox (which are in high esteem today) were regarded by almost all subsequent philosophers as isolated curiosities. Medieval logicians are the important exceptions. Their highly structured system of scholarly debate encouraged attention to logic and language. Contrary to stereotype, they approached many philosophical issues with an escape artist's mix of imagination and rigor. Pseudo-Scotus' paradox of validity and Jean Buridan's variants of the liar have found their way back into academic publications – and not just for their antiquarian value.

Currently, the dominant view in psychology is that perceptual illusions are more than an entertaining sideline. As Hermann von Helmholtz said a century ago, illusions are anomalies that provide clues as to how normal perception originates. Each sense is a package of rough and ready modules that evolved to achieve collective reliability, not individual reliability. Illusions arise when experimenters isolate systems under laboratory conditions or venture into environments and circumstances alien to our hunter-gatherer heritage. If the detection of validity is also a 'bag of tricks,' then fallacies should provide clues about normal reasoning. And indeed, psychologists have had much to say about 'cognitive illusions' (ignoring base rates, confirmation bias in the four card selection task, the Monty Hall problem). Yet they have had little to say about logical paradoxes.

The silence of the psychologists is an anomaly for those who believe that there is only a quantitative difference between brainteasers of recreational logic and deep logical paradoxes. Logical pedagogy reflects this gradualist sentiment. Raymond Smullyan's *What is the name of this book?* starts at the shallow end of the cognitive pool and then moves cheerfully and continuously into deeper waters. There is no sharp line between non-philosophical puzzles that have algorithmic solutions and philosophical problems in which we only have a hazy idea of what would even count as progress.

The gradualist might venture a random walk theory of profundity. If we are restricted to trial and error, it is statistically easier to solve an n step problem than an n + 1 step problem. The probability of solution is a geometrical rather than an arithmetic function of the number of steps needed for a solution. Consequently, a problem that is a few more steps from resolution will have a much lower probability of solution. These recalcitrant problems are apt to be perceived as qualitatively different. In addition to being more difficult to solve, the more complex problems will be less detectable. Lewis Carroll produced thousands of unmemorable logic exercises but discovered only a handful of logical paradoxes.

Gradualism breeds optimism about the ultimate solubility of philosophical problems. If profound problems differ only in degree from solvable brainteasers, then philosophical progress is probable. The appearance of stagnation would be best explained as a sampling illusion: as soon as philosophical problems get solved, they get exported to some other field.

However, this optimism about philosophical progress comes at the price of deflationism about philosophy. If philosophical problems are just highly complicated brainteasers, then why expect their solution to be more illuminating than the solution of a highly complex brainteaser? Lewis Carroll's immense corpus of puzzles contains counterexamples to the thesis that sheer complexity always generates an appearance of profundity. To divert his Victorian mind from unwelcome thoughts, the sleepless Carroll carried certain puzzle genres to awesome lengths. One genre involves inconsistent story telling in which a contradiction can be derived from n statements in the story but not any n – 1 of those statements. Carroll has stories in which n = 25 and even one in which n = 50. These look like clerical feats rather than deep thinking.

To constitute a paradox, a problem must be an apparent counterexample to an attractive principle. Arthur Prior's runaway inference ticket, 'tonk,' is paradoxical because it is a counterexample to the conventionalist thesis that the meaning of the logical connectives is dictated by the truth-tables. Nelson Goodman's new riddle of induction is a paradox because it refutes the assumption that induction is topic neutral. Hilary Putnam contends that Heisenberg's uncertainty principle refutes the principle of distribution $(A \& (B \vee C) \supset (A \vee B) \& (A \vee C))$.

Anti-gradualists are apt to interpret the silence of the psychologists as a sign that there is a qualitative difference between logical paradoxes and the shallower fare of howlers, forehead slappers, and joke demonstrations that $1 = 0$. Perhaps shallow logical errors are just performance errors. The subjects tested by psychologists only have a short time to answer the test questions and are vulnerable to distraction, memory overload, etc. Logical paradoxes, in contrast, have withstood the scrutiny of motivated, leisurely study by experts. The error occurs at the level of theory rather than implementation. Or perhaps the deep logical paradoxes reflect some master cognitive flaw – something akin to the transcendental illusion (of applying phenomenal categories to noumena) that Immanuel Kant postulated to explain the antinomies of space and time.

Possibly, the logical paradoxes will help us discover a single grand truth that explains the mix of anomalies. The random walk theory makes the opposite prediction that the paradoxes will only have the coincidental patterns that one normally finds in a well-shuffled deck of cards.

