Chapter 1

Numerical Adjectives and the Type of Sets

In this chapter I discuss the analysis of numerical expressions inside noun phrases. More particularly, I will be interested in the semantics of noun phrases like *the three boys*, *the more than seven girls*, *the exactly ninety kids*. Let us go back in time to the period around 1980 when Generalized Qantifier Theory was established in the work of Barwise and Cooper and others (Barwise and Cooper 1981; Keenan and Faltz 1985; Keenan and Stavi 1986).

Barwise and Cooper provided a semantics for noun phrases of the form *the n NOUN*, with *n* a number expression. I will slightly generalize their analysis to noun phrases of the form *the r n NOUN*, with *r* an expression denoting a numerical relation, like *more than*, *less than*, *at least*, *at most*, *exactly*, or Ø (where *the n* is *the Ø n*). On Barwise and Cooper's analysis, *the r n* forms a partial determiner (of generalized quantifier type <d,<d,t>>, where d is the type of expressions denoting individuals), which gets its interpretation according to the following schema:

$$\textit{the r n} \rightarrow \lambda Q.\begin{cases}\lambda P.\forall x[Q(x) \rightarrow P(x)] & \text{if } |Q| \text{ r n} \\ \text{undefined} & \text{otherwise}\end{cases}$$

> The function which takes a noun interpretation Q and gives the set of all properties that every Q has, if the cardinality of Q stands in relation r to n, and is undefined otherwise.

Thus, *the at most three boys* has the same interpretation as *every boy* if there are at most three boys ($|BOY| \leq 3$), and is undefined otherwise. In general, when defined, *the r n NOUN* has the same interpretation as *every NOUN*. The conditions under which it is defined are constrained by *r* and *n*.

With the above schema, we can define:

the boy = *the exactly one boys*
the boys = *the at least one boys*
both boys= *the exactly two boys*

This account is very successful in dealing with the partiality of definite noun phrases, the conditions under which definite noun phrases are defined: *the boy* is defined if and only if there is exactly one boy, *the boys* iff there are boys, *the at most three boys* iff there are at most three boys (which includes the possibility of no boys), etc. The pragmatic assumption that noun phrases should only be used when they are defined leads to the correct presuppositions for the use of these noun phrases: i.e. the felicitous use of *the at most three boys* presupposes that there are at most three boys.

The analysis is less successful in other respects. It does not incorporate a semantic singular–plural distinction, and does not deal with distinctions between distributive and collective readings: the above account only deals with distributive readings (*the*, when defined, is *every*). Also, and this is the aspect that concerns us most here, the analysis assumes that in *the r n NOUN*, *the r n* is a determiner which combines with the noun: we have a determiner schema which generates an infinite set of determiners. A similar assumption is dominant in the work of Keenan and his co-authors (Keenan and Faltz 1985, Keenan and Stavi 1986). This aspect of the analysis has been challenged, for instance by Rothstein (1988): there are several reasons to think that the constituent structure of these noun phrases is $[[_{\text{DET}}$ *the* $]\ [_{\text{NP}}$ *r n NOUN* $]]$, and not $[[_{\text{DET}}$ *the r n*$]$ $[_{\text{NP}}$ *NOUN*$]]$ (see Rothstein 1988).

The first of these structures is supported by very strong evidence. While numerical phrases in predicate or argument indefinites must be initial in the noun phrase (i.e. they cannot mingle with adjectives), this is not so inside the nominal domain, i.e. not in the noun phrases that we are looking at here (a similar argument has been made by de Jong 1983):

(1)a. *Fifty ferocious lions* were shipped to Artis.
b. #*Ferocious fifty lions* were shipped to Artis.

(2)a. The animals in the shipment were *fifty ferocious lions*.
b. #The animals in the shipment were *ferocious fifty lions*.

(3)a. We shipped the *fifty ferocious lions* to Blijdorp, and the *thirty meek lions* to Artis.
b. We shipped the *ferocious fifty lions* to Blijdorp, and the *meek thirty lions* to Artis.

Of course, there are subtle and hard to pinpoint interpretation differences between the cases in (3a) and (3b). However, it seems that most of these can be attributed to contextual interpretation factors that we know are operative in the adjectival domain anyway, like focus, contrast, comparison set, etc. That is, we find such interpretation differences also when we consider strings of normal adjectives. The point about (3b) is the contrast with (1b) and (2b): (1b) and (2b) are crashingly bad, while (3b) is not.

A complex determiner analysis can only account for these facts if it not only allows numericals to be part of the complex determiners, but adjectives as well.

While Keenan (1987) seems prepared to make the latter assumption, it is not clear that this should be the default choice, if we can make the alternative analysis work. The alternative analysis assumes that numerical phrases (like *at most three*) are part of the noun phrase structure like adjectives. I will call the assumption that numerical phrases (and more generally, indefinites) are adjectives, at least semantically, the Adjectival Theory of Indefinites:

The Adjectival Theory of Indefinites:
Indefinites have the semantics of intersective adjectives.

To give form to the adjectival theory, we turn to the analysis of plurality originating in the work of Sharvy (1980) and Link (1983).