3 Do Logical Paradoxes Exist?

Kant believed there were no logical paradoxes. This is evident from his preface to the second edition of the *Critique of Pure Reason*. In Kant's opinion, Aristotle had successfully grasped the basic truths of logic in his theory of the syllogism just as Euclid had grasped the basic truths of geometry with his axiomization. There are geometrical sophisms and questions of application. But there are no anomalies within the theory itself. After Aristotle, logic "has not had to retrace a single step, unless we choose to consider as improvements the removal of some unnecessary subtleties or the clearer exposition of its doctrine, both of which refer to the elegance rather than to the solidity of the science. It is remarkable also, that to the present day it has not been able to advance a step and is thus to all appearance complete and perfect."

The history of logic refutes Kant. I mean to include the history that preceded Kant (especially the medieval era). Plus the era to which he belonged. But most of all, Kant is refuted by the history that followed him. Logic made great strides after the nineteenth century, often under the stimulus of paradox.

Despite the historical record, there remain strangely rich grounds for doubting the existence of logical paradoxes. For instance, most theories of belief imply that no one can believe a contradiction (Sorensen 1996). Since the paradigm cases of logical paradoxes involve belief in contradictions, the very existence of logical paradoxes is itself paradoxical.

Another difficulty is taxonomic. Paradoxes are classified in terms of the propositions they contain. Olber's paradox of why the night sky is dark is an astronomical paradox because its constituent propositions are astronomical. Presumably, a logical paradox contains logical propositions. However, there are conceptions of 'proposition' and 'logic,' which preclude the existence of logical propositions. In the *Tractatus*, Wittgenstein reserves 'proposition' for statements reporting contingent states of affairs.

Gilbert Ryle regarded logical laws as rules of inference. Since rules *prescribe* how we ought to behave rather than describe how things are, logical laws are neither true nor false. Logic can only be relevant to paradoxes as a means of relating propositions to each other. Adjusting a rule of inference might resolve the paradox in the sense of dissolving the appearance of inconsistency. But there are no logical propositions that can serve as members of a paradox. Or so one might infer.

The sharp distinction between inference rules and premises appears to undermine the notion of a logical paradox. Ironically, this distinction was itself drawn in response to a logical paradox. In 1895, Lewis Carroll published a dialogue in *Mind* between Achilles and Tortoise. The Tortoise will grant Achilles any premise he wishes. But the Tortoise insists that Achilles link the premises to the conclusion via a further premise, which states that if the premises are true, then the conclusion is true. Since this extra conditional is itself a premise, adding it to the premise set forces the addition of a new linking premise. The common solution to this paradox is to deny that any extra premises are needed to link the premises and the conclusion. They are instead linked by an inference rule.

Lewis Carroll's puzzle about Achilles and Tortoise does dramatize the need to distinguish between premises and inference rules. However, it does not refute the basic interchangeability of premises and inference rules. The axiom that p can be considered as an inference rule lets us introduce p without any premises. Carroll's puzzle does show that a system that contains just axioms cannot have any deductions. However, natural deduction systems are practical examples of how a system may have deductions without any axioms.

Once we agree that there are logical paradoxes, there remains the question of which propositions are logical propositions. The sentence-level answer appeals to logical words. Certain statements are guaranteed to have a truth-value by virtue of vocabulary found in logical theories. This vocabulary uncontroversially includes the following: and, or, not, all, some, is. These words are topic neutral, appearing across all domains of discourse – not just in physics or tennis or algebra.

This does little to relieve the surprising amount of disagreement over what qualifies as a logical word. There is consensus that all the vocabulary of first order predicate logic with identity qualifies as logical. There is a debate over whether the introduction of predicate variables (to obtain second order logic) is just disguised set theory. Other marginal examples of logical words tend to be diplomatically treated 'as if' they were logical words. The modal logician declares he will treat 'necessary' as a logical word and thereby obtains a supplemental logic. The same is done for temporal logic (earlier than), mereology (part of), deontic logic (permissible), epistemic logic (know), etc. The longer the list of logical words, the greater the number and variety of logical paradoxes.

It is more natural to characterize logical paradoxes at the theory-level. Given that logic is the *theory* of what follows from what, there will be propositions about propositions. These meta-propositions about the consequence relation will sometimes be individually plausible and yet jointly inconsistent. This conception of a logical paradox accommodates the tendency to include meta-logical paradoxes as logical paradoxes. The Lowenheim-Skolem paradox makes essential use of words that are about logic but which are not logical words.