We assume that our interpretation domain for expressions of type d is a complete atomic Boolean algebra. I will be short here; for more details, see Landman (1991, 2000) (note that, for my purposes here, I use complete join and meet as the basic operations, instead of the standard two place operations).

Complete atomic Boolean algebras:
A **complete atomic Boolean algebra** is a structure $\mathbf{B} = \langle B, \sqcup, \sqsubseteq \rangle$, where B is a set, partially ordered by **part-of** relation $\sqsubseteq$, and for every $X \subseteq B$: $\sqcup X \in B$, where $\sqcup X$ is the **sum of** X, the smallest element of B such that for every $x \in X$: $x \sqsubseteq \sqcup X$.

Furthermore, the structure satisfies postulates (1)–(3) below, which use some of the following definitions:

Definitions:
Let $X \subseteq B$, $a,b \in B$:
$\sqcap X = \sqcup\{c \in B\text{: for every } x \in X\text{: } c \sqsubseteq x\}$
$a \sqcup b = \sqcup\{a,b\}$, $a \sqcap b = \sqcap\{a,b\}$
$0 = \sqcup\varnothing$; $1 = \sqcup B$
$\neg b = \sqcup\{c \in B\text{: } b \sqcap c = 0\}$
$\text{ATOM} = \{c \in B\text{: } c \neq 0 \text{ and for no } d \in B-\{0,c\}\text{: } d \sqsubseteq c\}$
(the set of atoms, elements that have only themselves and 0 as part)
$(b] = \{c \in B\text{: } c \sqsubseteq b\}$, the ideal generated by b (the set of all b's parts)
$[b) = \{c \in B\text{: } b \sqsubseteq c\}$, the filter generated by b (the set of all elements that b is part of)
$\text{ATOM}(b) = (b] \cap \text{ATOM}$
(the set of all b's atomic parts)
$|b| = |\text{ATOM}(b)|$
(the cardinality of element b is the cardinality of the set of its atomic parts)

Conditions:
1. **Distributivity**: if $a \sqsubseteq b \sqcup c$ then $a \sqsubseteq b$ or $a \sqsubseteq c$ or for some $b_1 \sqsubseteq b$ and some $c_1 \sqsubseteq c$: $a = b_1 \sqcup c_1$.
   (if a is part of a sum $b \sqcup c$, then it is either fully part of b or fully part of c, or the sum of some part of b and some part of c)

2. **Witness**: if $a \sqsubseteq b$ and $a \neq 0$ and $a \neq b$ then for some $c \sqsubseteq b$: $c \neq 0$ and $a \sqcap c = 0$.
(if a is a proper non-zero part of b, then there is another proper non-zero part c of b such that a and c have no non-zero part in common)
3. **Atomicity**: For every $b \in B–\{0\}$: $\text{ATOM}(b) \neq \varnothing$
(every non-zero element has atomic parts)

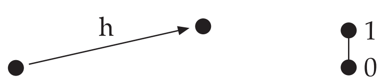
An atomic **mereology** is a complete atomic Boolean algebra with the bottom element 0 removed. It is good to point out that, while in the past I have been using atomic mereologies in the semantics of plurality, in this book it will be essential that the structures be full Boolean algebras.

Every complete atomic Boolean algebra has 2^α elements for some cardinality α, and per cardinality α there is, up to isomorphism, exactly one Boolean algebra with 2^α elements. One of the most instructive properties of these structures is their decomposition:

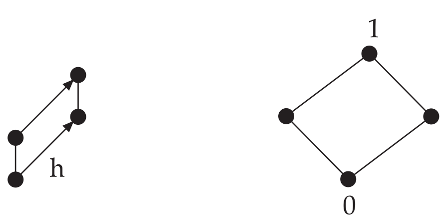
Decomposition Theorem:
Let **B** be a Boolean algebra and a an atom in B.
$\mathbf{[a)}$ and $\mathbf{(\neg a]}$ form non-overlapping isomorphic Boolean algebras (with the operations of **B** restricted to $\mathbf{[a)}$ and $\mathbf{(\neg a]}$ respectively).
Let h be an isomorphism from $\mathbf{[a)}$ into $\mathbf{(\neg a]}$.
$\mathbf{B = [a) \cup (\neg a],}$ ordered by the transitive closure of relation:
$\sqsubseteq_{[a)} \cup \sqsubseteq_{(\neg a]} \cup \{<c,h(c)>:c \in [a)\}$.

Thus, every complete atomic Boolean algebra can be decomposed into two isomorphic Boolean algebras: for any atom a: the filter generated by a, and the ideal generated by dual atom $\neg a$. Since in a finite Boolean algebra of cardinality 2^{n+1} for atom a, the cardinality of $[a)$ (and hence of $(\neg a]$) is 2^n, this theorem gives us an instructive method for generating each finite Boolean algebra:

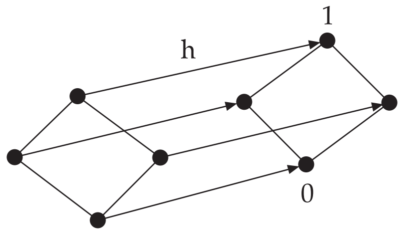
- Two one-element Boolean algebras and an isomorphism give you the two-element Boolean algebra:

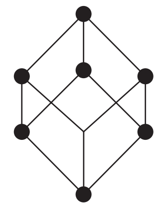


- Two two-element Boolean algebras and an isomorphism give you the four-element Boolean algebra:

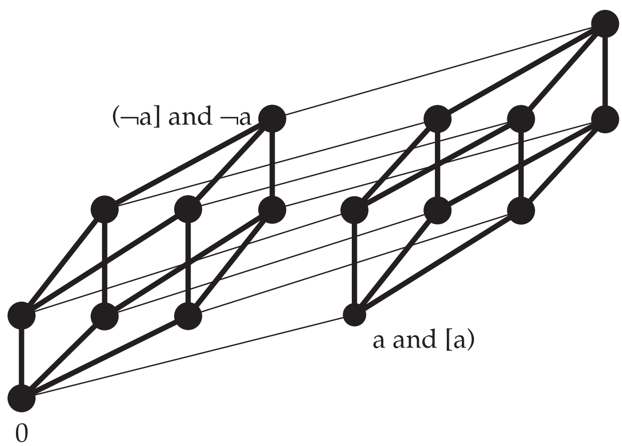


- Two four-element Boolean algebras and an isomorphism give you the eight-element Boolean algebra:





- Two eight-element Boolean algebras and an isomorphism give you the sixteen-element Boolean algebra:



As you can see in the diagram, the eight-element Boolean algebra on the left provides the 0 and three of the atoms of the sixteen-element Boolean algebra, while its 1 becomes a dual atom ¬a. The eight-element Boolean algebra on the right provides the 1 and the remaining three dual atoms of the sixteen-element Boolean algebra, while its 0 becomes the fourth atom a of the sixteen-element structure. And this happens at every level: from two Boolean algebras with 2^{α} elements and α atoms, we form a Boolean algebra with $2^{\alpha+1}$ elements and $\alpha+1$ atoms: α atoms come from the Boolean algebra on the left, one more atom is the reinterpretation of the 0 element of the Boolean algebra on the right.

Thus the structure of type d of individuals is a complete atomic Boolean algebra of singular, atomic individuals and their plural sums. We now come to the interpretation of (count) nouns. Nouns are interpreted as expressions of type <d,t>, sets of individuals. We assume that in languages such as English singular nouns lexically select singular individuals, i.e. atoms – singular nouns denote sets of atoms:

Singular nouns:
boy $\rightarrow$ BOY of type <d,t> BOY $\subseteq$ ATOM

We assume an operation of semantic pluralization (*) which we take to be closure under sum:

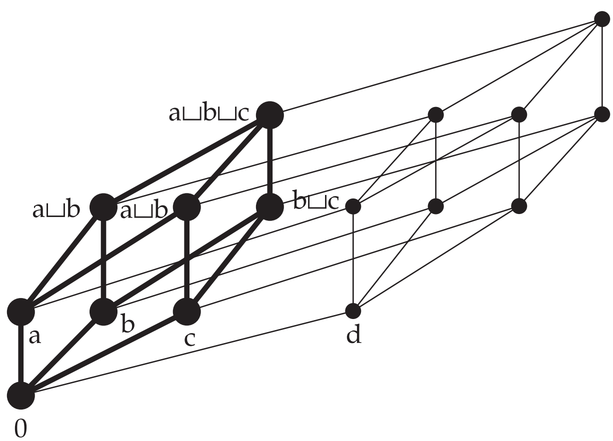
Pluralization:
$*P = \{x \in D: \exists Z \subseteq P: x = \sqcup Z\}$

And we assume that in languages like English plural morphology on nouns is (by and large) interpreted as semantic pluralization.

Plural nouns:
boys → *BOY

These assumptions, of course, go back to Link (1983).

Thus, if we assume that ATOM = {a,b,c,d} and BOY = {a,b,c}, this gives us: $*BOY = \{0,a,b,c,a\sqcup b,a\sqcup c,b\sqcup c,a\sqcup b\sqcup c\}$:



*P is the closure under sum of P. It is important to note that I use here the standard notion of closure for Boolean algebras (and not a notion modified for mereologies). On this notion, for each subset X of P, $\sqcup X \in *P$. Since one of these subsets is Ø, $\sqcup Ø \in *P$. Since $\sqcup Ø = 0$, $0 \in *P$. Thus, on the standard notion of closure under sum, $0 \in *P$.

Now, the standard assumption for intersective adjectives is that they intersect with the noun:

Intersective adjectives:
$[_{NP}$ ADJ NP $]$ → ADJ ∩ NP $(\lambda x.ADJ(x) \wedge NP(x))$

If we assume that an adjective like *young* also denotes a set of atoms, then we can make either one of two assumptions for a plural noun phrase like *young* *boys*: the adjective combines with the noun, and pluralization applies to the whole:

*(YOUNG ∩ BOY)

or the noun is pluralized and agreement triggers semantic pluralization of the adjective:

*YOUNG ∩ *BOY.