The sentence-level conception of paradox accommodates the inclusion of puzzles that inadvertently involve a logical truth or logical falsehood. The doctrine of the trinity is sometimes described as a logical paradox because it is a paradox (to many Christians) that involves violations of the law of identity.

Theory-level paradoxes arise out of logical doctrines and intuitions. For instance, we believe logic must handle every possible state of affairs and hence it cannot imply the existence of anything. We also believe in logical laws such as the principle that everything is identical to itself. But since the quantifiers in standard logic have existential import $(x)(x = x)$ entails $(\exists y)(y = y)$. Thus the empty universe is excluded as *logically* impossible. So at least one of these propositions must be false. Which? Metaphysical intuitions have little standing against science or mathematics. Why should logic be any more deferential to metaphysics? Good bookkeeping requires the rejection of empty universe.

4 Imagination Overflows Logical Possibility

Logic is unnervingly forthcoming with respect to the philosophical question "Why is there something rather than nothing?" But logic has a way of building up your nerve. Intuitions and imposing theories about what is possible have both been challenged on logical grounds.

Russell's theory of definite descriptions shows how, in Ludwig Wittgenstein's words, "A cloud of philosophy is condensed into a drop of grammar." Philosophy has no monopoly on fog. Wittgenstein would see an analogy between the realm opened by Dr. Seuss's trans-Z letters and Georg Cantor's 'paradise' of transfinite numbers. The theologian-mathematician Cantor was trying to solve mathematical paradoxes involving counting. On the one hand, there seem to be more natural numbers than even numbers because the even numbers are properly included amongst the naturals. Yet it is possible to put the natural numbers into a one to one correspondence with the even numbers. This mapping indicates the number of even numbers *equals* the number of natural numbers. Instead of dismissing this correspondence as revealing that there is something wonky in the notion of infinity, Richard Dedekind boldly defined 'infinite set' in terms of this paradoxical property of having a proper subset that is as large as itself. Cantor took set theory much further. His innovative diagonal argument showed that the set of real numbers is larger than the set of natural numbers. The argument generalizes to reveal a hierarchy of infinities that obey strange but elegant laws of addition, subtraction, and so forth. Most of those who become familiar with this transfinite arithmetic emerge with Russell's conviction that Zeno's paradoxes now have a mathematical solution. Set theory was speedily erected into a grand unifying theory of mathematics.

What is the difference between 'Cantor's paradise' and the realm the older boy offers Conrad Cornelius o'Donald o'Dell? There cannot be any letters beyond Z because 'The letters from A to Z exhaust the alphabet' is an analytic truth. Of course, one could invent another alphabet in which Z is not the last letter. But then the older boy would not be *correcting* Conrad's initial impression that Z is the last letter. For young Conrad was talking about the standard English alphabet. The suggestion that young Conrad is representationally deprived rests on an equivocation between the established alphabet and a pseudo-alternative. Similarly, Wittgenstein balked at the suggestion that Cantor had discovered numbers that had been previously overlooked by conceptually immature predecessors.

Philosophers have shrunk from Wittgenstein's intimation that Cantor's paradise is as mythical as Meinong's slum of nonexistents. They dwell on the ways in which paradoxes expand our horizons. Russell writes warmly of how Cantor's theory helps us distinguish between contradictions and possibilities that merely contradict our prejudice for thinking in finite terms. W. V. Quine advises us to abandon the quest to translate names into definite descriptions and to instead think directly in terms of Russell's logical notation for definite descriptions. Like any fluent speaker, we no longer need to translate back into our native tongue. Russell has enriched our minds with a new tool of thought. This kind of conceptual advance occurs in all fields. The scale of the effect varies with the centrality of the subject-matter. Since logic is at the center of the web of belief, the implications are wide indeed.

Although one may resist the Wittgensteinian assimilation of Cantor's transfinite numbers to Dr. Seuss's trans-Z letters, all must concede the general point that a cognitive advance often takes the form of an eliminated possibility. Children search their drawers for lost dogs. They scuttle toward the opposite end of the bathtub to avoid being sucked down the drain. This kind of open-mindedness needlessly alarms them and slows their searches. As children mature, their conception of what is possible more closely aligns with what is genuinely possible.

Some paradoxes rest on bogus possibilities. Consider the barber who shaves all and only those who do not shave themselves. Does the barber shave himself? If he shaves himself, then he is amongst those he does not shave. But if he does not shave himself, then he is amongst those he shaves. Contradiction. The universally accepted solution is that we should not assume that it is possible for there is to be a barber who shaves all and only those he does not shave. We need to rein in our imagination.