These are in essence alternative theories about where number takes its effect, but we do not at this point need to choose between them, since, by the Boolean structure,

*(YOUNG ∩ BOY) = *YOUNG ∩ *BOY

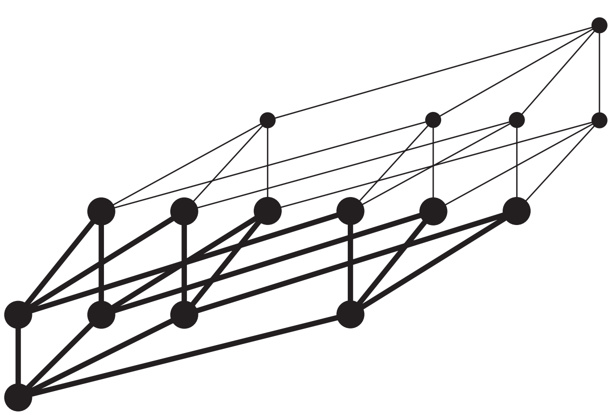
Thus, if YOUNG = {b,c,d}, *YOUNG = {0,b,c,d,b⊔c,b⊔d,c⊔d,b⊔c⊔d}, and *YOUNG ∩ *BOY = {0,b,c,b⊔c}. So *young boys* denotes the set of sums each of whose singular constituents is a young boy. All that this shows is that the standard account of intersective adjectival modification can straightforwardly be extended to the plural case.

Now we come to the adjectival theory of numericals. This theory is the assumption that numericals have the semantics of intersective adjectives. This means that they denote sets, like intersective adjectives, and that they combine with the noun through intersection. This gives us the following interpretation schema for numerical phrases:

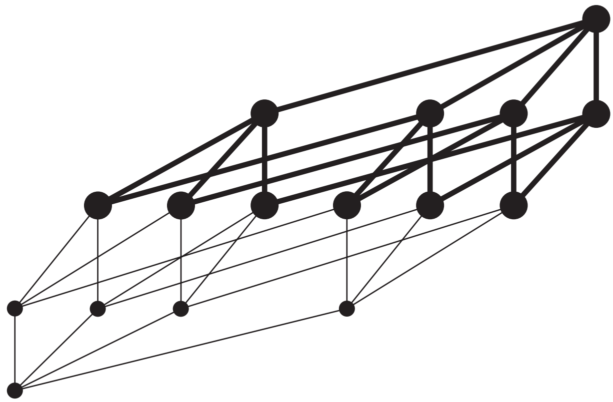
r n → $\lambda x.|x|$ r n   of type <d,t>
the set of sums whose cardinality stands in relation r to number n.

With *at most* → ≤, *at least* → ≥, *exactly* → =, we get:

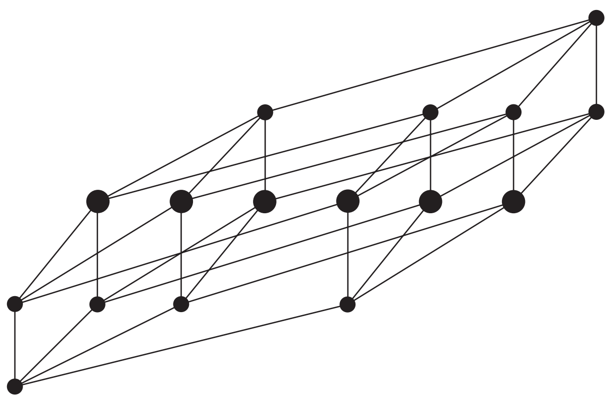
at most two → $\lambda x.|x| \leq 2$



at least two → $\lambda x.|x| \geq 2$



and *exactly two* → $\lambda x.|x| = 2$

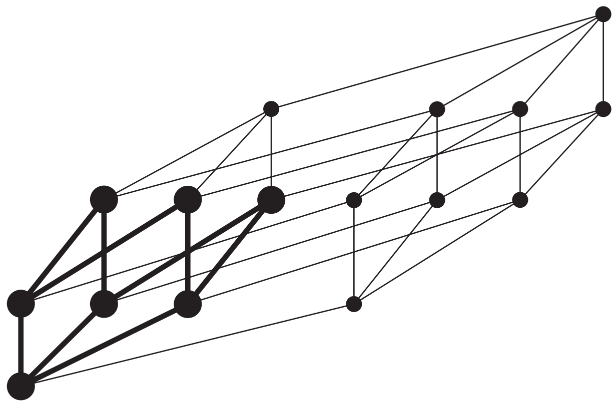


The diagrams illustrate that we can fruitfully define notions of upward and downward closure for sets of pluralities:

> Let **Y** be a Boolean algebra on a subset of **B** and let $X \subseteq Y$.
> X is **upward closed**, **UC** on **Y** iff
> if $x \in X$ and $y \in Y$ and $x \sqsubseteq y$ then $y \in X$
> X is **downward closed**, **DC** on **Y** iff
> if $x \in X$ and $y \in Y$ and $y \sqsubseteq x$ then $y \in X$