And not just about barbers. Our imaginations systematically run afoul of J. F. Thomson's theorem:

> Let S be any set and R any relation defined at least on S. Then no element of S has R to all and only those S-elements which do not R to themselves. (Thomson 1962: 104)

If we let S be the collection of men, then this set contains no man who bears the relation of shaving all and only those men who not shave themselves. That dissolves the barber paradox.

Thomson goes on to show how his 'small theorem' is at the root of Kurt Grelling's paradox about 'heterological.' The lesson is that there is no predicate that applies to all only those predicates that do not apply to themselves. This reveals a sobering limit to stipulative definitions. We cannot make the heterological predicate exist by fiat. Thomson could have gone up from predicates to larger linguistic units. In particular, the liar paradox can be seen as a logically impossible sentence (or proposition or thought).

Anyone who takes these rebuffs to intuition seriously will be more disposed to accept a logician's curt answer to "Why is there something rather than nothing?" They will be inclined to assimilate the possibility of an empty universe to the possibility of a barber who shaves all and only those who do not shave themselves. Under this analogy, revising logic to save the possibility of an empty universe is like revising logic to spare the possibility of Russell's barber.

Logically conservative responses to other paradoxes are repressive in some respects and liberating in others. The mix of liberation and suppression can be subtly accomplished at the level of notation. Bertrand Russell's notation in *Principia Mathematica* was intended to explain the possibility of mathematical knowledge by reducing mathematics to logic and set theory (which Russell regarded as a branch of logic). Russell tends to dwell on the doors opened by this notation. But he also gleefully observed that the ontological argument for the existence of God cannot even be formulated in *Principia* notation. This double-edged effect is natural because theories need to show us which possibilities are genuine and which are bogus. A theory is expressively incomplete only when it stops us from saying what we *want* to say.

5 Paradoxes Evoke Logical Analogies

The theme of repression and liberation can also be extended to styles of reasoning. The arbitrary individuals which populate pre-twentieth-century proofs were long known to have conflicts with laws of logic. For instance, an arbitrary number is neither odd nor even and yet an arbitrary number has the disjunctive property of being either odd or even! The anomalies were tolerated for lack of a better alternative. But once Gottlob Frege developed an adequate (though more complicated) quantification theory, arbitrary individuals were unceremoniously jettisoned.

However, the dominant trend has been in the direction of liberation. The paradox's potential for innovation is pregnant in the common definition of a paradox as an argument from incontestable premises to an unacceptable conclusion via an impeccable rule of inference. In Quine's (1966) terminology, some paradoxes are "veridical": their conclusions are true – just surprisingly so. These arguments have promising futures as instructive proofs.

A promising future is not destiny. The true conclusion of the veridical paradox does not guarantee that the argument is sound. Quine neglects the historical point that many veridical paradoxes are fallacious.

The Pythagoreans argued that the earth was a revolving, rotating sphere. Their conclusion is true and was as absurd to their contemporaries as Nicholas Copernicus' conclusion was to his contemporaries. But unlike Copernicus, the Pythagoreans argued fallaciously for their surprising truth. Typically, the brilliant argument for the initially absurd conclusion is only the beginning of a successful proof. The valuable part of the argument is its broad outlines, not its details. In other cases, the brilliant proof is only accidentally correct and is of no lasting value whatsoever.

Even a sound veridical paradox may have flaws. Some are circular. Others are vulnerable to refutation by logical analogy. The basic argument that all identities are necessary truths was regarded as sophistry before Saul Kripke championed it in *Naming and Necessity*. Almost all philosophers believed that physicists had established numerous contingent identities (such as Water = H_2O) and that the curious argument just paralleled Frege's deliberately absurd arguments against unbelieved identities. Kripke rehabilitated this ignored argument that all identities are necessary by offering an attractive alternative interpretation of scientific identities and raising doubts about the logical analogy.

Quine's distinction between veridical and falsidical paradoxes is also non-exhaustive. Consider Frank Ramsey's proof that there are exactly two Londoners who have exactly the same number of hairs on their heads. Ramsey notes that there are fewer than a million hairs on any one's head and there are more than a million Londoners. The conclusion follows by the pigeonhole principle: if there are more pigeons than pigeonholes, then at least two pigeons must share a hole. The existence of like-haired Londoners is not surprising. Even before hearing Ramsey's argument, Londoners agree that there is a high chance that two Londoners have the same number of hairs. What is paradoxical about Ramsey's proof is the *connection* between the premises and the conclusion. Londoners who are unfamiliar with the pigeonhole principle accept the premises and the conclusion but deny that the conclusion is entailed by the premises.