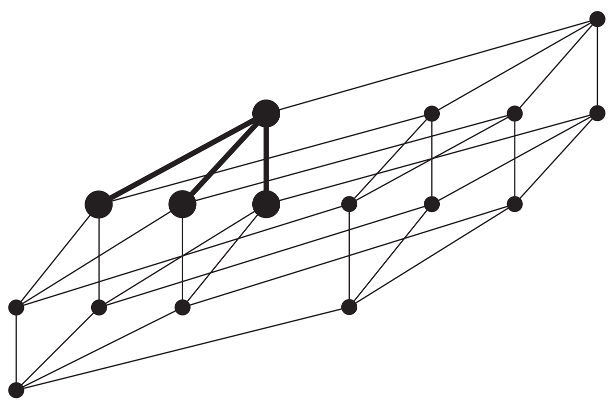
Clearly, *at least two* is UC on D, *at most two* is DC on D, *exactly two* is neither. Intersecting these three numerical phrases with *BOY gives:

at most two boys → λx.*BOY(x) ∧ |x| ≤ 2



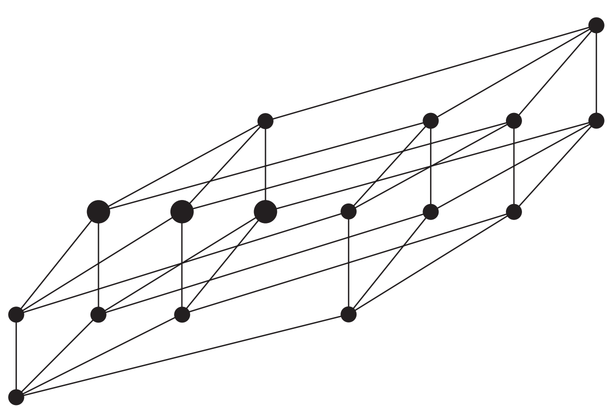
At most two boys is DC on D, and hence also on *BOY.

at least two boys → λx.*BOY(x) ∧ |x| ≥ 2



We see that *at least two boys* is not UC on D, but it is UC on *BOY. (Compare it with the diagram of the denotation of *BOY, given earlier.)

Finally, *exactly two boys* → λx.*BOY(x) ∧ |x| = 2



Exactly two boys is of course neither UC nor DC on D, or on *BOY.

So, what we can call the **polarity signature** is already determined by the numeral phrase. In fact, since the number *n* only determines the height of the interpretation in the Boolean algebra, the polarity signature is in fact determined by the numerical relation *r*. So we can define notions like downward and upward closure for numerical relations: ≤ is downward closed, ≥ is upward closed, and = is neither.

We now come to the definite article. We need to capture the presuppositional behavior of the different definite noun phrases that was captured in the schema derived from Barwise and Cooper (1981).

Link (1983) proposes the sum operator ⊔ as the interpretation of the definite article. This is not quite right for the definite article *the* in English, because it doesn't have the right presuppositional behavior (though it may be an option in certain cases where there is assumed to be an implicit definiteness operation). In later work, Link takes over the operation already proposed in Sharvy (1980), which is now usually notated as operation σ:

$$\sigma = \lambda Q.\begin{cases} \sqcup(Q) & \text{if } \sqcup(Q) \in Q \\ \text{undefined} & \text{otherwise} \end{cases}$$

The function which takes a noun interpretation Q and maps it onto the sum of the Qs if the sum of the Qs is in Q, and is undefined otherwise.

With this definition we get the following results:

the boy → σ(BOY)
BOY = {a,b,c}, ⊔BOY = a⊔b⊔c, a⊔b⊔c ∉ BOY,
Hence σ(BOY) is undefined.

If we let GIRL = {d}, then ⊔GIRL = ⊔{d} = d. d ∈ {d}, hence σ(GIRL) = d. For singular predicate P, σ(P) is defined iff P is a singleton set. Thus, for singular predicates, σ coincides with the iota operator.

the boys → σ(*BOY)
⊔(*BOY) = a⊔b⊔c, a⊔b⊔c ∈ *BOY, hence σ(*BOY) = a⊔b⊔c.
In a full Boolean algebra, if P = Ø, *P = {0} and σ(*P) = 0.

We will discuss the meaning of this later, but it will have the consequence that felicitous use of *the boys* implicates that there are boys.

Let **Y** again be a sub-Boolean algebra of **B**, X ⊆ B.
X shows **variety** on **Y** iff ⊔(X ∩ Y) = ⊔Y.

Equivalently, we can say that X shows variety on Y if each atom in Y is part of some element of X ∩ Y. (Because in a complete atomic Boolean algebra

$\sqcup(X \cap Y) = \sqcup(Y)$ iff $\cup$ {ATOM(b): $b \in X \cap Y$} = $\cup$ {ATOM(b): $x \in Y$}.) Variety is a consequence of a more general notion of **quantitativity** that I will not define here. But the intuition is as follows. Numerical phrases are quantitative restrictors. They restrict a set Y to a set $X \cap Y$ in which all the elements satisfy a certain quantitative profile, and this means that each element in $X \cap Y$ stands in a certain quantitative relation to its atoms. Quantitative means that the identity of the object and its atoms is irrelevant: any other object in Y that stands in the same quantitative relation to its atoms has the same quantitative profile and hence is in $X \cap Y$ as well.

Suppose that $b_1 \in X \cap Y$. $b_1 = \sqcup$ATOM(b_1). Let $a_1 \in$ ATOM(b_1) and $a_2 \notin$ ATOM(b_1). Now look at $b_2 = \sqcup((\text{ATOM}(b_1)-\{a_1\})\cup\{a_2\})$ (intuitively, the result of replacing in b_1 atomic part a_1 by atomic part a_2). Intuitively, b_2 has the same quantative profile as b_1, and hence b_2 is in $X \cap Y$ as well (if X is a quantitative restrictor). Since this argument can be made for any atom in Y, it follows that quantitative restrictors show variety on Y: while they kick out elements from the denotation of Y, the elements they leave in are built from the full variety of atoms in Y, since the restriction is quantitative and not qualitative.

It is easy to check that *at most one*, *at most two*, *at most three*, *at least one*, *at least two*, *at least three*, *exactly one*, *exactly two*, *exactly three* show variety on *BOY. This means that any of these noun phrases *the r n boys* will denote $\sqcup$*BOY (= a$\sqcup$b$\sqcup$c) if a$\sqcup$b$\sqcup$c is in their denotation, and will be undefined otherwise. This means that in the model given, all of the following are undefined:

the at most one boy, *the at most two boys*, *the exactly one boy*, *the exactly two boys*.

And it means that in the model given, all of the following denote a$\sqcup$b$\sqcup$c, the sum of the boys:

the at most three boys, *the at least one boy*, *the at least two boys*, *the at least three boys*, *the exactly three boys*.

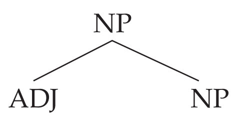
Thus we see that Link's theory of singular and plural predicates – with Sharvy's theory of the definite article as an operation with picks the maximal element out of a set, while presupposing that the set has a maximal element – and the Adjectival Theory of numericals provides the correct semantics for numerical definites.

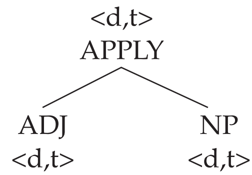
This is an appealingly simple and conceptually elegant theory:

The Adjectival Theory as part of the theory of plurality:
Singularity on predicates is atomicity.
Plurality on predicates is closure under sum.
The definite article is a maximalization operator.
Numerical phrases are adjectives.

Let's look at the compositional structure of numerical definite noun phrases in a bit more detail. Up to now, I assume the following compositional analysis:

the $\rightarrow \lambda Q.\sigma(Q)$	Q a variable of type <d,t>
NOUN $\rightarrow$ N	N of type <d,t>
r n $\rightarrow \lambda x.\|x\|$ r n	of type <d,t>

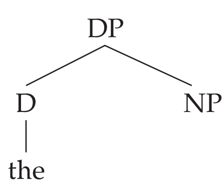


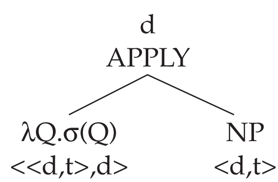


TYPE SHIFTING OPERATION ADJUNCT:
ADJUNCT: $<d,t> \rightarrow <<d,t>,<d,t>>$
ADJUNCT$[\alpha] = \lambda P\lambda x.P(x) \wedge \alpha(x)$

(With ADJUNCT, the type mismatch in APPLY[ADJ,NP] is resolved as:

APPLY[ADJUNCT[ADJ], NP]) =
APPLY[$\lambda P\lambda x.P(x) \wedge \alpha(x)$, NP] =
$((\lambda P\lambda x.P(x) \wedge \alpha(x))(NP))$ =
$\lambda x.NP(x) \wedge \alpha(x)$ of type <d,t>.





We haven't yet looked at the internal semantic structure of the numerical adjective. That is, we have treated *r n* as an unanalyzed whole.

In analyzing the numerical phrase as an adjective, we have tried to give the expressions involved – the determiner, the noun, and the adjective – the simplest and lowest possible interpretation. We now want to apply this same strategy of analysis to the internal analysis of the numerical phrase.

For that, I want to make the analysis a little bit more general. I have already suggested one way of doing that. Even though in a noun phrase like *the three boys* there is no numerical relation morphologically realized, I have given a general schema for noun phrases of the form *the r n boys*, subsuming *the three boys* under that case. This means that really I am assuming a morphologically null numerical relation Ø and a structure [*the* [[Ø *three*] [*boys*]]], with Ø interpreted as =.

I will now go one step further. Intersective plural numerical phrases like *at least three* in *the at least three boys* are part of a larger class of measure phrases like *at least three pounds (of)* in *the at least three pounds (of) sugar/boys*. Measure phrases pattern with the numerical phrases discussed here in that their semantics is **intersective**: *at least three pounds of sugar* denotes (sums of) sugar to the amount of at least three pounds. Intersective, here, means that it does in fact denote sugar. And this is, of course, true for *three boys* as well: *three boys* denotes (sums of) boys with three atoms. (While measures are intersective, classifiers in general are not. For a little more discussion, see chapter 11.)