Given the logical interchangeability of propositions and inference rules, one could convert any inferential paradox into a propositional paradox. The paradoxical proposition is the conditional whose antecedent is the conjunction of the premises and whose consequent is the conclusion. Or one could do the reverse, turning propositional surprises into inference surprises. Perhaps, on the model of natural deduction systems, one could turn all propositional paradoxes into inference paradoxes. But since any system that allows deductions must have inference rules, there is an extra obstacle to a universal reduction to propositional paradoxes. In practice, we use systems that will ensure that some paradoxes are propositional while others are inferential.

Some of the most interesting paradoxes are both propositionally and inferentially paradoxical. The epistemicist argument that vague predicates have sharp thresholds has an intrinsically surprising conclusion and a further surprise that there could be a connection with such trivial facts such as 'Bald men are logically possible.'

Fallacious paradoxes are often instructive disasters. They suggest analogous arguments that avoid a critical mis-step while retaining some of the power of the original paradox.

The liar paradox has been especially fertile. Kurt Gödel's incompleteness theorems are self-conscious, delicately re-moldings of the Richard paradox. Alan Turing's first example of an uncomputable function, the halting problem, was based on the liar. Gregory Chaitin's (1986) theorem that a computer cannot fully predict its own performance was based on Berry's paradox. The liar paradox contains a powerful style of reasoning that does not inevitably ignite into contradiction. Like engineers using dangerous explosives to safely demolish buildings, meticulous thinkers gingerly titrate the paradoxical reasoning in their refutations of completeness or computability or predictability.

When Russell (1917) was calculating how many things are in the universe, he was led to a set that included everything. The number of things in this set must be the largest number because there is nothing further to add! Russell therefore accused Cantor of committing some subtle fallacy in his proof that there is no largest number.

A resemblance gave Russell second thoughts. The self-referential aspect of the universal set evokes a liar paradoxical set – a set that includes all and only those sets that do not include themselves as members. If this set contains itself as a member, then it does not contain itself as a member. But if it does not contain itself as a member, then it does include itself as a member. Contradiction. The set cannot exist! Accordingly, Russell repudiated his objection to Cantor's proof.

Contradictions hurt. Russell sent news of the paradox to Gottlob Frege just as Frege's magnum opus on arithmetic was going to press. Frege hastily inserted a patch-up appendix. After this debacle, Frege never contributed anything of significance. Frege thought that we had infallible access to logical truths by intuition. Russell's paradox shows that we can have a clear intuition that something is possible even though it is demonstrably impossible.

Russell's paradox shows that naive set theory must be revised in a way that restricts the formation of sets. Accordingly, mathematicians have developed powerful set theories that unintuitively restrict the formation of sets. In particular, Zermelo-Fraenkel set theory has achieved the main objectives envisaged by the founders of set theory. But it has been a stop and go exploration. Each spurt ahead is accompanied by a look-about for unexpected trouble.

6 An Implication about the Nature of Paradox

An alternative definition of 'paradox' is as a set of individually plausible but jointly inconsistent propositions. This definition needs size limits to avoid counting Lewis Carroll's clerical inconsistencies as paradoxes. The immense scale of belief systems guarantees many such inconsistencies. As the number of propositions in a set increases, the number of conjunctions that can be formed from those propositions grows exponentially. This ensures that consistency checking is an NP-complete problem. Consequently, even a futuristic computer must eventually be overwhelmed and fail to detect many inconsistencies.

The infeasibility of the consistency check may explain why people tolerate large-scale inconsistency. However, their tolerance may also issue from their use of acceptance rules. People believe propositions to which they assign a negligibly small chance of falsehood. Small chances of error accumulate so the same people also believe the negation of the conjunction of their beliefs. Henry Kyburg's lottery paradox crisply formulates this anti-agglomerative pattern of belief formation in his lottery paradox. Large-scale inconsistency will also be precipitated by meta-beliefs. Meta-beliefs are a distinct but closely related source of inconsistency. Given that I really have first order beliefs, my belief that some of my beliefs are false is enough to ensure that not all of my beliefs can be true. For if all my first order beliefs are true, then my second order belief is not true.