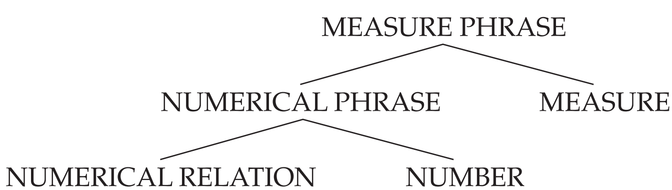
This means that the more general form of the numerical phrase is:

> *r n m* = numerical relation – number – measure
> where the count measure Ø is again morphologically not realized.

Whether you want to represent these empty elements in the syntax is not so much the point. While for clarity that is what I will be assuming, you can just as well give a syntax in which these empty elements are just not there. The point is a semantic one:

> The semantics of the numerical phrase is built from **three** semantic ingredients, even if only one is visible in *Ø three Ø*:
> **a numerical relation r**, **a number n**, **and a measure m**.

Thus, I assume the following compositional structure:



What I want to do now is provide as simple and natural as possible an interpretation for these structures.

We start with the category NUMBER. Let's assume we have a type n for numbers. Then the semantic interpretation is:

NUMBER	n
zero	0
one	1
two	2
. . .	. . .

Next, the category NUMERICAL RELATION. Obviously, as the name expresses, the simplest assumption is that expressions of this category denote relations between numbers, i.e. of type <n,<n,t>>.

NUMERICAL RELATION	<n,<n,t>>
at most	$\leq$
less than	$<$
at least	$\geq$
more than	$>$
exactly	$=$
Ø	$=$
. . .	. . .

So these are relations between numbers, nothing more: $\leq$ is the relation that 5 stands in to 7, but not to 4.

To combine the numerical relation and the number, we can follow the simplest assumption: since the types match for application, the semantics is just application:

NUMERICAL PHRASE		<n,t> APPLY	
NUMERICAL RELATION	NUMBER	r <n,<n,t>>	n n

Thus, *at most five* $\rightarrow$ APPLY($\leq$,5) = ($\leq$(5)) = $\lambda n.n \leq 5 = \{0,1,2,3,4,5\}$ (on the domain of natural numbers, of course). Hence *at most five* is a number predicate, it denotes a set of numbers of type <n,t>.

Now we come to the measures. The simplest account of measures is, obviously, to assume that they are functions from objects to numbers: the objects may be mass objects for mass measures, the numbers may be numbers on a particular scale, but that's not so important for our present purposes. What is important is the type of measures: functions in type <d,n>:

MEASURES	<d,n>	
liter	LITER	
pound	POUND	
Ø	C	where $C = \lambda x.\|x\|$
. . .		

This gives the following situation:

MEASURE PHRASE <d,t>

NUMERICAL PHRASE MEASURE (r(n)) M
<n,t> <d,n>

In this case, the simplest operation that would give an interpretation at type <d,t> from inputs <n,t> and <d,n> is not functional application, but **function composition**, so we assume that the numerical phrase composes with the measure:

<d,t>
MEASURE PHRASE COMPOSE

NUMERICAL PHRASE MEASURE (r(n)) M
<n,t> <d,n>

COMPOSE[(r(n)), M] = (r(n)) o M = λx.([r(n)]([M(x)]))

In this expression r is a relation between M(x) and n. Thus, in relational notation, we could write this as λx.r(M(x),n). We will use, in fact, infix notation for this relation, and write: λx. M(x) r n.

Thus, the measure phrase Ø *three* Ø gets the semantics: (=(3)) o C = λn.n=3 o λx.|x|.

This is: λx.([λn.n=3] ([[λy.|y|](x)])) =
λx. [λn.n=3 (|x|)] =
λx.|x|=3 of type <d,t>.

In other words, the simplest assumptions give the correct results by the simplest means: the measure phrase is built by applying a numerical relation to a number, forming a numerical predicate, and composing the numerical predicate with a measure.

The power of the adjectival theory is that it provides a simple and elegant analysis of numerical phrases in the nominal domain. And, as we will see at various points in this book, it provides a solid basis for simple compositional analyses of a variety of other nominal constructions.

I have defended the analysis [DET [$_{NP}$ NUM NOUN]] for numericals. I will end this chapter by discussing one case where I think the evidence goes the other way, and where we need to assume an analysis [[$_{DET}$ DET NUM] [$_{NP}$ NOUN]]. It concerns expressions like *every three* as in (4):

(4) *Every three lions* are sold to Artis.

Note first that, unlike in the case of determiner *the*, the numerical cannot mingle with adjectives:

(5)a. Every three ferocious lions are sold to Artis.
b. #Every ferocious three lions are sold to Artis.

A second peculiarity concerns the number. As is well known, *every* requires singular number on its head noun (6b), and triggers singular verb agreement (6c):

(6)a. Every lion is sold to Artis.
b. #Every lions is sold to Artis
c. #Every lion are sold to Artis.

(4), on the other hand, has a plural head noun *lions*, and triggers plural agreement. Semantically, *every* quantifies over atomic individuals, *every lion* quantifies over singular lions; but *every three lions* quantifies over groups of lions. That is, (4) can be roughly paraphrased as (7):

(7) The lions were sold to Artis in threes.

This means that, if we assume a structure [*every* [*three lions*]] we are going to violate just about everything we can think of about the syntax and semantics of *every*. On the other hand, if we assume a structure [*every three* [*lions*]] we can easily make all these facts fall into place.

First, obviously, the facts in (5) fall out: *three* cannot mingle with the adjectives. Secondly, we assume a standard semantics for *every*:

every $\rightarrow \lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)]$

Every requires its head noun to be a set of atoms, hence singular, and triggers singular agreement. This means, that combining it with a noun *lions* is going to be infelicitous, and similarly, combining it with *three lions* is infelicitous. *Three* on the other hand is a plural predicate, a property of non-atomic sums:

three $\rightarrow \lambda x.|x|=3$

Let's now assume that we have a complex determiner *every three*. The natural assumption will be that it is formed by composition of *every* and *three*:

$\text{COMPOSE}[\lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)], \lambda x.|x|=3]$

There is a mismatch here that needs to be resolved. One part is straightforward, we can shift $\lambda x.|x|=3$ with ADJUNCT to its modifier interpretation: $\text{ADJUNCT}[\lambda x.|x|=3]$

$\text{COMPOSE}[\lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)], \lambda Z\lambda x.Z(x) \wedge |x|=3]]$

Now, if we apply Z to a variable U – as we do in composition – we get something that has the right type to apply the interpretation of *every* to, but not the right interpretation, because *every* requires a singular predicate, a predicate of atoms, and the predicate we get, $\lambda x.U(x) \wedge |x|{=}3$, is a plural predicate, a predicate of sums. So we need to shift this predicate to a predicate of atoms. The technique for doing this will be worked out in detail in chapter 11. Here I will only sketch the basic idea. Link 1984 introduces a **group-formation** operator $\uparrow$, which maps pluralities onto group-atoms: whereas $a \sqcup b$ denotes the sum of a and b, $\uparrow(a \sqcup b)$ denotes "a and b as a group", which is an atom in its own right. Landman (1989) also introduces an inverse operation of **membership specification** $\downarrow$, which maps a group atom like $\uparrow(a \sqcup b)$ onto the sum that makes up that group (i.e. onto $a \sqcup b$). With this, we can introduce an operation that maps a set of sums onto a set of corresponding atoms:

$$^{\uparrow}P = \{\uparrow(x)\colon x \in P\}$$

With this, we resolve the interpretation mismatch as follows:

$$\text{COMPOSE}[\lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)],\ \lambda Z\lambda x.Z(x) \wedge |x|{=}3] =$$

$$\lambda U.\text{APPLY}[\lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)],\ {}^{\uparrow}(\lambda Z\lambda x.Z(x) \wedge |x|{=}3(U))] =$$

$$\lambda U.\ \text{APPLY}[\lambda Q\lambda P.Q \subseteq \text{ATOM} \wedge \forall x[Q(x) \rightarrow P(x)],\ {}^{\uparrow}(\lambda x.U(x) \wedge |x|{=}3)] =$$

$$\lambda U\lambda P.\ {}^{\uparrow}(\lambda x.U(x) \wedge |x|{=}3) \subseteq \text{ATOM} \wedge \forall x[{}^{\uparrow}(\lambda y.U(y) \wedge |y|{=}3)(x) \rightarrow P(x)] =$$
(because indeed $^{\uparrow}(\lambda x.U(x) \wedge |x|{=}3) \subseteq \text{ATOM}$)

$$\lambda U\lambda P.\ \forall x[{}^{\uparrow}(\lambda y.U(y) \wedge |y|{=}3)(x) \rightarrow P(x)].$$

Equivalently:

every three $\rightarrow \lambda U\lambda P.\forall a[\text{ATOM}(a) \wedge U(\downarrow(a)) \wedge |\downarrow(a)|{=}3 \rightarrow P(a)]$
The relation that holds between properties P and U if every group consisting of a sum of three Us has property P.

We see that this new determiner no longer has the requirement that the noun be singular, on the contrary, it maps a **plural noun** like *lions* onto the set of properties that every group correlate of sums of three lions has. Note that the universal quantification is over groups, not over sums, hence it can be contextually restricted to relevant groups (e.g. Landman 1989). Thus, the complex determiner *every three* takes as input semantically plural predicates, so it is not a surprise that it wants morphologically plural nouns:

every three lions $\rightarrow \lambda P.\forall a[{*}\text{LION}(\downarrow a) \wedge |\downarrow a|{=}3 \rightarrow P(a)]$
The set of properties that every group consisting of a sum of three lions has.

The generalized quantifier *every three lions* denotes a set of properties of groups. This gives the correct semantics. Since the plural noun *lions* is the head noun, the whole noun phrase is morphologically plural, and it triggers plural agreement.

Thus, in this case, the assumption that we have a complex determiner *every three* makes perfect semantic and morphological sense. But, of course, that only strengthens the case against imposing the very same analysis onto the other noun phrases: when we have a complex determiner, as we do here, almost everything in the grammar jumps up and down to signal that we do, indicating that we can rest reasonably assured that in the other cases we don't. Thus the exceptional nature of this complex provides support for the adjectival analysis of the other cases.