People find small-scale inconsistencies painful – the smaller the set, the more intense the pain. Consequently, most paradoxes are formulated as a set of between three and five propositions. More propositions may be involved but only as lemmas leading up to the key members of the paradox. The inverse relationship between size and pain also explains why the best known arguments have so few premises. The argument-based definition of 'paradox' requires a small size constraint for the same reasons required by the set-based definition. A set of n jointly inconsistent propositions can be turned into n valid arguments by using the remaining n – 1 members of the set as premises.

The set-based definition is often read as having individually consistent members. Indeed, they tend to be pictured as having the stronger property that each member is compatible with any non-exhaustive conjunction of the remaining members (like the inconsistent Carroll stories). The idea is that the victim of the paradox has exactly n ways to regain consistency corresponding to the n ways of rejecting a member of the paradox. Various '-isms' correspond to each solution (Rescher 1985). Logic has a role in structuring this menu of solutions. But it cannot dictate which member of paradox should be rejected.

Pierre Duhem gained fame for a similar thesis in science. Logic may dictate that we cannot believe both the theory and the conclusion based on an experiment. But it cannot tell us whether we should abandon the theory or the experiment. The physicist must instead rely on his 'good sense.'

Such thoughts provide a congenial environment for Gilbert Harman's (1986) distinction between proof and reasoning. Only reasoning concerns revision of one's beliefs and plans. Someone who believes 'If p then q' and then learns p need not conclude q. He could instead revise his belief that 'If p then q.'

The assumption of piecemeal consistency undergirds the hope that human inconsistency can be understood with a divide and conquer strategy. The divide and conqueror says that the inconsistency of a self-deceived person is the result of believing and disbelieving the same proposition in different ways (implicitly vs. explicitly, intuitively vs. theoretically, etc.). Thus the self-deceived widower *unconsciously* believes he is too old to marry his 18-year-old nanny but *consciously* believes he is not too old to marry his 18-year-old nanny. Another divide and conquer strategy is to analyze inconsistency as disagreement between parts of a person. This hope is not restricted to philosophy. When modular psychologists attribute inconsistency, they assume that there is a disagreement between self-consistent homunculi.

The divide and conquer strategy systematically fails for logical paradoxes. The belief that there is a barber who shaves all and only those who do not shave themselves is a logical contradiction that is not a conjunction of opposed propositions. Ditto for the massive family of paradoxes that involve violations of Thomson's theorem.

Many logical contradictions at the level of sentence logic are divisible. Human beings are comfortable with conjunction and negation, and so tend to couch propositions in a form amenable to the divide and conquer strategy. Since all sentential truth functions can be expressed in terms of conjunction and negation, one might hope to reduce all sentential contradictions to divisible contradictions. This seems psychologically unrealistic for belief in ostensibly non-conjunctive contradictions such as $\sim(P \supset P)$. The anthrocentricism of the reduction is also disturbing. Consider a Neanderthal who comfortably wields the Sheffer dagger function but can only fumble along with negation and conjunction. He can reduce all the contradictions of sentence logic to ones involving the dagger function. The Neanderthal's contradictions are not amenable to the divide and conquer strategy. Thus a human reduction of sentence contradictions to ones involving negation and conjunction would not show anything universal about the nature of contradictory belief.

A graver objection is that some logical contradictions are at the level of predicate logic. Many of these are clearly indivisible. When a contradiction has only variables and a single quantifier binds those variables, the contradiction is indivisible. Three illustrations: $(\exists x)(Fx \ \& \sim Fx)$, $(x)(x \neq x)$, $(\exists x)(y)(Cxy \ \& \sim Cxy)$. Any paradox that contains a logical falsehood as a member (or premise) is a logical paradox in the sentence-level sense that it contains a logical proposition.

Logical paradoxes are unique counterexamples to the principle that logic alone never implies a solution to a paradox. When one of the members of the paradox is a logical falsehood, logic *does* dictate what must be rejected. Since the inference to a logical truth is premiseless, the conclusion cannot be avoided by rejecting a premise.

Can logic itself be rejected? In *Beyond the Limits of Thought*, Graham Priest (1995) contends that the liar paradox shows that some contradictions are both true and false. He bridles against the limits of thought by rejecting standard inference rules.

If Priest is right, Duhem is wrong. For Duhem believed that standard logic structures the issues by specifying all the responses that have at least a bare chance of being true. If Priest is correct, then Duhem overlooked further true alternatives that a rational scientist might adopt. Thus Priest offers the scientist more freedom than Duhem. But is this the enhanced intellectual sweep of man who has dropped a false presupposition? Or is it the pseudo-liberty offered to Conrad Cornelius o'Donald o'Dell?

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